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Mathematica Bohemica, Vol. 131 (2006), No. 3, 279–290

Persistent URL: http://dml.cz/dmlcz/134140

131 (2006) MATHEMATICA BOHEMICA No. 3, 279–290

SOME CHARACTERIZATIONS OF THE PRIMITIVE OF STRONG HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS

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(Received September 13, 2005)

Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. In this paper we give some complete characterizations of the primitive of strongly Henstock-Kurzweil integrable functions which are defined on $\mathbb{R}^m$ with values in a Banach space.

Keywords: strong Henstock-Kurzweil integral, inner variation, SL condition

MSC 2000: 26A39

1. Introduction

It is well-known that the primitive F of a real-valued Henstock-Kurzweil integrable function f defined on a compact interval $[a,b] \subset \mathbb{R}$ is ACG*, and F is differentiable almost everywhere on $[a,b]$ and $F'(x) = f(x)$. The reverse implication also holds. That is, if a function F is ACG* and $F'(x) = f(x)$ almost everywhere on $[a,b]$, then f is Henstock-Kurzweil integrable on $[a,b]$ and F is the primitive of f. This fact is also valid for the strong Henstock-Kurzweil integral of Banach-space valued functions defined on 1-dimensional interval $[a,b]$, see [1, Theorem 4.5 in Chapter 7]. The question how to describe the primitive of a Banach-space valued Henstock-Kurzweil integrable function defined on a multidimensional interval $I_0 \subset \mathbb{R}^m$ arises naturally. However, the above well-known characterization of the 1-dimensional Henstock-Kurzweil integral in [1] relies heavily on the order structure of the real line, so it does not permit direct extension to the multidimensional Henstock-Kurzweil integral. Since the main

The work was supported by the Foundation No. 2084/40401109 of the Hohai University.

tool in the proof of the above characterization on the real line is the Vitali covering theorem which requires regularity, we cannot succeed in higher-dimensional spaces. For the strong McShane integral of Banach-valued function defined on a higher-dimensional Euclidean space some full characterizations were given using variational measure in [1], [2]. In this paper, we first use the methods from [3], [5], [6] to discuss the SHK derivative of strong Henstock-Kurzweil integral, based on inner variation, then we make use of the derivative, inner variation and the SL (strong Lusin) condition in [3], [5] to give some complete characterizations of the primitive of a strongly Henstock-Kurzweil integrable function mapping an interval I_0 in $\mathbb{R}^m$ into a Banach space. This work is closely related to Section 5 in Chapter 7 of [1].

2. Basic definitions and theorems

Throughout this paper X will denote a real Banach space, I_0 is a compact interval in $\mathbb{R}^m$ and Σ is the family of subintervals of I_0. Let $I \subset I_0$, its Lebesgue measure being denoted by $\mu(I)$. For $x \in \mathbb{R}^m$ with $x = (x_1, x_2, \dots, x_m)$, the norm $\|x\|$ is defined by $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_m|\}$. Given $\delta > 0$, $B(x, \delta)$ denotes the set $\{y \in \mathbb{R}^m \colon \|y - x\| < \delta\}$.

A partial partition D of I_0 is a finite family of interval-point pairs

$$D = \{(I_i, x_i);\ x_i \in I_i, i = 1, 2, \dots, m\}$$

with the intervals non-overlapping, and their union a subset of I_0. If a partial partition D is such that the union of the intervals in D is I_0, then we call D a partition of I_0.

Given a positive function $\delta\colon I_0 \to (0, +\infty)$ (a gauge) an interval-point pair (I, x) is said to be δ-fine if $I \subset B(x, \delta(x))$. A partition D of I_0 is said to be δ-fine if each interval-point pair in D is δ-fine.

Let $f\colon I_0 \to X$ and $\delta\colon I_0 \to (0, +\infty)$. Let $D = \{(I_i, x_i)\}_{i=1}^m$ be a δ-fine partition of I_0. The Riemann sum corresponding to f and D is written as $\sum\limits_{i=1}^{m} f(x_i)\mu(I_i)$.

In the sequel, a partition $D = \{(I_i, x_i)\}_{i=1}^m$ will be often written as $D = \{(I, x)\}$ in which (I, x) represents the typical interval-point pair in D. The corresponding Riemann sum will be written shortly in the form $(D)\sum f(x)\mu(I)$.

Definition 2.1. A function $f\colon I_0 \to X$ is said to be *Henstock-Kurzweil integrable* on I_0 if there is an additive interval function F with the following property: for every $\varepsilon > 0$, there exists a gauge δ on I_0 such that

$$\Bigl\|(D)\sum[f(x)\mu(I) - F(I)]\Bigr\| < \varepsilon$$

for every δ-fine partition $D = \{(I, x)\}$ of I_0. The function F is called the *primitive* of f on I_0.

$F(I_0) = (\mathrm{HK}) \int_{I_0} f \, \mathrm{d}t$ is the Henstock-Kurzweil integral of f over I_0.

Definition 2.2. A function $f\colon I_0 \to X$ is said to be strongly Henstock-Kurzweil integrable on I_0 if f is Henstock-Kurzweil integrable on I_0 with the primitive F such that for every $\varepsilon > 0$ there exists a gauge δ on I_0 such that

$$(D) \sum \|f(x)\mu(I) - F(I)\| < \varepsilon$$

for every δ-fine partition $D = \{(I, x)\}$ of I_0.

We denote $F(I_0) = (\mathrm{SHK}) \int_{I_0} f \, \mathrm{d}t$ in this case.

Denote further by $\mathrm{SHK} = \mathrm{SHK}(I_0; X)$ the set of functions $f\colon I_0 \to X$ which are strongly Henstock-Kurzweil integrable on I_0.

An additive interval function F and a point function correspond in a straightforward way uniquely to each other (see [1]). So, if there is no confusion, we use the same symbol F for an additive interval function on Σ and also for the corresponding point function on I_0.

Now we introduce some notations and concepts using the ideas from [3], [5], [6].

For each positive function δ on I_0 and each real number $\eta > 0$, let $\Gamma(\delta, \eta)$ be a family of δ-fine interval-point pairs (I, x) with I a subinterval of I_0 and $x \in I_0$.

Assume that for a fixed δ we have $\Gamma(\delta, \eta_1) \subset \Gamma(\delta, \eta_2)$ if $\eta_2 \leqslant \eta_1$ and for a fixed η, $\Gamma(\delta_1, \eta) \subset \Gamma(\delta_2, \eta)$ if $\delta_1(x) \leqslant \delta_2(x)$. A family $\Gamma(\delta, \eta)$ is called an inner cover of $E \subset I_0$ if for each $x \in E$, there is at least one $(I, x) \in \Gamma(\delta, \eta)$.

Assume that for a fixed δ, $\Gamma(\delta, \eta)$ is an inner cover of $E \subset I_0$ if η is small enough. Let us introduce the following concept.

Definition 2.3. Let G be a Banach space-valued function defined on the family of all interval-point pairs (I, x) with $I \subset I_0$ and let E be a subset of I_0. Then E is said to be of inner G-variation zero with respect to $\Gamma(\delta, \eta)$ as given above if for each $\varepsilon > 0$ there exists a positive function δ such that for every δ-fine partial partition $D = \{(I, x)\}$ of I_0 with $x \in E$ and $D \subset \Gamma(\delta, \eta)$, we have

$$(D) \sum_{x \in E} \|G(I, x)\| < \varepsilon.$$

If $G(I, x)$ represents the volume of I ($G(I, x) = \mu(I)$), then E is said to have inner variation zero with respect to $\Gamma(\delta, \eta)$.

It is obvious that if a set E is of measure zero then it is of inner variation zero with respect to $\Gamma(\delta, \eta)$, and the following propositions hold.

Proposition 2.1. *Let E be a subset of an interval $I_0 \subset \mathbb{R}^m$. If E is of inner variation zero with respect to $\Gamma(\delta,\eta)$, then any subset E' of E is of inner variation zero with respect to $\Gamma(\delta,\eta)$.*

Proposition 2.2. *Let $E_k, k = 1,2,\dots$ be a sequence of disjoint subsets of I_0 and let each E_k be of inner variation zero with respect to $\Gamma(\delta,\eta)$. Then $E = \bigcup_{k=1}^{\infty} E_k$ is of inner variation zero with respect to $\Gamma(\delta,\eta)$.*

The proofs are trivial, we omit them.

3. Derivatives of strong Henstock-Kurzweil integrals

Definition 3.1. An interval function F on I_0 is said to be SHK differentiable at $x \in I_0$ with the SHK derivative $D_{\rm SHK}F(x)$ if for every $\varepsilon > 0$ there exists a gauge δ such that whenever (I,x) is δ-fine with $x \in I$, we have

$$\|F(I) - D_{\rm SHK}F(x)\mu(I)\| < \varepsilon\mu(I).$$

F is said to be SHK differentiable on I_0 if F is SHK differentiable at each point x in I_0.

Note that in fact the SHK derivative $D_{\rm SHK}F(x)$ is introduced by Henstock-Kurzweil interval-point pairs in the above Definitions 2.1–2.2. In order to discuss the derivatives of strong Henstock-Kurzweil integral, we need to specify $\Gamma(\delta,\eta)$ introduced above.

Let $f\colon I_0 \to X$ and let F be an X-valued interval function on I_0. For each $\delta(x) > 0$ and each $\eta > 0$, define

$$\Gamma(f,F,\delta,\eta) = \{(I,x);\ x \in I_0, \|F(I) - f(x)\mu(I)\| \geqslant \eta\mu(I) \text{ and } (I,x) \text{ is } \delta\text{-fine}\}. \tag{3.1}$$

Then $\Gamma(f,F,\delta,\eta)$ is a family of δ-fine interval-point pairs. From now on, we write $\Gamma(\delta,\eta)$ instead of $\Gamma(f,F,\delta,\eta)$ from (3.1) if it is obvious that we are discussing the case of fixed f and F; and we take the inner variation with respect to this specific family $\Gamma(\delta,\eta) = \Gamma(f,F,\delta,\eta)$, when we are discussing differentiation.

Let

$$E(f,F,\delta,\eta) = \{x \in I_0;\ \text{there exists } I \text{ such that } x \in I \text{ and } (I,x) \in \Gamma(f,F,\delta,\eta)\},$$
$$E(f,F) = \bigcup_{\eta}\bigcap_{\delta} E(f,F,\delta,\eta).$$

The set $E(f,F) \subset I_0$ consists of points x where $D_{\rm SHK}F(x) \neq f(x)$ or $D_{\rm SHK}F(x)$ does not exist, and while $\Gamma(f,F,\delta,\eta)$ need not be a Vitali cover of $E(f,F)$, but it is an inner cover and satisfies all conditions imposed on $\Gamma(\delta,\eta)$ mentioned above.

For convenience we denote $E(f,F)$ by E_0, i.e.,

$$E_0 = E(f,F) = \bigcup_{\eta} \bigcap_{\delta} E(f,F,\delta,\eta). \tag{3.2}$$

Theorem 3.1. *Let $f\colon I_0 \to X$ be a strongly Henstock-Kurzweil integrable function on I_0 with the primitive F. Then $D_{\rm SHK}F(x) = f(x)$ except at points of the set E_0 in* (3.2) *with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$.*

Proof. We only need to prove that E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$.

Let η be any positive real number and $\varepsilon > 0$. Since F is the primitive of the strongly Henstock-Kurzweil integrable function f, there is a gauge δ of I_0 such that for any δ-fine partition $D = \{(I,x)\}$ of I_0 we have

$$(D)\sum \|f(x)\mu(I) - F(I)\| < \varepsilon \cdot \eta.$$

Then for any δ-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E_0$ and $D \subset \Gamma(\delta,\eta)$ we have by (3.1) the inequality

$$\eta(D)\sum \mu(I) < (D)\sum \|f(x)\mu(I) - F(I)\| < \varepsilon\eta.$$

So,

$$(D)\sum \mu(I) < \frac{1}{\eta}(D)\sum \|f(x)\mu(I) - F(I)\| < \frac{1}{\eta}\cdot \varepsilon\eta = \varepsilon.$$

The proof is complete. □

4. The primitive of strong Henstock-Kurzweil integral

In order to obtain a characterization of the primitive of a strongly Henstock-Kurzweil integrable function, we introduce the following concept.

Definition 4.1. An interval function F is said to satisfy the SL (strong Lusin) condition with respect to $\Gamma(\delta,\eta)$ on a set $E \subset I_0$ if for every $\varepsilon > 0$ there exists a gauge δ on E such that for any δ-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E$ and $D \subset \Gamma(\delta,\eta)$, we have

$$(D)\sum \|F(I)\| < \varepsilon\mu(I).$$

Theorem 4.1. *Let $f\colon I_0 \to X$ be a strongly Henstock-Kurzweil integrable function on I_0 with the primitive F. Then for every $\eta > 0$ the function F satisfies the* SL *condition with respect to* $\Gamma(\delta,\eta)$ *on E_0 from* (3.2).

P r o o f. By Theorem 3.1 we know that E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$. Let

$$E_n = \{x \in E_0\colon\ n-1 \leqslant \|f(x)\| < n\}, n = 1, 2, \ldots$$

Then $E_0 = \bigcup\limits_n E_n$. Since for every $\eta > 0$, E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$, E_n is of inner variation zero with respect to $\Gamma(\delta,\eta)$ by Proposition 2.1. That is, for every $\varepsilon > 0$ there is a gauge δ_n on E_n such that for any δ_n-fine partial partition $D_n = \{(I,x)\}$ with $x \in E_n$ and $D_n \subset \Gamma(\delta_n,\eta)$, we have

$$(D_n)\sum \mu(I) < \frac{\varepsilon}{2^{n+1}n}. \tag{4.1}$$

Since f is a strongly Henstock-Kurzweil integrable function, for given $\varepsilon > 0$ there is a gauge δ' of I_0 such that for any δ'-fine partition $D = \{(I,x)\}$ of I_0 we have

$$(D)\sum \|f(x)\mu(I) - F(I)\| < \frac{\varepsilon}{2}. \tag{4.2}$$

Define δ on I_0 as follows: $\delta(x) = \min\{\delta_n(x), \delta'(x)\}$ if $x \in E_n$, $n = 1, 2, \ldots$ and $\delta(x) = \delta'(x)$ if $x \in I_0 \setminus E_0$. Let $D = \{(I,x)\}$ be a δ-fine partial partition of I_0 with $x \in E_0$ and $D \subset \Gamma(\delta,\eta)$ and $D_n = \{(I,x) \in D;\ x \in E_n\}$. Then (4.1) holds for this $D_n = \{(I,x)\}$, therefore, by (4.1) and (4.2), we obtain

$$\begin{aligned}(D)\sum \|F(I)\| &\leqslant (D)\sum \|F(I) - f(x)\mu(I)\| + (D)\sum \|f(x)\mu(I)\| \\ &< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}(D_n)\sum \|f(x)\|\mu(I) \\ &< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}\cdot n}\cdot n = \varepsilon.\end{aligned} \tag{4.3}$$

That is, F satisfies the SL condition with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ on E_0 and the proof is complete. □

Theorem 4.2. *Let $f\colon I_0 \to X$ and let F be an additive interval function on Σ. If $D_{\mathrm{SHK}}F(x) = f(x)$ except at points of the set E_0 with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ and F satisfies the* SL *condition with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ on E_0, then the function f is strongly Henstock-Kurzweil integrable on I_0 with the primitive F.*

Proof. Let $\varepsilon > 0$ be an arbitrary real number. Suppose $\eta < \varepsilon/2\mu(I_0)$. If $x \in I_0 \setminus E_0$ then F is differentiable at x and its derivative is $f(x)$. Hence there is a positive function $\delta_0(x)$ on $I_0 \setminus E_0$ such that

$$\|F(I) - f(x)\mu(I)\| < \eta\mu(I) \tag{4.4}$$

whenever (I,x) is δ_0-fine.

On the other hand, let

$$E_n = \{x \in E_0\colon\ n-1 \leqslant \|f(x)\| < n\}, n = 1,2,\ldots.$$

Since E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$, then for each $n \in \mathbb{N}$, E_n is of inner variation zero with respect to $\Gamma(\delta,\eta)$. So, for given $\varepsilon > 0$, there exists a positive function δ_n on E_n, $n = 1,2,\ldots$, such that

$$(D)\sum \mu(I) < \frac{\varepsilon}{2^{n+1}n} \tag{4.5}$$

for any δ_n-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E_n$ and $D \subset \Gamma(\delta_n,\eta)$. On E_0, we define $\delta'(x) = \delta_n(x)$ if $x \in E_n$, $n = 1,2,\ldots$; then for any δ'-fine partial partition $D = \{(I,x)\}$ with $x \in E_0$ and $D \subset \Gamma(\delta',\eta)$, we have by (4.5)

$$(D)\sum \|f(x)\|\mu(I) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}n} n = \varepsilon. \tag{4.6}$$

Recall that

$$\|F(I) - f(x)\mu(I)\| \geqslant \eta\mu(I)$$

for all $(I,x) \in D \subset \Gamma(\delta',\eta)$ with $x \in E_0$. Suppose (I,x) is $\delta'(x)$-fine with $x \in E_0$ and $(I,x) \notin \Gamma(\delta',\eta)$, then

$$\|F(I) - f(x)\mu(I)\| < \eta\mu(I). \tag{4.7}$$

Since F satisfies the SL condition with respect to $\Gamma(\delta,\eta)$ on E_0, there exists a gauge δ'' on E_0 such that

$$(D)\sum \|F(I)\| < \varepsilon \tag{4.8}$$

for any δ''-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E_0$ and $D \subset \Gamma(\delta'',\eta)$.

Now we define a $\delta(x)$ on I_0 as follows: $\delta(x) = \delta_0(x)$ if $x \in I_0 \setminus E_0$, and $\delta(x) = \min\{\delta'(x), \delta''(x)\}$ if $x \in E_0$. Then for any δ-fine partition $D = \{(I, x)\}$ of I_0, by (4.4),

$$(4.9) \qquad (D) \sum_{x \in I_0 \setminus E_0} \|f(x)\mu(I) - F(I)\| < \eta \cdot (D) \sum_{x \in I_0 \setminus E_0} \mu(I) \leqslant \eta |I_0|.$$

On the other hand, a δ-fine partial partition $D = \{(I, x)\}$ of I_0 with all $x \in E_0$ can be decomposed into D' and D'', where

$$D' = \{(I, x) \in D;\ x \in E_0, (I, x) \notin \Gamma(\delta, \eta)\},$$
$$D'' = \{(I, x) \in D;\ x \in E_0, (I, x) \in \Gamma(\delta, \eta)\}.$$

Then D' satisfies (4.7) and D'' satisfies (4.6) and (4.8). Thus

$$(4.10) \qquad (D') \sum \|f(x)\mu(I) - F(I)\| < \eta |I_0|,$$

$$(4.11) \qquad (D'') \sum \|f(x)\|\mu(I) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1} n} n = \varepsilon$$

and

$$(4.12) \qquad (D'') \sum \|F(I)\| < \varepsilon.$$

Hence, by (4.9)–(4.12), we have

$$\begin{aligned}(4.13) \quad (D) \sum \|f(x)\mu(I) - F(I)\| &= (D) \sum_{x \in I_0 \setminus E_0} \|f(x)\mu(I) - F(I)\| + (D') \sum_{x \in E_0} \|f(x)\mu(I) - F(I)\| \\ &\quad + (D'') \sum_{x \in E_0} \|f(x)\mu(I) - F(I)\| \\ &\leqslant 2\eta |I_0| + (D'') \sum_{x \in E_0} \|f(x)\|\mu(I) + (D'') \sum_{x \in E_0} \|F(I)\| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

Therefore f is a strongly Henstock-Kurzweil integrable function on I_0 and F is its primitive. □

By Theorems 3.1, 4.1 and 4.2 we obtain the following complete characterization of the primitive of a strong Henstock-Kurzweil integrable function mapping an interval I_0 in $\mathbb{R}^m$ into a Banach space X.

Theorem 4.3. *A function $f\colon I_0 \to X$ is strongly Henstock-Kurzweil integrable on I_0 if and only if there exists an additive interval function F such that $D_{\rm SHK}F(x) = f(x)$ except at points of the set E_0 with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$, and F satisfies the* SL *condition on E_0 with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$.*

To establish another characterization of the primitive of a strongly Henstock-Kurzweil integrable function, we recall the concept ACG_δ^{**} in [7], [8], which can also be found in [1].

Definition 4.2 [1], [7], [8]**.** Let I_0 be an interval in $\mathbb{R}^m$ and $M \subset I_0$. An interval function F defined on Σ is said to be *$AC_\delta^{**}(M)$* if for every $\varepsilon > 0$ there exist a gauge $\delta\colon I_0 \to (0,\infty)$ and $\eta > 0$ such that for any two δ-fine partitions $D_1 = \{(t_i, I_i)\}$, $D_2 = \{(s_j, J_j)\}$ with tags $t_i, s_j \in M$ such that any interval J_j lies in some interval I_i, we have

$$\sum_{D_1\setminus D_2} \mu(I) < \eta \Longrightarrow \sum_{D_1\setminus D_2} \|F(I)\|_X < \varepsilon$$

where $D_1 \setminus D_2 = \{(t_i, I_i \setminus \bigcup_{j, J_j\subset I_i} J_j)\}$. If $I = I_i \setminus \bigcup_{j,J_j\subset I_i} J_j$ then $F(I) = F(I_i \setminus \bigcup_{j,J_j\subset I_i} J_j) = F(I_i) - \sum_{j,J_j\subset I_i} F(J_j)$ and $\mu(I) = \mu(I_i \setminus \bigcup_{j,J_j\subset I_i} J_j) = \mu(I_i) - \sum_{j,J_j\subset I_i} \mu(J_j)$.

Furthermore, F is $\mathrm{ACG}_\delta^{**}(I_0)$ if $I_0 = \bigcup_{i=1}^{\infty} M_i$ and F is $AC_\delta^{**}(M_i)$ for each $i \in \mathbb{N}$.

It is known that the primitive F of a strongly Henstock-Kurzweil integrable function f is ACG_δ^{**} (see [7]). Further, we prove the following theorem.

Theorem 4.4. *Let $f\colon I_0 \to X$ and let an additive $\mathrm{ACG}_\delta^{**}(I_0)$ interval function F be given.*

If $D_{\rm SHK}F(x) = f(x)$ except at points of a set E_0 with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$, then f is a strongly Henstock-Kurzweil integrable function on I_0 and F is its primitive.

P r o o f. Let $\varepsilon > 0$ be an arbitrary real number. Suppose $\eta < \varepsilon/2\mu(I_0)$.

Since $D_{\rm SHK}F(x) = f(x)$ for each $x \in I_0 \setminus E_0$, there is a gauge $\delta_{I_0\setminus E_0}$ on $I_0 \setminus E_0$ such that for any $\delta_{I_0\setminus E_0}$-fine partial partition $D_{I_0\setminus E_0} = \{(I,x)\}$ of I_0 with $x \in I_0 \setminus E_0$, we have

$$(4.14)\qquad (D_{I_0\setminus E_0})\sum \|f(x)\mu(I) - F(I)\| < \eta|I_0|.$$

Since F is $\mathrm{ACG}_\delta^{**}(I_0)$, $I_0 = \bigcup_{i=1}^{\infty} E_i$ and F is $AC_\delta^{**}(E_i)$. We assume that $E_i \cap E_j = \emptyset$ for any $i \neq j$. Then for any given $\varepsilon > 0$ there is a gauge $\tilde\delta_i$ on each E_i and

$0 < \eta_i \leqslant \varepsilon 2^{-i}$ $\left(\sum\limits_{i=1}^{\infty} \eta_i \leqslant \varepsilon\right)$, such that for any $\tilde{\delta}_i$-fine partial partition $D_i = \{(I, x)\}$ of I_0 with $x \in E_i$, we have

$$(D_i)\sum \mu(I) < \eta_i \Rightarrow (D_i)\sum \|F(I)\| < \frac{\varepsilon}{2^i}. \tag{4.15}$$

Let $X_i = E_0 \cap E_i$, $Y_n = \{x \in I_0\colon\ n-1 \leqslant \|f(x)\| < n\}$ and $X_{in} = X_i \cap Y_n$. Then

$$E_0 = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty} X_{in} \text{ and also } E_0 = \bigcup_{n=1}^{\infty}(E_0 \cap Y_n). \tag{4.16}$$

Since E_0 is of inner variation zero with respect to $\Gamma(\delta, \eta)$, each X_{in} is of inner variation zero with respect to $\Gamma(\delta, \eta)$. So, for $\eta_{in} = \eta_i/n2^n$, there is a $\delta_{in} \leqslant \tilde{\delta}_i$ on X_{in} such that for any δ_{in}-fine partial partition $D_{in} = \{(I, x)\}$ of I_0 with $x \in X_{in}$ and $D_{in} \subset \Gamma(\delta_{in}, \eta)$, we have

$$(D_{in})\sum \mu(I) < \frac{\eta_i}{n2^n}. \tag{4.17}$$

Now define a gauge δ_{E_0} on E_0 as follows: $\delta_{E_0}(x) = \delta_{in}(x)$ if $x \in X_{in}$, $i, n = 1, 2, \ldots$. Then for any δ_{E_0}-fine partial partition $D_i = \{(I, x)\}$ of I_0 with $x \in X_i$ and $D_i \subset \Gamma(\delta_{E_0}, \eta)$, by (4.16) and (4.17), we have

$$(D_i)\sum \mu(I) = \sum_{n=1}^{\infty}(D_i)\sum_{x \in X_{in}} \mu(I) < \sum_{n=1}^{\infty} \frac{\eta_i}{2^n} = \eta_i \tag{4.18}$$

and by (4.15) and (4.18), we obtain

$$(D_i)\sum \|F(I)\| < \frac{\varepsilon}{2^i}. \tag{4.19}$$

Let us now define a gauge $\delta(x)$ on I_0 in the following way: $\delta(x) = \delta_{E_0}(x)$ if $x \in E_0$ and $\delta(x) = \min\{\delta_{I_0\setminus E_0}(x), \tilde{\delta}_i(x)\}$ if $x \in (I_0 \setminus E_0) \cap E_i$, $i = 1, 2, \ldots$. Then for any δ-fine partition $D = \{(I, x)\}$ of I_0 (similarly to the proof of (4.13)), by (4.14), (4.17) and (4.19), we conclude

$$
\begin{aligned}
&(D)\sum \|f(x)\mu(I)-F(I)\| \\
&\quad = (D)\sum_{x\in I_0\setminus E_0} \|f(x)\mu(I)-F(I)\| + (D)\sum_{x\in E_0} \|f(x)\mu(I)-F(I)\| \\
&\quad \leqslant 2(D)\sum_{x\in I_0\setminus E_0} \eta\mu(I) + (D)\sum_{x\in E_0,(I,x)\in\Gamma(\delta,\eta)} \|f(x)\|\mu(I) \\
&\qquad + (D)\sum_{x\in E_0,(I,x)\in\Gamma(\delta,\eta)} \|F(I)\| \\
&\quad \leqslant 2\eta|I_0| + \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(D)\sum_{x\in X_{in},(I,x)\in\Gamma(\delta,\eta)} \|f(x)\|\mu(I) + \sum_{i=1}^{\infty}(D)\sum_{x\in X_i}\|F(I)\| \\
&\quad < \varepsilon + \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} n\frac{\eta_i}{n2^n} + \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i} \\
&\quad < \varepsilon+\varepsilon+\varepsilon = 3\varepsilon.
\end{aligned}
$$

The proof is complete. □

By Theorem 4.4 and Theorem 4.1 of [7], it is easy to obtain the following theorem giving another complete characterization of the primitive of strongly Henstock-Kurzweil integrable functions.

Theorem 4.5. *Let $f\colon I_0\to X$, and let F be an additive interval function defined on Σ. Let E_0 be as in (3.2). Then f is strongly Henstock-Kurzweil integrable on I_0 with the primitive F if and only if E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta>0$ and F is ACG_δ^{**} on I_0.*

5. Acknowledgment

I consider it a great honour to be invited to contribute to the issue of Mathematica Bohemica dedicated to Prof. J. Kurzweil's anniversary.

During my stay in the Mathematical Institute of the Academy of Science of the Czech Republic, I was so happy as to share an office with Professor Kurzweil and I spent a pleasant time there. I learned a lot from him and his colleagues in the Institute. I would like to express my sincere thanks for everything they did for me.

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