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# A PRIORI BOUNDS FOR SOLUTIONS OF PARABOLIC PROBLEMS AND APPLICATIONS

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*Abstract.* We review some recent results concerning a priori bounds for solutions of superlinear parabolic problems and their applications.

*Keywords*: a priori estimate, blow-up rate, periodic solution, multiplicity

*MSC 2000*: 35B45, 35K60, 35J65

## 1. Introduction

In this paper we study mainly parabolic problems of the form

$$(1.1)\qquad \begin{cases} u_t - \Delta u = f(x,u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \Gamma,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega$ is a domain in $\mathbb{R}^n$ with a smooth compact boundary $\Gamma$ and $f$ is a Carathéodory function which is superlinear in $u$. Some generalizations and modifications of (1.1) are also considered.

It is well known that under suitable assumptions on $f$ the problem (1.1) is well posed in an appropriate Banach space $X$ ($X = L_\infty(\Omega)$, for example). Denote by $u(t,u_0)$ the solution of this problem and let $T_{\max}(u_0)$ be its maximal existence time. Assume $\delta > 0$. Our main aim is to show that for a large class of nonlinearities, the norm of $u(t,u_0)$, $t \in [0, T_{\max}(u_0) - \delta)$, can be bounded by a constant which depends only on $\delta$ and on the norm of the initial condition $u_0$. In other words, we are interested in the estimate

$$(1.2)\qquad \|u(t,u_0)\|_X \leqslant C(\delta, c_0) \quad \begin{cases} \text{for any } u_0 \in X \text{ with } \|u_0\|_X \leqslant c_0, \\ \text{and any } t < T_{\max}(u_0) - \delta, \end{cases}$$

where $T_{\max}(u_0) - \delta = \infty$ and $C(\delta, c_0)$ does not depend on $\delta$ if $T_{\max}(u_0) = \infty$. Note that under some circumstances global solutions are bounded even if the estimate (1.2) does not hold for these solutions, see V. Galaktionov and J. L. Vázquez [16] or M. Fila and P. Poláčik [13]. For a survey on the boundedness of global solutions we refer to [12].

We shall also mention some results on universal bounds of the form

$$\|u(t, u_0)\|_X \leqslant C(\delta_1, \delta_2) \qquad \text{for any } t \in (\delta_1, T_{\max}(u_0) - \delta_2), \tag{1.3}$$

where the constant $C(\delta_1, \delta_2)$ does not depend on $u_0$ at all.

The bound (1.2) has several important consequences. It implies the continuity of the maximal existence time $T_{\max}\colon X \to (0, \infty]$ and plays a crucial role in establishing the blow-up rate of blowing-up solutions, in the study of domains of attraction of stable equilibria and connecting orbits between various equilibria. It can also be used for the proof of existence of multiple stationary and periodic solutions.

Let us first discuss the model case $f(x, u) = |u|^{p-1}u$, $p > 1$, $\Omega \subset \mathbb{R}^n$ bounded. Set

$$p_S := (n+2)/(n-2) \text{ if } n \geqslant 3, \quad p_S := \infty \text{ otherwise.}$$

The bounds (1.2) and (1.3) and their proofs are strongly related to the a priori estimates for positive stationary solutions of (1.1) which were proved in the subcritical case $p < p_S$ by D. G. de Figueiredo, P.-L. Lions and R. D. Nussbaum [11] and B. Gidas and J. Spruck [17] (partial results were obtained earlier by R. E. L. Turner [36], R. D. Nussbaum [26], H. Brézis and R. E. L. Turner [5]). Due to the result of S. I. Pohozaev [27], the condition $p < p_S$ is optimal in these estimates (at least if $\Omega$ is starshaped). The bound (1.2) for the time-dependent solutions of this model problem was derived for any $p < p_S$ by the author in [28] under the assumption $T_{\max}(u_0) = \infty$. Partial results requiring a stronger condition on $p$ and/or nonnegativity of $u$ were previously obtained by W.-M. Ni, P. E. Sacks and J. Tavantzis [25], T. Cazenave and P.-L. Lions [6] and Y. Giga [18]. The condition $p < p_S$ is optimal again.

Considering a general superlinear function $f$, the results on a priori estimates for positive stationary solutions mentioned above are far from satisfactory: they require either $\Omega$ to be convex or various technical conditions on $f$ (either monotonicity of $u \mapsto f(x, u)u^{-p_S}$ in [11] or a precise asymptotic behavior of $f(x, u)$ as $u \to +\infty$ in [17]). From this point of view it is interesting to know to what extent one can generalize the results of [28] concerning the estimate (1.2) for the time-dependent solutions. The approach in [28] is based on a bootstrap argument, interpolation, energy and maximal regularity estimates. It turns out that the assumption $T_{\max}(u_0) = \infty$ and the precise asymptotic behavior of the nonlinearity $f$ as $|u| \to \infty$ are not important for this approach. Moreover, the results remain true for more general differential operators, boundary conditions and nonlinearities.

In Section 2 we discuss the estimate (1.2) for (1.1) and some of its consequences (including continuity of $T_{\max}$ and existence of nontrivial equilibria) in the case of a bounded spatial domain $\Omega$. In Section 3 we study the unbounded domain case. Section 4 is devoted to time-dependent nonlinearities and the existence of periodic solutions. In Section 5 we briefly mention some results on the universal bound (1.3) and initial and final blow-up rates. In Sections 6, 7 and 8 we deal with nonlinear boundary conditions, nonlocal problems and problems involving measures, respectively. For one-dimensional problems we refer to [31, Section 6] and [7, Section 5].

## 2. Bounded domains

Denote $F(x,u) := \int_0^u f(x,v)\,\mathrm{d}v$ and assume that there exist positive constants $p_1, p_2, d_1, d_2, d_3, d_4, \beta, r$ and nonnegative functions

$$a_1 \in L_{(p_1+1)/p_1}(\Omega), \quad a_2 \in L_{(p_2+1)/p_2}(\Omega), \quad a_3 \in L_1(\Omega), \quad a_4 \in L_\beta(\Omega) \tag{2.1}$$

such that

$$1 < p_1 \leqslant p_2 < p_S, \quad d_3 > 2, \quad \beta > n/2, \quad r < p_S, \tag{2.2}$$

$$|f(x,u)| \leqslant d_2|u|^{p_2} + a_2(x), \tag{2.3}$$

$$f(x,u)\mathrm{sign}(u) \geqslant d_1|u|^{p_1} - a_1(x), \tag{2.4}$$

$$f(x,u)u \geqslant d_3F(x,u) - a_3(x), \tag{2.5}$$

$$|f(x,u) - f(x,v)| \leqslant d_4\big(a_4(x) + |u|^{r-1} + |v|^{r-1}\big)|u - v|. \tag{2.6}$$

Assume also that either $p_2 < p_{CL}$ or

$$p_2 - p_1 < \kappa_1(p_2), \tag{2.7}$$

where $\kappa_1\colon (1, p_S) \to (0,\infty)$ is defined in [31] (cf. Figures 1 and 2 below) and

$$p_{CL} := (3n+8)/(3n-4) \text{ if } n \geqslant 2, \quad p_{CL} := \infty \text{ if } n = 1.$$

Set

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x.$$

Then we have the following theorem (see [31] and [32]).

**Theorem 2.1.** *Consider the problem* (1.1). *Let* $\Omega$ *be a smoothly bounded domain in* $\mathbb{R}^n$. *Assume* (2.1)–(2.6) *and either* $p_2 < p_{CL}$ *or* (2.7). *Set* $X := H_0^1(\Omega)$. *Then the estimate* (1.2) *is true,* $T_{\max}\colon X \to (0,\infty]$ *is continuous and*

$$E\big(u(t,u_0)\big) \to -\infty \quad \text{as} \quad t \to T_{\max}(u_0)- \qquad \text{if } T_{\max}(u_0) < \infty. \tag{2.8}$$

*If, in addition,* $\beta > n$, $f(\cdot,0) \in L_\beta(\Omega)$, $u_s$ *is an asymptotically stable equilibrium of* (1.1) *in* $X$ *and* $D_A$ *denotes its domain of attraction,*

$$D_A = \{u_0 \in X\colon\ u(t,u_0) \text{ exists globally, } u(t,u_0) \to u_s \text{ in } X \text{ as } t \to \infty\},$$

*then there exist stationary solutions* $u_+, u_-, \tilde u \in \partial D_A$ *of* (1.1) *such that* $u_+ > u_s > u_-$ *and* $\tilde u - u_s$, $\tilde u - u_+$, $\tilde u - u_-$ *change sign.*

Remarks 2.1. (i) The condition (2.7) in Theorem 2.1 seems to be of technical nature. In fact, if

$$f(x,u)u \leqslant d_5 F(x,u) + a_5(x), \qquad d_5 > 0,\ a_5 \in L_1(\Omega),$$

then this assumption can be replaced by

$$p_2 - p_1 < \kappa_2(p_2), \tag{2.9}$$

where $\kappa_2\colon (1,p_S) \to (0,\infty)$ is defined in [31], $\kappa_2 > \kappa_1$ (see Figures 1 and 2). The same is true for all assertions in the subsequent sections.

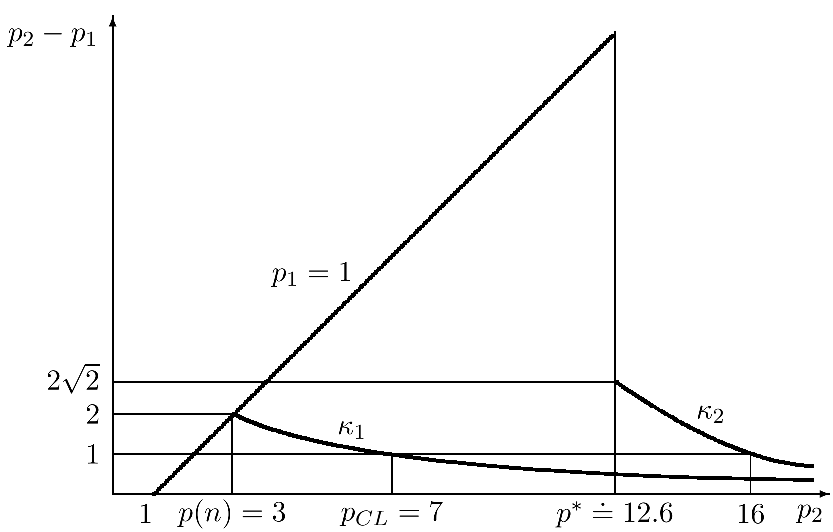


Figure 1. Functions $\kappa_1$, $\kappa_2$ for $n = 2$

In Figures 1 and 2 we set $p(n) := 1 + 4/n$,

$$p^* := \begin{cases} \dfrac{9n^2 - 4n + 16\sqrt{n(n-1)}}{(3n-4)^2} & \text{if } n \geqslant 2, \\ +\infty & \text{if } n = 1. \end{cases}$$

Note that the condition (2.7) or (2.9) is superfluous if $p \leqslant p(n)$ or $p < p^*$, respectively.

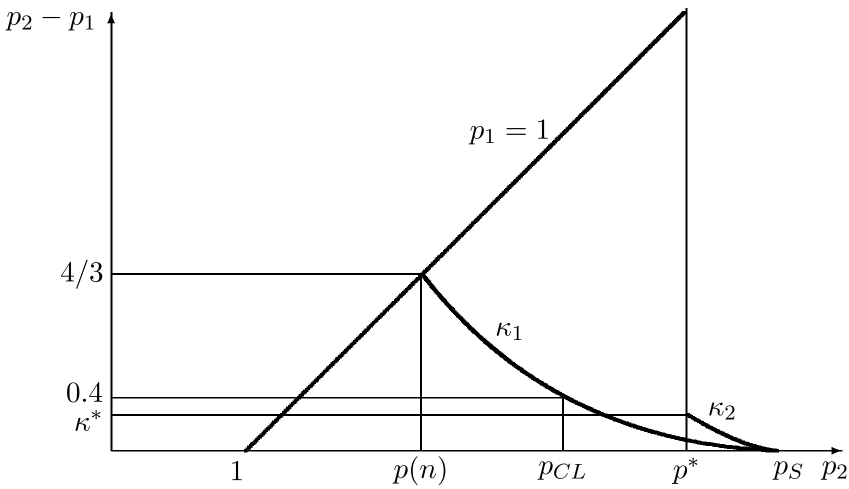


Figure 2. Functions $\kappa_1$, $\kappa_2$ for $n = 3$: $p(n) = 2 + 1/3$, $p_{CL} = 3.4$, $p^* \doteq 4.3$, $p_S = 5$, $\kappa^* \doteq 0.27$

(ii) The property (2.8) plays an important role in the proof of complete blow-up, see [2]. This property was proved earlier by H. Zaag [37] for the model case $f(x, u) = |u|^{p-1}u$ under additional assumptions $p(3n - 4) < 3n + 8$ or $u \geqslant 0$.

(iii) Continuity of $T_{\max}$ for nonnegative solutions, bounded domains $\Omega$ and convex functions $f = f(u)$ with subcritical growth was previously proved by P. Baras and L. Cohen [2]. Note that the function $T_{\max}$ need not be continuous in the supercritical case, due to a result of V. Galaktionov and J. L. Vázquez [16]. More precisely, the set $\{u_0\colon T_{\max}(u_0) = \infty\}$ need not be closed.

(iv) If $u_s = 0$ in Theorem 2.1 then this theorem guarantees the existence of a sign-changing equilibrium $\tilde{u}$ of (1.1) lying on $\partial D_A$. Similar assertions (without the information $\tilde{u} \in \partial D_A$) were proved by variational and topological methods by many authors: see the discussion in [32], for example.

## 3. Unbounded domains

Let $F$ and $E$ be the same as in Section 2. Assume that there exist positive constants $p_1, p_2, d_1, d_2, d_3, d_4, \beta, r$ satisfying (2.2) and nonnegative constants $e_1, C_1$ such that

$$|f(x,u)| \leqslant d_2(|u|^{p_2} + |u|) + a_2(x), \tag{3.1}$$

$$f(x,u)\operatorname{sign}(u) \geqslant d_1|u|^{p_1} - e_1|u| - a_1(x), \tag{3.2}$$

$$f(x,u)u \geqslant d_3F(x,u) + C_1u^2 - a_3(x), \tag{3.3}$$

$$|f(x,u) - f(x,v)| \leqslant \big(a_4(x) + d_4\big(1 + |u|^{r-1} + |v|^{r-1}\big)\big)|u-v|, \tag{3.4}$$

$$f(\cdot,0) \in L_\beta(\Omega), \tag{3.5}$$

where $a_1, a_2, a_3, a_4$ satisfy (2.1). Notice that the assumptions (3.1)–(3.4) are equivalent to (2.3)–(2.6) if $\Omega$ is bounded. The conditions above guarantee, in particular, that the problem (1.1) is well posed in $H_0^1(\Omega)$. Denote by $T_{\max}(u_0)$ the maximal existence time of the solution in $H_0^1(\Omega)$. Then we have the following theorem (see [31]).

**Theorem 3.1.** *Let $\Omega \subset \mathbb{R}^n$ have a smooth compact boundary (or let $\Omega$ be a half-space). Assume (2.1)–(2.2), (3.1)–(3.5) and (2.7). Set $X := H_0^1(\Omega) \cap L_{(p_2+1)/p_2}(\Omega) \cap L_\infty(\Omega)$, assume $u_0 \in X$ and let*

$$T^X_{\max}(u_0) := \sup\{t \in [0, T_{\max}(u_0))\colon\ u(\tau) \in X \text{ for } \tau \leqslant t\}.$$

*Then the following holds:*

*(i) $T^X_{\max}(u_0) = T_{\max}(u_0)$, $T_{\max}\colon X \to (0,\infty]$ is continuous and (2.8) is true.*

*(ii) Let $C_1 > 0$ in (3.3) and let there exist constants $d_6, \lambda > 0$, $\alpha \in (1, p_2)$, a nonnegative function $a_6 \in L_{(p_2+1)/p_2}(\Omega)$ and a bounded measurable function $V\colon \Omega \to [\lambda, \infty)$ such that*

$$|f(x,v) + V(x)v| \leqslant d_6\big(|v|^{p_2} + |v|^\alpha\big) + a_6(x).$$

*Let $u_0 \in X$ and $T_{\max}(u_0) = \infty$. Then there exists a constant $C = C(\|u_0\|_X)$ such that*

$$\|u(t)\|_X \leqslant C \qquad \text{for any } t \geqslant 0. \tag{3.6}$$

Remarks 3.1. (i) We are not able to show the bound (1.2) if $T_{\max}(u_0) < \infty$. Consequently, the proof of continuity of $T_{\max}$ requires some additional arguments (using a refinement of the concavity method due to H. A. Levine [23]). Note that all previous results concerning the estimate (3.6) and the continuity of $T_{\max}$ required a stronger assumption on the growth of $f$ or were restricted to nonnegative solutions and nonlinearities with a precise asymptotic behavior (see C. Fermanian Kammerer, F. Merle and H. Zaag [10], for example).

(ii) If $\lambda > 0$ and $1 < p < p_S$ then $f(x,u) := |u|^{p-1}u - \lambda u$ satisfies all assumptions of Theorem 3.1 (ii).

## 4. Periodic solutions

In this section we study a priori estimates of solutions and existence of positive periodic solutions of the problem

$$\begin{cases} u_t - \Delta u = m(t)f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \Gamma,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$, $m > 0$ is $T$-periodic and $f(u) = |u|^{p-1}u$, $1 < p < p_S$. We refer to [32] for the case of a general superlinear function $f = f(u)$ and to [22] for the case $f = f(x,u)$.

**Theorem 4.1** (see [32]). *Let $\Omega \subset \mathbb{R}^n$ be smoothly bounded, let $m \in W^1_\infty(\mathbb{R})$ be $T$-periodic, $m(t) \geqslant m_0 > 0$ for any $t$, $f(u) = |u|^{p-1}u$, $1 < p < p_S$. Set $X := H^1_0(\Omega)$.*

(i) *Let $u$ be the solution of* (4.1), *$T_{\max}(u_0) \geqslant T + \delta$, $\delta > 0$. Then there exists a constant $C = C(\|u_0\|_X, \delta, T)$ such that*

$$\|u(t)\|_X \leqslant C \quad \text{for any } t \in [0,T].$$

(ii) *Assume*

$$\frac{\big(m'(t)\big)^-}{m(t)} < \frac{2n - (n-2)(p+1)}{r^2(\Omega)} \qquad \text{for a.a. } t \in (0,T), \tag{4.2}$$

*where $r(\Omega)$ is the radius of the smallest ball containing $\Omega$ and $a^- := \max(0,-a)$. Then there exists at least one positive $T$-periodic solution of* (4.1) *and there exists $C > 0$ such that any positive $T$-periodic solution of* (4.1) *satisfies*

$$\|u(t)\|_X \leqslant C \quad \text{for any } t \in [0,T].$$

Remarks 4.1. (i) The technical assumption (4.2) is superfluous if $p(n-2) < n$.

(ii) Existence of positive periodic solutions of (4.1) with $f(u) = |u|^{p-1}u$ (and more general nonlinearities) was obtained earlier by M. J. Esteban in [8] and [9] under the additional assumptions $(3n-4)p < 3n+8$ and $p(n-2) < n$, respectively.

(iii) Assertion (i) in Theorem 4.1 is based on the fact that the functional

$$V\big(u(t)\big) = \frac{1}{2}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x - m(t)\int_\Omega F\big(u(t)\big)\,\mathrm{d}x$$

is "almost" a Lyapunov functional for (4.1). A Pohozaev's type identity plays a significant role in the proof of (ii) (cf. [11] in the elliptic case).

(iv) A different approach to problems without variational structure can be found in [33].

## 5. Universal bounds and blow-up rates

In this section we are interested in the universal bound (1.3) for positive solutions of (1.1) (note that this bound cannot be true for all solutions, in general). The following theorem follows from the results in [35].

**Theorem 5.1.** *Consider the problem* (1.1) *with* $\Omega \subset \mathbb{R}^n$ *being (smoothly) bounded and convex,* $f(x,u) = |u|^{p-1}u$, $1 < p < p_S$, $u_0 \geqslant 0$. *Let* $p(n-3) < n-1$ *if* $n \geqslant 5$ *and* $T_{\max}(u_0) \geqslant T_0 > 0$. *Set* $X := L_\infty(\Omega)$. *Then there exist* $C(p,\Omega,T_0) > 0$ *and* $\alpha = \alpha(n,p) > 0$ *such that*

$$\|u(t)\|_X \leqslant C(p,\Omega,T_0)\big(1 + t^{-\alpha} + (T_{\max}(u_0) - t)^{-1/(p-1)}\big)$$

*for any* $t \in \big(0, T_{\max}(u_0)\big)$, *where* $(T_{\max}(u_0) - t)^{-1/(p-1)} := 0$ *if* $T_{\max}(u_0) = \infty$.

Remarks 5.1. (i) The convexity of $\Omega$ is needed only for the estimate of $u(t)$ close to $T_{\max}(u_0)$. The assumption $p < (n-1)/(n-3)$ for $n \geqslant 5$ seems to be of technical nature.

(ii) If $p < 1 + 2/(n+1)$ then one can choose $\alpha = (n+1)/2$ in Theorem 5.1 and this choice is optimal. Note that this initial blow-up rate exponent is different from the corresponding exponent for the homogeneous Neumann problem (see [35]).

(iii) Due to the result of M.-F. Bidaut-Véron in [4] concerning the Cauchy problem, one can conjecture that the choice $\alpha = 1/(p-1)$ should be possible (and optimal) for $p \geqslant 1 + 2/(n+1)$ but this seems to be an open problem.

(iv) The (final) blow-up rate estimate

$$\|u(t)\|_X \leqslant C(p,\Omega,u_0)(T_{\max}(u_0) - t)^{-1/(p-1)} \tag{5.1}$$

(where $C$ depends on $u_0$!) is true also for sign-changing solutions and any $p \in (1, p_S)$ if $\Omega = \mathbb{R}^n$. This follows from a very recent result of Y. Giga, S. Matsui and S. Sasayama based on the approach in [28]. If $p(3n-4) < 3n+8$ or $u_0 \geqslant 0$ and $p < p_S$ then (5.1) was proved by Y. Giga and R. V. Kohn [19] for both unbounded and bounded convex domains. On the other hand, it is known that such an estimate fails, in general, for $p \geqslant p_S$, see the results of S. Filippas, M. A. Herrero and J. J. L. Velázquez in [15], [20] and [21]. Concerning universal blow-up rate estimates for positive solutions in unbounded domains we refer to J. Matos and Ph. Souplet [24].

(v) First results concerning universal bounds for global positive solutions of (1.1) with $f(x,u) = |u|^{p-1}u$ and $\Omega$ bounded were obtained by M. Fila, Ph. Souplet, F. Weissler in [14] and the author in [30].

## 6. Nonlinear boundary conditions

In this section we study a priori estimates for global solutions of the problem

$$\begin{cases} u_t = \Delta u - au, & x \in \Omega,\ t \in (0,\infty), \\ u_\nu = |u|^{q-1}u, & x \in \Gamma,\ t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{6.1}$$

where $a > 0$, $q > 1$, $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$ and $\nu$ denotes the outer unit normal on the boundary $\Gamma$. Since we study only global solutions, the bounds (1.2) and (1.3) have the form

$$\|u(t)\|_X \leqslant C(\|u_0\|_X) \qquad \text{for any } t > 0, \tag{6.2}$$

$$\|u(t)\|_X \leqslant C(\delta) \qquad \text{for any } t > \delta. \tag{6.3}$$

The following result is proved in [34].

**Theorem 6.1.** *Consider the problem* (6.1). *Let* $X := H^1(\Omega)$ *and* $q(n-2) < n$.

(i) *Let* $T_{\max}(u_0) = \infty$. *If* $u_0 \geqslant 0$ *or* $q < q^*$, *where*

$$q^* = \begin{cases} +\infty & \text{if } n = 1, \\ \bigl(9n^2 - 22n + 24 + 8\sqrt{4n^2 - 10n + 8}\bigr)/(3n-4)^2 & \text{if } n > 1, \end{cases}$$

*then the bound* (6.2) *is true.*

(ii) *Assume* $q(n-4) < n-3$ *if* $n \geqslant 7$. *Then the bound* (6.3) *is true for all global nonnegative solutions of* (6.1).

R e m a r k s 6.1. (i) The value $q_S := n/(n-2)$ is the limiting exponent for which the trace operator maps $H^1(\Omega)$ into $L_{q+1}(\Gamma)$. Unlike the case of the homogeneous Dirichlet boundary condition, it is not clear whether the subcriticality condition $q < q_S$ is necessary for the a priori bounds mentioned above.

(ii) The assumptions $q < q^*$ and $q < (n-3)/(n-4)$ for $n \geqslant 7$ seem to be of technical nature.

(iii) The validity of (1.2) or (1.3) for non-global solutions is an open problem.

## 7. Nonlocal problems

As already mentioned in the introduction, the estimate (1.2) can be derived for more general problems than (1.1). For example, in [31] we considered two nonlocal problems, which were frequently studied from the point of view of blow-up and global existence in the past decade (see the references in [31]). For both these problems we derived the estimate (1.2) and the continuity of the blow-up time.

The first problem has the form

$$\begin{aligned} u_t - \Delta u &= f\big(x, u(x,t)\big) - \frac{1}{|\Omega|}\int_\Omega f\big(x,u(x,t)\big)\,\mathrm{d}x, && x \in \Omega,\ t > 0,\\ u_\nu &= 0, && x \in \Gamma,\ t > 0,\\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}$$

where $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$ and $f(x,\cdot)$ is a superlinear function (in particular, one can choose $f(x,u) = |u|^{p-1}u$, $p_S > p > 1$).

The second nonlocal problem has the form

$$\begin{aligned} u_t - \Delta u &= \varphi\Big(\int_\Omega F(u)\,\mathrm{d}x\Big) f(u), && x \in \Omega,\ t > 0,\\ u &= 0, && x \in \Gamma,\ t > 0,\\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}$$

where $f = F'$, $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$ and either

$$F(u) = \frac{1}{p+1}|u|^{p+1}, \quad \varphi(s) = (s+1)^{-\alpha}, \quad 1 < p < p_S, \quad 0 \leqslant \alpha < \frac{p-1}{p+1},$$

or

$$F(u) = \mathrm{e}^u, \quad \varphi(s) = s^{-q}, \quad 0 < q < 1, \quad n = 1.$$

Notice that the assumption (2.3) in Section 2 requires $f(\cdot,0) \in L_{(p_2+1)/p_2}(\Omega)$ and that even a stronger assumption on the integrability of $f(\cdot,0)$ is required in the second part of Theorem 2.1. If $f(\cdot,0)$ is less regular then we can still expect similar results as in Theorem 2.1 provided we restrict the range for the exponent $p_2$. Consider, for example, the model problem

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u + a\mu, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \Gamma,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{8.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, $\mu$ is a positive bounded Radon measure on $\Omega$, $a > 0$ and $1 < p$, $p(n-2) < n$. The restriction on $p$ is necessary for the local solvability of (8.1).

It is known (see [3] or [1]) that

$$a^* := \sup\{a > 0\colon\ (8.1) \text{ has a positive equilibrium}\} > 0.$$

Set $X := \{u \in W_q^z(\Omega)\colon\ u = 0 \text{ on } \Gamma\}$, where

$$-\frac{n}{p} \leqslant z - \frac{n}{q} < 2 - n, \qquad q > 1, \quad z \geqslant 0, \quad z \neq 1/q.$$

The following result from [29] is restricted to global solutions of (8.1), but we believe that a complete analogue to Theorem 2.1 can be proved.

**Theorem 8.1.** *Let $\Omega$, $n$, $p$, $\mu$, $a^*$, $X$ be as above and let $0 < a < a^*$. Let $u$ be a global solution of* (8.1). *Then $\|u(t)\|_X \leqslant C(\|u_0\|_X)$.*

*Let $u_s$ be the minimal positive stationary solution of* (8.1). *Then there exist stationary solutions $u_+, u_-, \tilde{u}$ of* (8.1) *such that $u_+ > u_s > u_-$ and the function $\tilde{u} - u_s$ changes sign.*

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*References*

[1] *H. Amann, P. Quittner*: Elliptic boundary value problems involving measures: existence, regularity, and multiplicity. Adv. Differ. Equ. *3* (1998), 753–813.

[2] *P. Baras, L. Cohen*: Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation. J. Funct. Anal. *71* (1987), 142–174.

[3] *P. Baras, M. Pierre*: Critère d'existence de solutions positives pour des équations semi-linéaires non monotones. Analyse Non Linéaire, Ann. Inst. H. Poincaré *2* (1985), 185–212.

[4] *M.-F. Bidaut-Véron*: Initial blow-up for the solutions of a semilinear parabolic equation with source term. Equations aux dérivées partielles et applications, articles dédiés à Jacques-Louis Lions, Gauthier-Villars, Paris. 1998, pp. 189–198.

[5] *H. Brézis, R. E. L. Turner*: On a class of superlinear elliptic problems. Commun. Partial Differ. Equations *2* (1977), 601–614.

[6] *T. Cazenave, P.-L. Lions*: Solutions globales d'équations de la chaleur semi linéaires. Commun. Partial Differ. Equations *9* (1984), 955–978.

[7] *M. Chipot, M. Fila, P. Quittner*: Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions. Acta Math. Univ. Comen. *60* (1991), 35–103.

[8] *M. J. Esteban*: On periodic solutions of superlinear parabolic problems. Trans. Amer. Math. Soc. *293* (1986), 171–189.

[9] *M. J. Esteban*: A remark on the existence of positive periodic solutions of superlinear parabolic problems. Proc. Amer. Math. Soc. *102* (1988), 131–136.

[10] *C. Fermanian Kammerer, F. Merle, H. Zaag*: Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. Math. Ann. *317* (2000), 347–387.

[11] *D. G. de Figueiredo, P.-L. Lions, R. D. Nussbaum*: A priori estimates and existence of positive solutions of semilinear elliptic equations. J. Math. Pures Appl. *61* (1982), 41–63.

[12] *M. Fila*: Boundedness of global solutions of nonlinear parabolic problems. Proc. of the 4th European Conf. on Elliptic and Parabolic Problems, Rolduc 2001. To appear.

[13] *M. Fila, P. Poláčik*: Global solutions of a semilinear parabolic equation. Adv. Differ. Equ. *4* (1999), 163–196.

[14] *M. Fila, P. Souplet, F. Weissler*: Linear and nonlinear heat equations in $L^q_\delta$ spaces and universal bounds for global solutions. Math. Ann. *320* (2001), 87–113.

[15] *S. Filippas, M. A. Herrero, J. J. L. Velázquez*: Fast blow-up mechanism for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity. R. Soc. Lond. Proc. Ser. A *456* (2000), 2957–2982.

[16] *V. Galaktionov, J. L. Vázquez*: Continuation of blow-up solutions of nonlinear heat equations in several space dimensions. Commun. Pure Applied Math. *50* (1997), 1–67.

[17] *B. Gidas, J. Spruck*: A priori bounds for positive solutions of nonlinear elliptic equations. Commun. Partial Differ. Equations *6* (1981), 883–901.

[18] *Y. Giga*: A bound for global solutions of semilinear heat equations. Commun. Math. Phys. *103* (1986), 415–421.

[19] *Y. Giga, R. V. Kohn*: Characterizing blowup using similarity variables. Indiana Univ. Math. J. *36* (1987), 1–40.

[20] *M. A. Herrero, J. J. L. Velázquez*: Explosion de solutions d'équations paraboliques semilinéaires supercritiques. C. R. Acad. Sci. Paris Sér. I Math. *319* (1994), 141–145.

[21] *M. A. Herrero, J. J. L. Velázquez*: A blow up result for semilinear heat equations in the supercritical case. Preprint.

[22] *J. Húska*: Periodic solutions in superlinear parabolic problems. Acta Math. Univ. Comen. To appear.

[23] *H. A. Levine*: Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. Arch. Rational Mech. Anal. *51* (1973), 371–386.
[24] *J. Matos, Ph. Souplet*: Universal blow-up estimates and decay rates for a semilinear heat equation. Preprint.
[25] *W.-M. Ni, P. E. Sacks, J. Tavantzis*: On the asymptotic behavior of solutions of certain quasilinear parabolic equations. J. Differ. Equations *54* (1984), 97–120.
[26] *R. D. Nussbaum*: Positive solutions of nonlinear elliptic boundary value problems. J. Math. Anal. Appl. *51* (1975), 461–482.
[27] *S. I. Pohozaev*: Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Soviet Math. Dokl. *5* (1965), 1408–1411.
[28] *P. Quittner*: A priori bounds for global solutions of a semilinear parabolic problem. Acta Math. Univ. Comen. *68* (1999), 195–203.
[29] *P. Quittner*: A priori estimates of global solutions and multiple equilibria of a superlinear parabolic problem involving measure. Electronic J. Differ. Equations *2001* (2001), no. 29, 1–17.
[30] *P. Quittner*: Universal bound for global positive solutions of a superlinear parabolic problem. Math. Ann. *320* (2001), 299–305.
[31] *P. Quittner*: Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems. Houston J. Math. To appear.
[32] *P. Quittner*: Multiple equilibria, periodic solutions and a priori bounds for solutions in superlinear parabolic problems. NoDEA, Nonlinear Differ. Equations Appl. To appear.
[33] *P. Quittner, Ph. Souplet*: A priori estimates of global solutions of superlinear parabolic problems without variational structure. Discrete Contin. Dyn. Systems. To appear.
[34] *P. Quittner, Ph. Souplet*: Bounds of solutions of parabolic problems with nonlinear boundary conditions. In preparation.
[35] *P. Quittner, Ph. Souplet, M. Winkler*: Initial blow-up rates and universal bounds for nonlinear heat equations. Preprint.
[36] *R. E. L. Turner*: A priori bounds for positive solutions of nonlinear elliptic equations in two variables. Duke Math. J. *41* (1974), 759–774.
[37] *H. Zaag*: A remark on the energy blow-up behavior for nonlinear heat equations. Duke Math. J. *103* (2000), 545–556.

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