Štefan Schwabik
Operator-valued functions of bounded semivariation and convolutions

*Mathematica Bohemica*, Vol. 126 (2001), No. 4, 745–777

Persistent URL: http://dml.cz/dmlcz/134117

# OPERATOR-VALUED FUNCTIONS OF BOUNDED SEMIVARIATION AND CONVOLUTIONS

Štefan Schwabik, Praha

(Received November 19, 1999)

*Abstract.* The abstract Perron-Stieltjes integral in the Kurzweil-Henstock sense given via integral sums is used for defining convolutions of Banach space valued functions. Basic facts concerning integration are preseted, the properties of Stieltjes convolutions are studied and applied to obtain resolvents for renewal type Stieltjes convolution equations.

*Keywords*: Kurzweil-Henstock integration, convolution, Banach space

*MSC 2000*: 26A45, 26A42, 46G12

Assume that $X$ is a Banach space and that $L(X)$ is the Banach space of all bounded linear operators $A\colon X \to X$ with the uniform operator topology. Defining the bilinear form $B\colon L(X) \times X \to X$ by $B(A,x) = Ax \in X$ for $A \in L(X)$ and $x \in X$, we obtain in a natural way the bilinear triple $\mathcal{B} = (L(X), X, X)$ because using the usual operator norm we have

$$\|B(A,x)\|_X \leqslant \|A\|_{L(X)}\|x\|_X.$$

Similarly, if we define the bilinear form $B^*\colon L(X) \times L(X) \to L(X)$ by the relation $B^*(A,C) = AC \in L(X)$ for $A, C \in L(X)$ where $AC$ is the composition of the linear operators $A$ and $C$ we get the bilinear triple $\mathcal{B}^* = (L(X), L(X), L(X))$ because we have

$$\|B^*(A,C)\|_{L(X)} \leqslant \|AC\|_{L(X)} \leqslant \|A\|_{L(X)}\|C\|_{L(X)}.$$

This work was supported by the grant 201/97/0218 of the Grant Agency of the Czech Republic

Assume that $[a,b] \subset \mathbb{R}$ is a bounded interval.

Given $A\colon [a,b] \to L(X)$, the function $A$ is of *bounded variation on* $[a,b]$ if

$$\operatorname*{var}_{[a,b]}(A) = \sup \left\{ \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \right\} < \infty,$$

where the supremum is taken over all finite partitions

$$D\colon\ a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a,b]$.

We denote the set of all functions $A\colon [a,b] \to L(X)$ with $\operatorname{var}_{[a,b]}(A) < \infty$ by $BV([a,b];L(X))$.

For $A\colon [a,b] \to L(X)$ and a partition $D$ of the interval $[a,b]$ define

$$V_a^b(A,D) = \sup \left\{ \left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j \right\|_X \right\},$$

where the supremum is taken over all possible choices of $x_j \in X, j = 1,\ldots,k$ with $\|x_j\|_X \leqslant 1$ and similarly

$$\overset{*}{V}{}_a^b(A,D) = \sup \left\{ \left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \right\},$$

where the supremum is taken over all possible choices of $C_j \in L(X)$, $j = 1,\ldots,k$ with $\|C_j\|_{L(X)} \leqslant 1$.

Let us set

$$(\mathcal{B}) \operatorname*{var}_{[a,b]}(A) = \sup V_a^b(A,D)$$

and

$$(\mathcal{B}^*) \operatorname*{var}_{[a,b]}(A) = \sup \overset{*}{V}{}_a^b(A,D)$$

where the suprema on the right hand sides are taken over all finite partitions $D$ of the interval $[a,b]$.

An operator valued function $A\colon [a,b] \to L(X)$ with $(\mathcal{B}) \operatorname{var}_{[a,b]}(A) < \infty$ is called a *function with bounded* $\mathcal{B}$*-variation on* $[a,b]$ (or a *function of bounded semi-variation,* cf. [4]), and similarly if $(\mathcal{B}^*) \operatorname{var}_{[a,b]}(A) < \infty$ then $A$ is of *bounded* $\mathcal{B}^*$*-variation on* $[a,b]$.

We denote by $(\mathcal{B})BV([a,b];L(X))$ the set of all functions $A\colon\ [a,b]\to L(X)$ with $(\mathcal{B})\operatorname{var}_{[a,b]}(A)<\infty$ and by $(\mathcal{B}^*)BV([a,b];L(X))$ the set of all functions $A\colon\ [a,b]\to L(X)$ with $(\mathcal{B}^*)\operatorname{var}_{[a,b]}(A)<\infty$.

Concerning these concepts the following proposition holds.

**1. Proposition.** *We have*

$$(\mathcal{B})BV([a,b];L(X))=(\mathcal{B}^*)BV([a,b];L(X))$$

*and if* $A\in(\mathcal{B})BV([a,b];L(X))$ *then*

$$(\mathcal{B})\operatorname*{var}_{[a,b]}(A)=(\mathcal{B}^*)\operatorname*{var}_{[a,b]}(A).$$

(See [9, Proposition 1.1] or [1, Proposition 2.1]).

## Regulated functions

Given $x\colon\ [a,b]\to X$, the function $x$ is called *regulated on* $[a,b]$ if it has one-sided limits at every point of $[a,b]$, i.e. if for every $s\in[a,b)$ there is a value $x(s+)\in X$ such that

$$\lim_{t\to s+}\|x(t)-x(s+)\|_X=0$$

and for every $s\in(a,b]$ there is a value $x(s-)\in X$ such that

$$\lim_{t\to s-}\|x(t)-x(s-)\|_X=0.$$

The set of all regulated functions $x\colon\ [a,b]\to X$ will be denoted by $G([a,b];X)$.

Similarly in the case of an unbounded interval, e.g. $[a,+\infty)$, we denote by $G([a,+\infty);X)$ the set of all $x\colon\ [a,+\infty)\to X$ such that for every $s\in[a,+\infty)$ there is a value $x(s+)\in X$ such that

$$\lim_{t\to s+}\|x(t)-x(s+)\|_X=0$$

and for every $s\in(a,+\infty)$ there is a value $x(s-)\in X$ such that

$$\lim_{t\to s-}\|x(t)-x(s-)\|_X=0.$$

The space $G([a,b];X)$ endowed with the norm

$$\|x\|_{G([a,b];X)}=\sup_{t\in[a,b]}\|x(t)\|_X,\ x\in G([a,b];X)$$

is a Banach space (see [4, Theorem 3.6]). Hence the uniform limit of a sequence $x_n \in G([a,b];X)$ belongs to $G([a,b];X)$.

The space $C([a,b];X)$ of continuous functions $x\colon [a,b] \to X$ is a closed subspace of $G([a,b];X)$, i.e.

$$C([a,b];X) \subset G([a,b];X).$$

Assume now that $\mathcal{B} = (L(X), X, X)$ is the bilinear triple of Banach spaces mentioned above.

A function $A\colon [a,b] \to L(X)$ is called *$\mathcal{B}$-regulated on* $[a,b]$ if for every $y \in X$, $\|y\|_X \leqslant 1$, the function $Ay\colon [a,b] \to X$ given by $t \in [a,b] \mapsto A(t)y \in X$ for $t \in [a,b]$ is regulated, i.e. $Ay \in G([a,b];X)$ for every $y \in X$, $\|y\|_X \leqslant 1$.

We denote by $(\mathcal{B})G([a,b];L(X))$ the set of all $\mathcal{B}$-regulated functions $A\colon [a,b] \to L(X)$.

A function $x\colon [a,b] \to X$ is called a *(finite) step function on* $[a,b]$ if there exists a finite partition

$$D\colon a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a,b]$ such that $x$ has a constant value in $X$ on $(\alpha_{j-1}, \alpha_j)$ for every $j = 1, \ldots, k$; similarly for operator valued functions.

The following result is well known for regulated functions.

**2. Proposition** (see e.g. [2, Theorem 3.1, p. 16]). *A function $x\colon [a,b] \to X$ is regulated ($x \in G([a,b];X)$) if and only if $x$ is the uniform limit of step functions.*

**3. Proposition.** *We have*

$$BV([a,b];L(X)) \subset (\mathcal{B})BV([a,b];L(X))$$

*and if $A \in BV([a,b];L(X))$, then*

$$(\mathcal{B}) \operatorname*{var}_{[a,b]}(A) \leqslant \operatorname*{var}_{[a,b]}(A)$$

*and*

$$BV([a,b];L(X)) \subset G([a,b];L(X)) \subset (\mathcal{B})G([a,b];L(X)).$$

(See [8, Proposition 1] and [9, Proposition 1.5]).

Remark. It is not difficult to see that if $A\colon [a,b] \to L(X)$ and the space $X$ is finite dimensional then $A \in (\mathcal{B})BV([a,b];L(X))$ if and only if $A \in BV([a,b];L(X))$ (cf. [8, Remark on p. 427]).

Therefore the concept of $\mathcal{B}$-variation of a function $A\colon [a,b]\to L(X)$ is relevant for infinite-dimensional Banach spaces $X$ only.

Similarly, if the Banach space $X$ is finite dimensional, then it is easy to check that a function $A\colon [a,b]\to L(X)$ is $\mathcal{B}$-regulated if and only if it is regulated.

## On some spaces of operators

Assume that $[a,b]\subset\mathbb{R}$ is a bounded interval.

If $A,B\in(\mathcal{B})BV([a,b],L(X))=(\mathcal{B})BV(L(X))$ then

$$\Bigl\|\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_{j-1})+B(\alpha_j)-B(\alpha_{j-1})]x_j\Bigr\|_X \leqslant \Bigl\|\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_{j-1})]x_j\Bigr\|_X+\Bigl\|\sum_{j=1}^{k}[B(\alpha_j)-B(\alpha_{j-1})]x_j\Bigr\|_X$$

for every partition $D\colon a=\alpha_0<\alpha_1<\ldots<\alpha_{k-1}<\alpha_k=b$ and any choice of $x_j\in X$, $\|x_j\|\leqslant 1$, $j=1,\ldots,k$. Hence

$$(\mathcal{B})\operatorname*{var}_{[a,b]}(A+B)\leqslant(\mathcal{B})\operatorname*{var}_{[a,b]}(A)+(\mathcal{B})\operatorname*{var}_{[a,b]}(B) \tag{1}$$

and $A+B\in(\mathcal{B})BV(L(X))$.

Similarly it can be shown that

$$(\mathcal{B})\operatorname*{var}_{[a,b]}(\lambda A)=|\lambda|(\mathcal{B})\operatorname*{var}_{[a,b]}(A) \tag{2}$$

for any $\lambda\in\mathbb{R}$ and therefore $(\mathcal{B})BV(L(X))$ *is a linear space.*

At the same time (1) and (2) show that

**A.** $(\mathcal{B})\operatorname{var}_{[a,b]}(\cdot)\colon (\mathcal{B})BV(L(X))\to\mathbb{R}$ *defines a seminorm on* $(\mathcal{B})BV(L(X))$.

Further we have

**B.** *If* $A\in(\mathcal{B})BV(L(X))$ *and* $(\mathcal{B})\operatorname{var}_{[a,b]}(A)=0$ *then* $A(t)=C\in L(X)$ *for every* $t\in[a,b]$.

To show this take $x\in X$, $x\neq 0$. Then for every $t_1,t_2\in[a,b]$ we have

$$\begin{aligned}\|A(t_1)x-A(t_2)x\|_X&=\|x\|_X\Bigl\|[A(t_1)-A(t_2)]\frac{x}{\|x\|_X}\Bigr\|_X\\&\leqslant\|x\|_X(\mathcal{B})\operatorname*{var}_{[a,b]}(A)=0\end{aligned}$$

and this yields $A(t_1) = A(t_2)$, i.e. $A(t) = C \in L(X)$ for every $t \in [a,b]$.

For $A \in (\mathcal{B})BV(L(X))$ define

$$\|A\|_{SV} = \|A(a)\|_{L(X)} + (\mathcal{B}) \operatorname*{var}_{[a,b]}(A). \tag{3}$$

Using the result given above we can see that

**C.** $\|\cdot\|_{SV}$ *defines a seminorm on* $(\mathcal{B})BV(L(X))$.

Moreover, if $A(t) = 0$ for every $t \in [a,b]$, then $\|A\|_{SV} = 0$ and if $\|A\|_{SV} = 0$ then $\|A(a)\| = 0$ and $(\mathcal{B})\operatorname{var}_{[a,b]}(A) = 0$. Hence $A(a) = 0$ and by **A.** also $A(t) = A(0) = 0$ for every $t \in [a,b]$.

This implies that

**D.** $\|\cdot\|_{SV}$ *is a norm on the linear space* $(\mathcal{B})BV(L(X))$.

If $A\colon [a,b] \to L(X)$ then for every $x \in X$, $t \in [a,b]$ we have

$$\|A(t)x\|_X \leqslant \|A(a)x\|_X + \|A(t) - A(a)x\|_X \leqslant \|A(a)\|_{L(X)}\|x\|_X + (\mathcal{B}) \operatorname*{var}_{[a,b]}(A)\|x\|_X,$$

and therefore

$$\|A(t)\|_{L(X)} \leqslant \|A\|_{SV} \quad \text{for every} \quad t \in [a,b]. \tag{4}$$

Looking at the inequality (4) we see that

$$\sup_{t\in[a,b]} \|A(t)\|_{L(X)} \leqslant \|A\|_{SV}. \tag{5}$$

If $A \in G([a,b]; L(X)) \cap (\mathcal{B})BV([a,b]; L(X))$ then (5) yields

$$\|A\|_{G([a,b];L(X))} = \sup_{t\in[a,b]} \|A(t)\|_{L(X)} \leqslant \|A\|_{SV}. \tag{6}$$

**4. Proposition.** *$(\mathcal{B})BV(L(X))$ is a Banach space when equipped with the norm $\|\cdot\|_{SV}$ given by* (3).

P r o o f. According to **D.** it is sufficient to prove that $(\mathcal{B})BV(L(X))$ is complete. Assume that $A_n \in (\mathcal{B})BV(L(X))$, $n \in \mathbb{N}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{SV}$.

Then for every $\varepsilon > 0$ there is $M \in \mathbb{N}$ such that for $m,n \geqslant M$ we have

$$\|A_n - A_m\|_{SV} < \varepsilon.$$

By (4) we get $\|A_n(t) - A_m(t)\|_{L(X)} < \varepsilon$ for every $t \in [a,b]$ and for $m, n \geqslant M$. Since $L(X)$ is a Banach space, for every $t \in [a,b]$ there exists $A(t) \in L(X)$ such that

$$\lim_{n\to\infty} \|A_n(t) - A(t)\|_{L(X)} = 0 \tag{7}$$

uniformly on $[a,b]$.

Let us consider $A\colon [a,b] \to L(X)$ given above.

If

$$D\colon a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

is a partition and $x_j \in X$, $\|x_j\| \leqslant 1$, $j = 1, \ldots, k$, then

$$\Big\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j\Big\|_X \leqslant \Big\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1}) - A_n(\alpha_j) + A_n(\alpha_{j-1})]x_j\Big\|_X + \Big\|\sum_{j=1}^{k}[A_n(\alpha_j) - A_n(\alpha_{j-1})]x_j\Big\|_X.$$

If we take into account that (7) holds and that the sequence of reals $\|A_n\|_{SV}$, $n \in \mathbb{N}$ is bounded ($\|A_n\|_{SV} \leqslant K$), we obtain for sufficiently large $n \in \mathbb{N}$

$$\Big\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1}) - A_n(\alpha_j) + A_n(\alpha_{j-1})]x_j\Big\|_X \leqslant 1$$

and because

$$\Big\|\sum_{j=1}^{k}[A_n(\alpha_j) - A_n(\alpha_{j-1})]x_j\Big\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[a,b]}(A_n) \leqslant \|A_n\|_{SV} \leqslant K,$$

we get

$$\Big\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j\Big\|_X \leqslant 1 + K.$$

By definition this implies $(\mathcal{B}) \operatorname{var}_{[a,b]}(A) < \infty$ and we get $A \in (\mathcal{B})BV(L(X))$. This result shows that $(\mathcal{B})BV(L(X))$ is complete. □

**Corollary.** *$G([a,b]; L(X)) \cap (\mathcal{B})BV([a,b]; L(X))$ is a closed subspace of the Banach space $(\mathcal{B})BV([a,b]; L(X))$ with the norm $\|\cdot\|_{SV}$.*

P r o o f. This follows immediately from (6) which holds for every

$$A \in G([a,b]; L(X)) \cap (\mathcal{B})BV([a,b]; L(X)).$$

□

We are using the concept of Perron-Stieltjes integral based on the Kurzweil-Henstock definition presented via integral sums (for a more detailed exposition see e.g. [5], [7], [8]). We recall this concept shortly.

A finite system of points

$$\{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

such that

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

and

$$\tau_j \in [\alpha_{j-1}, \alpha_j] \quad \text{for} \quad j = 1, \ldots, k$$

is called a *P-partition* of the interval $[a, b]$.

Any positive function $\delta\colon [a, b] \to (0, \infty)$ is called a *gauge on* $[a, b]$.

For a given gauge $\delta$ on $[a, b]$ a $P$-partition

$$\{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

of $[a, b]$ is called *δ-fine* if

$$[\alpha_{j-1}, \alpha_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) \quad \text{for} \quad j = 1, \ldots, k.$$

**5. Definition.** Assume that functions $A, C\colon [a, b] \to L(X)$ and $x\colon [a, b] \to X$ are given.

The Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists if there is an element $J \in X$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that we have

$$\Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x(\tau_j) - J \Big\|_X < \varepsilon$$

provided $D$ is a $\delta$-fine $P$-partition of $[a, b]$. We denote $J = \int_a^b \mathrm{d}[A(s)]x(s)$. For the case $a = b$ it is convenient to set $\int_a^b \mathrm{d}[A(s)]x(s) = 0$ and if $b < a$, then $\int_a^b \mathrm{d}[A(s)]x(s) = -\int_b^a \mathrm{d}[A(s)]x(s)$.

Analogously we say that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]C(s)$ exists if there is an element $J \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that we have

$$\Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]C(\tau_j) - J \Big\|_X < \varepsilon$$

provided $D$ is a $\delta$-fine $P$-partition of $[a,b]$.

Similarly we can define the Stieltjes integral $\int_a^b A(s)\mathrm{d}[C(s)]$ using Stieltjes integral sums of the form $\sum_{j=1}^{k} A(\tau_j)\big[C(\alpha_j)-C(\alpha_{j-1})\big]$.

**6. Proposition.** *If functions $A\colon\ [a,b]\to L(X)$ and $x\colon\ [a,b]\to X$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists then*

$$\Big\| \int_a^b \mathrm{d}[A(s)]x(s)\Big\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[a,b]}(A).\ \sup_{s\in[a,b]} \|x(s)\|_X.$$

(See [8, Proposition 10]).

**7. Proposition.** *Assume that $A\colon\ [a,b]\to L(X)$ and $x\colon\ [a,b]\to X$ where $A\in(\mathcal{B})G([a,b],L(X))\cap(\mathcal{B})BV([a,b],L(X))$ and $x\in G([a,b],X)$.*

*Then the integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists.*

(See [8, Proposition 15]).

**8. Corollary.** *If*

$$A\in(\mathcal{B})G([a,b],L(X))\cap(\mathcal{B})BV([a,b],L(X)),$$

*$C\in G([a,b],L(X))$ then the integral $\int_a^b \mathrm{d}[A(s)]C(s)$ exists.*

Assume that $U,V\colon\ [0,\infty)\to L(X)$ and $x\colon\ [0,\infty)\to X$ are given.

Let us define convolutions

$$(U*x)(t)=\int_0^t \mathrm{d}[U(s)]x(t-s)$$

and

$$(U*V)(t)=\int_0^t \mathrm{d}[U(s)]V(t-s)$$

for $t\in[0,\infty)$.

Denote by $(\mathcal{B})BV_{\mathrm{loc}}([0,\infty),L(X))$ the set of all $U\colon\ [0,\infty)\to L(X)$ for which $U\in(\mathcal{B})BV([0,b],L(X))$ for every $b>0$.

Assume that $U\in G([0,\infty),L(X))\cap(\mathcal{B})BV_{\mathrm{loc}}([0,\infty),L(X))$.

If $x\in G([0,\infty),X)$ and $V\in G([0,\infty),L(X))$ then the convolutions $(U*x)(t)$ and $(U*V)(t)$ are well defined for every $t\in[0,\infty)$ because the corresponding integrals exist by Proposition 7 and Corollary 8.

Indeed, if for a fixed $t \in [0,\infty)$ we set $\tilde{x}(s) = x(t-s)$ for $s \in [0,t]$ then $\tilde{x} \in G([0,t],X)$, $\|x\|_{G([0,t],X)} = \|\tilde{x}\|_{G([0,t],X)}$ and the integral $\int_0^t \mathrm{d}[U(s)]\tilde{x}(s)$ exists by Proposition 7. We have

$$\int_0^t \mathrm{d}[U(s)]x(t-s) = \int_0^t \mathrm{d}[U(s)]\tilde{x}(s) = (U * x)(t)$$

and $(U * x)(t)$ makes sense for every $t \in [0,\infty)$. Hence Proposition 6 yields

$$\|(U * x)(t)\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[0,t]}(U) \cdot \|\tilde{x}\|_{G([0,t],X)} = (\mathcal{B}) \operatorname*{var}_{[0,t]}(U) \cdot \|x\|_{G([0,t],X)}. \tag{8}$$

Similarly the convolution $(U * V)(t)$ exists for every $t \in [0,\infty)$ by Corollary 8 and

$$\|(U * V)(t)\|_{L(X)} \leqslant (\mathcal{B}) \operatorname*{var}_{[0,t]}(U) \cdot \|V\|_{G([0,t],L(X))}. \tag{9}$$

If $U, V \in G([0,\infty),L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty),L(X))$ and $t \in [0,\infty)$ then take $b > t$. Using the definition of the norm $\|\cdot\|_{SV}$ given in (3) for $(\mathcal{B})BV([0,b],L(X))$ and (6) we obtain from (9) also

$$\|(U * V)(t)\|_{L(X)} \leqslant \|U\|_{SV} . \|V\|_{SV} \tag{10}$$

which holds for every $t \in [0,b]$.

**9. Lemma.** *Assume $U \in G([0,\infty),L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty),L(X))$ and*

$$f_n, f \in G([0,\infty),X),\ f_n \to f \ \text{ in } \ G([0,b],X)$$

*for every $b > 0$,*

$$F_n, F \in G([0,\infty),L(X)),\ F_n \to F \ \text{ in } \ G([0,b],L(X))$$

*for every $b > 0$. Then for every $b > 0$*

$$(U * f_n)(t) \to (U * f)(t) \ \text{ uniformly in } \ [0,b]$$

*and*

$$(U * F_n)(t) \to (U * F)(t) \ \text{ uniformly in } \ [0,b]$$

*and therefore also*

$$(U * f_n)(t) \to (U * f)(t) \ \text{ for all } \ [0,\infty)$$

*and*

$$(U * F_n)(t) \to (U * F)(t) \quad \textit{for all} \quad [0, \infty).$$

P r o o f. Assume that $b \in [0, \infty)$ is given. By Proposition 7 the convolutions $U * f_n$, $U * f$ exist and for $t \subset [0, b]$ we get by (8)

$$\begin{aligned} \|(U * (f_n - f))(t)\|_X &\leqslant (\mathcal{B}) \operatorname*{var}_{[0,t]}(U) \cdot \|f_n - f\|_{G([0,t],X)} \\ &\leqslant (\mathcal{B}) \operatorname*{var}_{[0,b]}(U) \cdot \|f_n - f\|_{G([0,b],X)}. \end{aligned}$$

Because $f_n \to f$ in $G([0, b], X)$ the first assertion of the Lemma holds. The second can be proved similarly. □

**10. Proposition.**

**I.** *If*

$$U \in G([0, \infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X))$$

*and* $V \in G([0, \infty), L(X))$, $x \in G([0, \infty), X)$ *then the convolutions* $(U * V)(t)$, $(U * x)(t)$ *exist for every* $t \in [0, +\infty)$ *and* $U * V \in G([0, \infty), L(X))$, $U * x \in G([0, \infty), X)$.

**II.** *If*

$$U \in C([0, \infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X))$$

*and* $V \in G([0, \infty), L(X))$, $x \in G([0, \infty), X)$ *then the convolutions* $(U * V)(t)$, $(U * x)(t)$ *exist for every* $t \in [0, +\infty)$ *and* $U * V \in C([0, \infty), L(X))$, $U * x \in C([0, \infty), X)$.

P r o o f. Let us first show part I of the statement. The existence of the convolution $(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t - s)$ for $t \in [0, +\infty)$ follows from Corollary 8.

Assume now that $0 \leqslant c < d < +\infty, V_0 \in L(X)$ and define

$$\begin{aligned} W(\tau) &= V_0 \quad \text{for} \quad \tau \in (c, d) \\ W(\tau) &= 0 \quad \text{for} \quad \tau \in [0, +\infty) \setminus (c, d). \end{aligned}$$

Evidently $W \in G([0, \infty), L(X))$ and

$$\begin{aligned} &(U * W)(t) = \int_0^t \mathrm{d}[U(s)]W(t - s) = 0 \quad \text{if} \quad t \leqslant c, \\ &(U * W)(t) = -\int_0^{t-c} \mathrm{d}[U(s)]V_0 = [U((t - c)-) - U(0)]V_0 \quad \text{if} \quad t \in (c, d), \\ &(U * W)(t) = [U((t - c)-) - U((t - d)+)]V_0 \quad \text{if} \quad t \geqslant d \end{aligned}$$

where we write $U(\sigma+) = \lim\limits_{\varrho\to\sigma+} U(\varrho)$ and $U(\sigma-) = \lim\limits_{\varrho\to\sigma-} U(\varrho)$. Now it is easy to see that the convolution $(U * W)(t)$ has onesided limits at every point $t \in [0, +\infty)$ and this means that $U * W \in G([0,\infty), L(X))$.

Similarly it can be shown that if $c \in [0, +\infty), V_0 \in L(X)$ and

$$W(\tau) = V_0 \quad \text{for} \quad \tau = c$$
$$W(\tau) = 0 \quad \text{for} \quad \tau \in [0, +\infty), \tau \neq c$$

then again $U * W \in G([0,\infty), L(X))$.

Assume now that $V \in G([0,\infty), L(X))$. Then for every $b > 0$ there exists a sequence of finite step functions $W_n\colon\ [0,b] \to L(X)$, $n \in \mathbb{N}$ such that

$$\|V - W_n\|_{G([0,b],L(X))} \to 0 \quad \text{for} \quad n \to \infty.$$

Since every finite step function is a finite combination of functions of the type $W$ considered above, we can conclude by the results stated above that for every $n$ the convolution $U * W_n$ belongs to $G([0,b], L(X))$.

Let us note that $b > 0$ can be taken arbitrarily large, e.g. larger than $t$ at which we look for the existence of onesided limits of the convolution $U * W_n$.

By Lemma 9 we have for $t \in [0,b]$

$$\|U * V - U * W_n\|_{G([0,b],L(X))} \to 0$$

if $n \to \infty$ and this means that on $[0,b]$ the convolution $U * V$ is the uniform limit of regulated functions $U * W_n$. Hence $U * V$ is regulated on the interval $[0,b]$. Since $b > 0$ can be taken arbitrarily large in this reasoning, we conclude easily that $U * V \in G([0,\infty), L(X))$ and the statement is proved.

Concerning the convolution $(U * x)(t)$ we can proceed analogously.

Concernig part II of the proposition we can check that for the function $W$ given by

$$W(\tau) = V_0 \in L(X) \quad \text{for} \quad \tau \in (c,d)$$
$$W(\tau) = 0 \quad \text{for} \quad \tau \in [0, +\infty) \setminus (c,d)$$

we get

$$(U * W)(t) = \int_0^t \mathrm{d}[U(s)]W(t-s) = 0 \quad \text{if} \quad t \leqslant c,$$
$$(U * W)(t) = -\int_0^{t-c} \mathrm{d}[U(s)]V_0 = [U(t-c) - U(0)]V_0 \quad \text{if} \quad t \in (c,d),$$
$$(U * W)(t) = [U(t-c) - U(t-d)]V_0 \quad \text{if} \quad t \geqslant d$$

and since $U \in C([0,\infty), L(X))$ we can see easily that $U * V$ is continuous. Similarly for the case when $W$ is nonzero at a single point $c \geqslant 0$.

Using Lemma 9 as above we can see that for the general case of a regulated $V$ we obtain that on every bounded interval $[0,b]$ the convolution $U * V$ is the uniform limit of continuous functions $U * W_n$ and therefore it is also continuous on this interval. $\square$

**11. Proposition.** *If $U, V \in G([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ and $V(0) = 0$ then $U * V \in G([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ and*

$$(\mathcal{B}) \operatorname*{var}_{[0,b]}(U * V) \leqslant (\mathcal{B}) \operatorname*{var}_{[0,b]}(U).(\mathcal{B}) \operatorname*{var}_{[0,b]}(V) \tag{11}$$

*holds for every $b > 0$.*

Proof. If $U, V \in G([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ then the convolution

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s) \in L(X)$$

is well defined for every $t \in [0,\infty)$.

By Proposition 9 we have $U * V \in G([0,\infty), L(X))$ and it remains to show that $U * V \in (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$.

Define

$$\widetilde{V}(\sigma) = V(\sigma) \ \text{ for } \ \sigma \geqslant 0 \ \text{ and } \ \widetilde{V}(\sigma) = 0 \ \text{ for } \ \sigma < 0.$$

Assume that $b \geqslant 0$ and let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$ be an arbitrary partition of $[0,b]$.

Using the definition of $\widetilde{V}$ we have for every $\alpha \in [0,b]$ the equality

$$\int_0^\alpha \mathrm{d}[U(s)]V(\alpha - s) = \int_0^b \mathrm{d}[U(s)]\widetilde{V}(\alpha - s)$$

(at this point the assumption $V(0) = 0$ is used) and therefore by Proposition 6 we obtain for any choice of $x_j \in X$, $\|x_j\|_X \leqslant 1$, $j = 1, \ldots, k$ the inequalities

$$\begin{aligned} &\Big\| \sum_{j=1}^k [(U * V)(\alpha_j) - (U * V)(\alpha_{j-1})]x_j \Big\|_X \\ &\quad = \Big\| \sum_{j=1}^k \Big[ \int_0^{\alpha_j} \mathrm{d}[U(s)]V(\alpha_j - s) - \int_0^{\alpha_{j-1}} \mathrm{d}[U(s)]V(\alpha_{j-1} - s) \Big] x_j \Big\|_X \\ &\quad = \Big\| \sum_{j=1}^k \int_0^b \mathrm{d}[U(s)][\widetilde{V}(\alpha_j - s) - \widetilde{V}(\alpha_{j-1} - s)]x_j \Big\|_X \\ &\quad \leqslant (\mathcal{B}) \operatorname*{var}_{[0,b]}(U) \sup_{s \in [0,b]} \Big\| \sum_{j=1}^k [\widetilde{V}(\alpha_j - s) - \widetilde{V}(\alpha_{j-1} - s)]x_j \Big\|_X . \end{aligned} \tag{12}$$

Since for every $s \in [0,b]$ the points

$$\alpha_0 - s < \alpha_1 - s < \ldots < \alpha_k - s$$

form a partition of $[-s, b-s]$ and if $\alpha_j - s \leqslant 0$ then $\widetilde{V}(\alpha_j - s) = 0$, we can state that the point 0 (at which $\widetilde{V}(0) = V(0) = 0$) with the points for which $\alpha_j - s > 0$ form a partition of $[0, b-s] \subset [0,b]$.

Hence

$$\Big\| \sum_{j=1}^{k} [\widetilde{V}(\alpha_j - s) - \widetilde{V}(\alpha_{j-1} - s)] x_j \Big\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[0,b-s]}(V) \leqslant (\mathcal{B}) \operatorname*{var}_{[0,b]}(V)$$

and by (12) we get

$$\Big\| \sum_{j=1}^{k} [(U*V)(\alpha_j) - (U*V)(\alpha_{j-1})] x_j \Big\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[0,b]}(U) \cdot (\mathcal{B}) \operatorname*{var}_{[0,b]}(V). \tag{13}$$

This inequality immediately yields

$$(\mathcal{B}) \operatorname*{var}_{[0,b]}(U*V) < \infty,$$

i.e. $U*V \in (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ because $b \geqslant 0$ can be taken arbitrary. Moreover, from the inequality (13) we also obtain that (11) holds for every $b \in [0,\infty)$. □

## Some special cases

Let us denote by $BV_{\mathrm{loc}}([0,\infty), L(X))$ the space of $U\colon [0,\infty) \to L(X)$ for which $U \in BV([0,b], L(X))$ for every $b \geqslant 0$.

In this more special space the following result analogous to Proposition 10 holds.

**12. Proposition.** *If $U, V \in BV_{\mathrm{loc}}([0,\infty), L(X))$ and $V(0) = 0$ then $U*V \in BV_{\mathrm{loc}}([0,\infty), L(X))$ and*

$$\operatorname*{var}_{[0,b]}(U*V) \leqslant \operatorname*{var}_{[0,b]}(U) . \operatorname*{var}_{[0,b]}(V) \tag{14}$$

*holds for every $b > 0$.*

Proof. Since by Proposition 3 we have

$$BV([0,b], L(X)) \subset G([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$$

for every $b \geqslant 0$, the convolution $(U * V)(t)$ is well defined for every $t \geqslant 0$.

Define $\widetilde{V}(\sigma) = V(\sigma)$ for $\sigma \geqslant 0$ and $\widetilde{V}(\sigma) = 0$ for $\sigma < 0$.

Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$ be an arbitrary partition of $[0, b]$.

Then

$$
\begin{aligned}
&\sum_{j=1}^{k} \|[(U * V)(\alpha_j) - (U * V)(\alpha_{j-1})]\|_{L(X)} \\
&= \sum_{j=1}^{k} \left\| \int_0^b \mathrm{d}[U(s)][\widetilde{V}(\alpha_j - s) - \widetilde{V}(\alpha_{j-1} - s)] \right\|_{L(X)} \\
&\leqslant \operatorname*{var}_{[0,b]}(U) \cdot \sup_{s \in [0,b]} \sum_{j=1}^{k} \|\widetilde{V}(\alpha_j - s) - \widetilde{V}(\alpha_{j-1} - s)\|_{L(X)} \leqslant \operatorname*{var}_{[0,b]}(U) \cdot \operatorname*{var}_{[0,b]}(V)
\end{aligned}
$$

because for every $s \in [0, b]$,

$$\alpha_0 - s < \alpha_1 - s < \ldots < \alpha_k - s$$

is a partition of $[-s, b - s]$ and for $\alpha_j - s \leqslant 0$ we have $\widetilde{V}(\alpha_j - s) = 0$. The points for which $\alpha_j - s > 0$ together with the point 0 (at which $\widetilde{V}(0) = V(0) = 0$) form a partition of $[0, b - s] \subset [0, b]$ and this yields the result similarly as in the proof of Proposition 10. □

Another special case is the case of the space

$$C([0, \infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X)).$$

We have the following result.

**13. Proposition.** *If $U, V \in C([0, \infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X))$ and $V(0) = 0$ then $U * V \in C([0, \infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X))$ and (11) holds for every $b > 0$.*

P r o o f. Since $C([0, \infty), L(X)) \subset G([0, \infty), L(X))$, Proposition 10 can be used and it remains to show that $U * V \in C([0, \infty), L(X))$.

Consider a compact interval $[0, b]$, $b \geqslant 0$. Assume that $t_1, t_2 \in [0, b]$, $t_1 < t_2$.

Then

$$
\begin{aligned}
(U * V)(t_2) - (U * V)(t_1) = &\int_0^{t_1} \mathrm{d}[U(s)][V(t_2 - s) - V(t_1 - s)] \\
&+ \int_{t_1}^{t_2} \mathrm{d}[U(s)]V(t_2 - s).
\end{aligned}
\tag{15}
$$

Since $U$ and $V$ are continuous on $[0,b]$, they are uniformly continuous on this compact interval and therefore for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $s_1, s_2 \in [0,b]$, $|s_1 - s_2| < \delta$ then $\|V(s_2) - V(s_1)\|_{L(X)} < \varepsilon$.

Assume that $|t_1 - t_2| < \delta$. Then by (15) we have

$$\begin{aligned}
&\|(U*V)(t_2) - (U*V)(t_1)\|_{L(X)} \\
&\qquad\leqslant \left\| \int_0^{t_1} \mathrm{d}[U(s)][V(t_2 - s) - V(t_1 - s)] \right\|_{L(X)} \\
&\qquad\quad + \left\| \int_{t_1}^{t_2} \mathrm{d}[U(s)]V(t_2 - s) \right\|_{L(X)} \\
&\qquad\leqslant (\mathcal{B}) \operatorname*{var}_{[0,t_1]}(U) \sup_{s\in[0,t_1]} \|V(t_2 - s) - V(t_1 - s)\|_{L(X)} \\
&\qquad\quad + (\mathcal{B}) \operatorname*{var}_{[t_1,t_2]}(U) \sup_{s\in[t_1,t_2]} \|V(t_2 - s)\|_{L(X)} \leqslant 2(\mathcal{B}) \operatorname*{var}_{[0,b]}(U) \cdot \varepsilon
\end{aligned}$$

and this yields the continuity of the convolution $U * V$. □

**14. Lemma.** *If $U, V \in C([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ and $V(0) = 0$ then for any $f \in G([0,\infty), X)$, $F \in G([0,\infty), L(X))$ we have*

$$(U * (V * f))(t) = ((U * V) * f)(t) \tag{16}$$

*and*

$$(U * (V * F))(t) = ((U * V) * F)(t). \tag{17}$$

P r o o f. The relation (16) holds for every finite step function.

To show this assume that $f(t) = x \in X$ for $t \in (c,d)$, $0 \leqslant c < d$ and $f(t) = 0$ for $t \notin (c,d)$.

Assume e.g. that $t > d$. We have

$$\begin{aligned}
&(V * f)(r) = 0 \quad \text{for} \quad r \leqslant c, \\
&(V * f)(r) = [V(r - c) - V(0)]x = V(r - c)x \quad \text{for} \quad r \in (c,d), \\
&(V * f)(r) = [V(r - c) - V(r - d)]x \quad \text{for} \quad r \geqslant d.
\end{aligned}$$

Hence

$$\begin{aligned}
&(V * f)(t - s) = 0 \quad \text{for} \quad s \geqslant t - c, \\
&(V * f)(t - s) = V(t - s - c)x \quad \text{for} \quad s \in (t - d, t - c), \\
&(V * f)(t - s) = [V(t - s - c) - V(t - s - d)]x \quad \text{for} \quad s \leqslant t - d
\end{aligned}$$

and

$$
\begin{aligned}
(U*(V*f))(t) &= \int_0^t \mathrm{d}[U(s)](V*f)(t-s) \\
&= \int_0^{t-d} \mathrm{d}[U(s)](V*f)(t-s) + \int_{t-d}^{t-c} \mathrm{d}[U(s)](V*f)(t-s) \\
&= \int_0^{t-d} \mathrm{d}[U(s)][V(t-s-c) - V(t-s-d)]x \\
&\quad + \int_{t-d}^{t-c} \mathrm{d}[U(s)]V(t-s-c)x \\
&= \int_0^{t-c} \mathrm{d}[U(s)]V(t-s-c)x - \int_0^{t-d} \mathrm{d}[U(s)]V(t-s-d)x \\
&= (U*V)(t-c)x - (U*V)(t-d)x
\end{aligned}
$$

since $V(0) = 0$.

On the other hand,

$$
\begin{aligned}
((U*V)*f))(t) &= \int_0^t \mathrm{d}[(U*V)(s)]f(t-s) \\
&= \int_{t-d}^{t-c} \mathrm{d}[(U*V)(s)]x = (U*V)(t-c)x - (U*V)(t-d)x
\end{aligned}
$$

and (16) holds for this choice of $f$ and $t > d$. Similarly we can show that (16) holds also for $0 \leqslant t \leqslant d$.

The relation (16) can be easily checked also for a function $f$ given by $f(c) = x \in X$ and $f(t) = 0$ for $t \neq c$ if $c \geqslant 0$.

Using these facts we see that (16) holds for every finite step function because these functions are finite linear combinations of functions of type given above.

For the general case of $f \in G([0,\infty), X)$ we take a $t \in [0,\infty)$, $b \geqslant t$ and use the fact that $f \in G([0,b], X)$ can be uniformly approximated by a sequence of finite step functions $f_n \in G([0,b], X)$ for which we have

$$
(U*(V*f_n))(t) = ((U*V)*f_n)(t).
$$

Using Lemma 11 we pass to the limits for $n \to \infty$ in this last equality and we get (16) for $f \in G([0,\infty), X)$.

The case of $F \in G([0,\infty), L(X))$ is analogous and therefore (17) holds, too. □

**15 A. Theorem.** *For every $b > 0$ the set of all $U\colon [0,b] \to L(X)$ with $U \in C([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution $U*V$ as multiplication and $(\mathcal{B})\operatorname{var}_{[0,b]}(U)$ as the norm.*

P r o o f. The set of all $U\colon [0,b] \to L(X)$ with

$$U \in C([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$$

and $U(0) = 0$ is a Banach space with the norm given by $(\mathcal{B})\operatorname{var}_{[0,b]}(U)$.

It remains to show that this set is an algebra with respect to the multiplication given by the convolution $U * V$.

It is evident by the linearity of the integral that for $U, V, W$ belonging to our set we have

$$U * (V + W) = U * V + U * W, \tag{18}$$

$$(U + V) * W = U * V + U * W, \tag{19}$$

$$\alpha(U * V) = (\alpha U) * V = U * (\alpha V) \quad \text{for} \quad \alpha \in \mathbb{R}. \tag{20}$$

From Lemma 14 we obtain the associativity of the product, i.e.

$$U * (V * W) = (U * V) * W, \tag{21}$$

and these relations show that our set is an algebra. By Proposition 11 we have (11) and this completes the proof of the theorem. □

**15 B. Theorem.** *For every $b > 0$ the set of all $U\colon [0,b] \to L(X)$ with $U \in C([0,b], L(X)) \cap BV([0,b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution $U * V$ as multiplication and $\operatorname{var}_{[0,b]}(U)$ as norm.*

P r o o f. The set of all $U\colon [0,b] \to L(X)$ with

$$U \in C([0,b], L(X)) \cap BV([0,b], L(X))$$

and $U(0) = 0$ is a Banach space with the norm given by $\operatorname{var}_{[0,b]}(U)$.

As in the proof of Theorem 15 A the relations (18)–(21) hold and our set is therefore an algebra. By Proposition 12 we have (14) and this completes the proof of the theorem. □

R e m a r k. It is interesting to mention that the set of all $U\colon [0,b] \to L(X)$ with $U \in G([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$ and $U(0) = 0$ is not a Banach algebra with the Stieltjes convolution $U * V$ as multiplication and $(\mathcal{B})\operatorname{var}_{[0,b]}(U)$ as the norm even if Proposition 10 and (18)–(20) hold. The problem is caused by the associativity relation (21) which is not valid for regulated $U, V, W$ which are not continuous.

In [11, Theorem 13] an integration by parts result was proved which in our situation reads as follows.

**16. Proposition.** *If* $U, W \in G([a,b], L(X)) \cap (\mathcal{B})BV([a,b], L(X))$ *then*

$$\begin{aligned}\int_a^b U(s)\mathrm{d}[W(s)] + \int_a^b \mathrm{d}[U(s)]W(s) &= U(b)W(b) - U(a)W(a)\\ &- \sum_{a\leqslant\tau<b} \Delta^+U(\tau)\Delta^+W(\tau) + \sum_{a<\tau\leqslant b} \Delta^-U(\tau)\Delta^-W(\tau),\end{aligned} \tag{22}$$

*where* $\Delta^+U(\tau) = U(\tau+) - U(\tau) = \lim_{s\to\tau+} U(s) - U(\tau)$, $\tau \in [a,b)$,

$$\Delta^-U(\tau) = U(\tau) - U(\tau-) = U(\tau) - \lim_{s\to\tau-} U(s), \tau \in (a,b]$$

*and similarly for* $W$.

Remark. Since $BV([a,b], L(X)) \subset G([a,b], L(X)) \cap (\mathcal{B})BV([a,b], L(X))$ by Proposition 3, the relation (16) holds also if $U, W \in BV([a,b], L(X))$.

**17. Proposition.** *If* $U, V \in G([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ *then*

$$\begin{aligned}(U*V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s) &= \int_0^t U(t-s)\mathrm{d}[V(s)] + U(t)V(0) - U(0)V(t)\\ &+ \sum_{0\leqslant\tau<t} \Delta^+U(\tau)\Delta^-V(t-\tau) - \sum_{0<\tau\leqslant t} \Delta^-U(\tau)\Delta^+V(t-\tau).\end{aligned} \tag{23}$$

Proof. Let us fix a $t \in [0,\infty)$.

Putting $a = 0$, $b = t$ and $W(s) = V(t-s)$ for $s \in [0,t]$ we can see that $U, W \in G([0,t], L(X)) \cap (\mathcal{B})BV([0,t], L(X))$ holds.

The integration by parts formula (16) can be used to get

$$\begin{aligned}\int_0^t U(s)\mathrm{d}_s[V(t-s)] + \int_0^t \mathrm{d}[U(s)]V(t-s) &= \int_0^t U(s)\mathrm{d}[W(s)]\\ &+ \int_0^t \mathrm{d}[U(s)]W(s) = U(t)W(t) - U(0)W(0)\\ &- \sum_{0\leqslant\tau<t} \Delta^+U(\tau)\Delta^+W(\tau) + \sum_{0<\tau\leqslant t} \Delta^-U(\tau)\Delta^-W(\tau).\end{aligned} \tag{24}$$

Since $W(s) = V(t-s)$ we have

$$W(\tau+) = \lim_{\varrho\to 0+} W(\tau+\varrho) = \lim_{\varrho\to 0+} V(t-(\tau+\varrho)) = V((t-\tau)-)$$

for $\tau \in [0,t)$ and

$$\Delta^+ W(\tau) = W(\tau+) - W(\tau) = V((t-\tau)-) - V(t-\tau) = -\Delta^- V(t-\tau).$$

Similarly for $\tau \in (0,t]$ we get

$$\Delta^- W(\tau) = W(\tau) - W(\tau-) = -\Delta^+ V(t-\tau)$$

and by (18) we obtain

$$\begin{aligned}\int_0^t U(s)\mathrm{d}_s[V(t-s)] + \int_0^t \mathrm{d}[U(s)]V(t-s) &= U(t)V(0) - U(0)V(t) \\ &+ \sum_{0\leqslant\tau<t} \Delta^+ U(\tau)\Delta^- V(t-\tau) - \sum_{0<\tau\leqslant t} \Delta^- U(\tau)\Delta^+ V(t-\tau).\end{aligned} \tag{25}$$

Using the substitution $t - s = \sigma$ we get

$$\int_0^t U(s)\mathrm{d}_s[V(t-s)] = \int_t^0 U(t-\sigma)\mathrm{d}[V(\sigma)] = -\int_0^t U(t-s)\mathrm{d}[V(s)]$$

and from (19) we easily obtain (17). □

Denote now

$$G^-([a,b],X) = \{x \in G([a,b],X);\ x(t-) = x(t), t \in (a,b]\}$$

and similarly e.g.

$$G^-([0,+\infty),X) = \{x \in G([0,+\infty),X);\ x(t-) = x(t), t \in (0,+\infty)\}$$

for infinite intervals.

**18. Corollary.** *Assume that*

$$U, V \in G^-([0,+\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X)).$$

*Then*

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s) = \int_0^t U(t-s)\mathrm{d}[V(s)] + U(t)V(0) - U(0)V(t) \tag{26}$$

*for every* $t \in [0, +\infty)$.

*Moreover, if* $U(0) = V(0) = 0$ *then*

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s) = \int_0^t U(t-s)\mathrm{d}[V(s)] \tag{27}$$

*for every* $t \in [0, +\infty)$.

Proof. If

$$U, V \in G^-([0, +\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X))$$

then clearly $\Delta^- U(\tau) = 0$ and $\Delta^- V(\tau) = 0$ for every $\tau > 0$. Therefore the sums on the right hand side of (23) vanish and (26) holds.

The additional assertion (27) is trivial. □

**19. Corollary.** *Assume that*

$$U, V \in C([0, +\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0, \infty), L(X)).$$

*Then* (26) *holds for every* $t \in [0, +\infty)$. *Moreover, if* $U(0) = V(0) = 0$ *then* (27) *is satisfied for every* $t \in [0, +\infty)$.

Proof. The statement follows easily from Corollary 18 and from the fact that

$$C([0, +\infty), L(X)) \subset G^-([0, +\infty), L(X)).$$

□

## $\eta$-variations

Assume that $[a, b] \subset \mathbb{R}$ is a bounded interval and that $\eta \geqslant 0$ is given. Define

$$\eta \operatorname*{var}_{[a,b]}(A) = \sup \left\{ \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \mathrm{e}^{-\eta(\alpha_{j-1} - a)} \right\}$$

where the supremum is taken over all finite partitions $D$ of the interval $[a, b]$.

Similarly define

$$V_a^b(\eta, A, D) = \sup \left\{ \left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j \mathrm{e}^{-\eta(\alpha_{j-1} - a)} \right\|_X \right\},$$

where the supremum is taken over all possible choices of $x_j \in X$, $j = 1, \ldots, k$ with $\|x_j\|_X \leqslant 1$, and set

$$\eta(\mathcal{B}) \operatorname*{var}_{[a,b]}(A) = \sup V_a^b(\eta, A, D)$$

where the supremum is taken over all finite partitions $D$ of the interval $[a, b]$.

Since for every $j = 1, \ldots, k$ we have

$$\mathrm{e}^{-\eta(b-a)} \leqslant \mathrm{e}^{-\eta(\alpha_{j-1}-a)} \leqslant 1$$

we get

$$\mathrm{e}^{-\eta(b-a)} \operatorname*{var}_{[a,b]}(A) \leqslant \eta \operatorname*{var}_{[a,b]}(A) \leqslant \operatorname*{var}_{[a,b]}(A) \tag{28}$$

and also

$$\mathrm{e}^{-\eta(b-a)} V_a^b(A, D) \leqslant V_a^b(\eta, A, D) \leqslant V_a^b(A, D).$$

The last inequalities lead immediately to

$$\mathrm{e}^{-\eta(b-a)} (\mathcal{B}) \operatorname*{var}_{[a,b]}(A) \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[a,b]}(A) \leqslant (\mathcal{B}) \operatorname*{var}_{[a,b]}(A). \tag{29}$$

Let us mention that evidently

$$0 \operatorname*{var}_{[a,b]}(A) = \operatorname*{var}_{[a,b]}(A) \text{ and } 0(\mathcal{B}) \operatorname*{var}_{[a,b]}(A) = (\mathcal{B}) \operatorname*{var}_{[a,b]}(A)$$

hold by the corresponding definitions.

It is well known that $BV([a, b]; L(X))$ with the norm

$$\|A\|_{BV} = \|A(a)\|_{L(X)} + \operatorname*{var}_{[a,b]}(A)$$

is a Banach space and by Proposition 4 we know that $(\mathcal{B})BV([a, b]; L(X))$ with the norm

$$\|A\|_{SV} = \|A(a)\|_{L(X)} + (\mathcal{B}) \operatorname*{var}_{[a,b]}(A)$$

is also a Banach space.

Taking into account the inequalities (28) and (29) we get the following statement.

**20. Proposition.** *For every $\eta \geqslant 0$ the space $BV([a, b]; L(X))$ with the norm*

$$\|A\|_{BV,\eta} = \|A\|_{L(X)} + \eta \operatorname*{var}_{[a,b]}(A)$$

*is a Banach space and the space* $(\mathcal{B})BV([a,b];L(X))$ *with the norm*

$$\|A\|_{SV,\eta} = \|A(a)\|_{L(X)} + \eta(\mathcal{B}) \operatorname*{var}_{[a,b]}(A)$$

*is also a Banach space.*

*The norms* $\|A\|_{BV,\eta}$ *and* $\|A\|_{BV}$ *are equivalent on* $BV([a,b];L(X))$ *and the norms* $\|A\|_{SV,\eta}$ *and* $\|A\|_{SV}$ *are equivalent on* $(\mathcal{B})BV([a,b];L(X))$.

**21. Lemma.** *Assume that*

$$U \in G([0,\infty),L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty),L(X)), \ f \in G([0,\infty),X)$$

*and that* $\eta \geqslant 0$ *is given.*

*Then the integral* $\int_0^b \mathrm{d}[U(s)]\mathrm{e}^{-\eta s} f(s) \in X$ *exists for every* $b > 0$ *and*

$$\left\| \int_0^b \mathrm{d}[U(s)]\mathrm{e}^{-\eta s} f(s) \right\|_X \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[0,b]}(U) \cdot \sup_{s\in[0,b]} \|f(s)\|_X \tag{30}$$

*holds.*

Proof. The existence of the integral $\int_0^b \mathrm{d}[U(s)]\mathrm{e}^{-\eta s} f(s)$ is clear because the function $\mathrm{e}^{-\eta s} f(s)$ is regulated on $[0,\infty)$ (cf. Proposition 7).

Assume that $b > 0$ is fixed. By the existence of the integral for any $\varepsilon > 0$ there is a gauge $\delta$ on $[0,b]$ such that for every $\delta$-fine $P$-partition

$$D = \{0 = \alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k = b\}$$

of $[0,b]$ the inequality

$$\left\| \int_0^b \mathrm{d}[U(s)]\mathrm{e}^{-\eta s} f(s) - \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]\mathrm{e}^{-\eta\tau_j} f(\tau_j) \right\|_X < \varepsilon$$

holds. Hence

$$\left\| \int_0^b \mathrm{d}[U(s)]\mathrm{e}^{-\eta s} f(s) \right\|_X < \varepsilon + \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]\mathrm{e}^{-\eta\tau_j} f(\tau_j) \right\|_X. \tag{31}$$

Let us choose a fixed $\delta$-fine $P$-partition $D$ of $[0,b]$ for which $\alpha_{j-1} < \tau_j$ for every $j = 1,\ldots,k$. Then

$$\begin{aligned}\Big\|\sum_{j=1}^{k}&[U(\alpha_j)-U(\alpha_{j-1})]\mathrm{e}^{-\eta\tau_j}f(\tau_j)\Big\|_X\\ &=\Big\|\sum_{j=1}^{k}[U(\alpha_j)-U(\alpha_{j-1})]\mathrm{e}^{-\eta\alpha_{j-1}}\mathrm{e}^{-\eta(\tau_j-\alpha_{j-1})}f(\tau_j)\Big\|_X\\ &=\Big\|\sum_{j=1}^{k}[U(\alpha_j)-U(\alpha_{j-1})]\mathrm{e}^{-\eta\alpha_{j-1}}\frac{\mathrm{e}^{-\eta(\tau_j-\alpha_{j-1})}f(\tau_j)}{\|f(\tau_j)\|_X}\|f(\tau_j)\|_X\Big\|_X\end{aligned}$$

and we have

$$\Big\|\frac{\mathrm{e}^{-\eta(\tau_j-\alpha_{j-1})}f(\tau_j)}{\|f(\tau_j)\|_X}\Big\|_X \leqslant 1$$

for $j = 1,\ldots,k$.

Hence

$$\begin{aligned}\Big\|\sum_{j=1}^{k}&[U(\alpha_j)-U(\alpha_{j-1})]\mathrm{e}^{-\eta\alpha_{j-1}}\frac{\mathrm{e}^{-\eta(\tau_j-\alpha_{j-1})}f(\tau_j)}{\|f(\tau_j)\|_X}\|f(\tau_j)\|_X\Big\|_X\\ &\leqslant \sup_{j=1,\ldots,k}\|f(\tau_j)\|_X\Big\|\sum_{j=1}^{k}[U(\alpha_j)-U(\alpha_{j-1})]\mathrm{e}^{-\eta\alpha_{j-1}}\frac{\mathrm{e}^{-\eta(\tau_j-\alpha_{j-1})}f(\tau_j)}{\|f(\tau_j)\|_X}\Big\|_X\\ &\leqslant \sup_{s\in[0,b]}\|f(s)\|_X\cdot\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(U)\end{aligned}$$

and this together with (31) gives (30) because $\varepsilon > 0$ can be taken arbitrarily small. □

In addition to Proposition 10 we will prove the following statement.

**22. Proposition.** *Assume that*

$$U,V\in G([0,\infty),L(X))\cap(\mathcal{B})BV_{\mathrm{loc}}([0,\infty),L(X))$$

*and that* $V(0)=0$.

*Then the convolution* $(U*V)(t)\in L(X)$ *is well defined for every* $t\in[0,\infty)$ *and for every* $b>0$, $\eta\geqslant 0$ *the inequality*

$$\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(U*V)\leqslant\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(U).\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(V) \tag{32}$$

*holds.*

P r o o f.

Define

$$\widetilde{V}(\sigma)=V(\sigma)\quad\text{for}\quad\sigma\geqslant 0$$

and

$$\widetilde{V}(\sigma)=0\quad\text{for}\quad\sigma<0.$$

Assume that $b\geqslant 0$ and let $0=\alpha_0<\alpha_1<\ldots<\alpha_k=b$ be an arbitrary partition of $[0,b]$.

Using the definition of $\widetilde{V}$ we have for every $\alpha\in[0,b]$ the equality

$$\int_0^{\alpha}\mathrm{d}[U(s)]V(\alpha-s)=\int_0^{b}\mathrm{d}[U(s)]\widetilde{V}(\alpha-s)$$

and therefore we obtain for any choice of $x_j\in X$, $\|x_j\|_X\leqslant 1$, $j=1,\ldots,k$ the equalities

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k}[(U*V)(\alpha_j)-(U*V)(\alpha_{j-1})]x_j\mathrm{e}^{-\eta\alpha_{j-1}}\Big\|_X\\
&=\Big\|\sum_{j=1}^{k}\Big[\int_0^{\alpha_j}\mathrm{d}[U(s)]V(\alpha_j-s)-\int_0^{\alpha_{j-1}}\mathrm{d}[U(s)]V(\alpha_{j-1}-s)\Big]x_j\mathrm{e}^{-\eta\alpha_{j-1}}\Big\|_X\\
&=\Big\|\sum_{j=1}^{k}\int_0^{b}\mathrm{d}[U(s)][\widetilde{V}(\alpha_j-s)-\widetilde{V}(\alpha_{j-1}-s)]x_j\mathrm{e}^{-\eta\alpha_{j-1}}\Big\|_X\\
&=\Big\|\int_0^{b}\mathrm{d}[U(s)]\mathrm{e}^{-\eta s}\sum_{j=1}^{k}[\widetilde{V}(\alpha_j-s)-\widetilde{V}(\alpha_{j-1}-s)]x_j\mathrm{e}^{-\eta(\alpha_{j-1}-s)}\Big\|_X.
\end{aligned}\tag{33}$$

The function

$$s\mapsto\sum_{j=1}^{k}[\widetilde{V}(\alpha_j-s)-\widetilde{V}(\alpha_{j-1}-s)]x_j\mathrm{e}^{-\eta(\alpha_{j-1}-s)}\in X$$

is regulated on $[0,b]$ because $V\in G([0,b],L(X))$ and therefore by Lemma 21 we obtain

$$\begin{aligned}
&\Big\|\int_0^{b}\mathrm{d}[U(s)]\mathrm{e}^{-\eta s}\sum_{j=1}^{k}[\widetilde{V}(\alpha_j-s)-\widetilde{V}(\alpha_{j-1}-s)]x_j\mathrm{e}^{-\eta(\alpha_{j-1}-s)}\Big\|_X\\
&\qquad\leqslant\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(U)\cdot\sup_{s\in[0,b]}\Big\|\sum_{j=1}^{k}[\widetilde{V}(\alpha_j-s)-\widetilde{V}(\alpha_{j-1}-s)]x_j\mathrm{e}^{-\eta(\alpha_{j-1}-s)}\Big\|_X.
\end{aligned}$$

On the other hand, for every $s \in [0,b]$ we have

$$\Big\| \sum_{j=1}^{k} [\widetilde{V}(\alpha_j - s) - \widetilde{V}(\alpha_{j-1} - s)] x_j \mathrm{e}^{-\eta(\alpha_{j-1}-s)} \Big\|_X \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[0,b]}(V)$$

and this gives

$$\Big\| \sum_{j=1}^{k} [(U*V)(\alpha_j) - (U*V)(\alpha_{j-1})] x_j \mathrm{e}^{-\eta\alpha_{j-1}} \Big\|_X \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[0,b]}(U) \cdot \eta(\mathcal{B}) \operatorname*{var}_{[0,b]}(V)$$

and by the definition we obtain (32). □

Analogously it can be proved that the following statement holds.

**23. Proposition.** *Assume that $U, V \in BV_{\mathrm{loc}}([0,\infty), L(X))$ and that $V(0) = 0$. Then the convolution*

$$(U*V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s) \in L(X)$$

*is well defined for every $t \in [0,\infty)$, and for every $b > 0$ the inequality*

$$\eta \operatorname*{var}_{[0,b]}(U*V) \leqslant \eta \operatorname*{var}_{[0,b]}(U) \cdot \eta \operatorname*{var}_{[0,b]}(V) \tag{28}$$

*holds.*

**24. Lemma.** *Assume that $A \in (\mathcal{B})BV([0,b], L(X))$ for some $b > 0$. Then for every $\eta \geqslant 0$ and $c \in (0,b]$ we have*

$$\eta(\mathcal{B}) \operatorname*{var}_{[0,b]}(A) \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[0,c]}(A) + \mathrm{e}^{-\eta c} \eta(\mathcal{B}) \operatorname*{var}_{[c,b]}(A). \tag{35}$$

P r o o f. Assume that $D$ is a partition of $[0,b]$ given by points

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$$

and that $x_j \in X$ with $\|x_j\|_X \leqslant 1$ for $j = 1, \ldots, k$. Then there is an index $l = 1, \ldots, k$ such that $c \in (\alpha_{l-1}, \alpha_l]$ and

$$\sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j \mathrm{e}^{-\eta\alpha_{j-1}} = \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})] x_j \mathrm{e}^{-\eta\alpha_{j-1}}$$
$$+ [A(\alpha_l) - A(\alpha_{l-1})] x_l \mathrm{e}^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j \mathrm{e}^{-\eta\alpha_{j-1}}.$$

Taking into account that

$$[A(\alpha_l) - A(\alpha_{l-1})]x_l \mathrm{e}^{-\eta\alpha_{l-1}} = [A(\alpha_l) - A(c)]x_l \mathrm{e}^{-\eta\alpha_{l-1}} + [A(c) - A(\alpha_{l-1})]x_l \mathrm{e}^{-\eta\alpha_{l-1}}$$

we obtain

$$\begin{aligned}
&\Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} \Big\|_X \\
&= \Big\| \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l \mathrm{e}^{-\eta\alpha_{l-1}} \\
&\quad + [A(\alpha_l) - A(c)]x_l \mathrm{e}^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} \Big\|_X \\
&\leqslant \Big\| \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l \mathrm{e}^{-\eta\alpha_{l-1}} \Big\|_X \\
&\quad + \Big\| [A(\alpha_l) - A(c)]x_l \mathrm{e}^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} \Big\|_X .
\end{aligned}$$

For the first term on the right hand side of this inequality we evidently have

$$\Big\| \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l \mathrm{e}^{-\eta\alpha_{l-1}} \Big\|_X \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[0,c]}(A)$$

and for the second

$$\begin{aligned}
&\Big\| [A(\alpha_l) - A(c)]x_l \mathrm{e}^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} \Big\|_X \\
&= \Big\| [A(\alpha_l) - A(c)]x_l \mathrm{e}^{-\eta\alpha_{l-1}} + \mathrm{e}^{-\eta c} \sum_{j=l+1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta(\alpha_{j-1} - c)} \Big\|_X \\
&\leqslant \mathrm{e}^{-\eta c} V_c^b(\eta, A, D_+) \leqslant \mathrm{e}^{-\eta c} \eta \operatorname*{var}_{[c,b]}(A)
\end{aligned}$$

($D_+$ is the partition of $[c, b]$ given by the points $c \leqslant \alpha_l < \ldots < \alpha_k = b$). Hence

$$\Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x_j \mathrm{e}^{-\eta\alpha_{j-1}} \Big\|_X \leqslant \eta(\mathcal{B}) \operatorname*{var}_{[0,c]}(A) + \mathrm{e}^{-\eta c} \eta \operatorname*{var}_{[c,b]}(A)$$

and the lemma is proved. □

Similarly it can be shown that the following statement is valid.

**25. Lemma.** *Assume that $A \in BV([0,b], L(X))$ for some $b > 0$. Then for every $\eta \geqslant 0$ and $c \in (0,b]$ we have*

$$\eta \operatorname*{var}_{[0,b]}(A) \leqslant \eta \operatorname*{var}_{[0,c]}(A) + \mathrm{e}^{-\eta c} \eta \operatorname*{var}_{[c,b]}(A). \tag{36}$$

## Resolvents and linear convolution equations

Using Propositions 20, 22 and Theorem 15 A we can now formulate the following result.

**26. Theorem.** *For every $b > 0$ the set of all $U\colon [0,b] \to L(X)$ with $U \in C([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution $U * V$ as multiplication and $\eta(\mathcal{B}) \operatorname{var}_{[0,b]}(U)$ as the norm.*

Similarly by Propositions 20, 23 and Theorem 15 B we get

**27. Theorem.** *For every $b > 0$ the set of all $U\colon [0,b] \to L(X)$ with $U \in C([0,b], L(X)) \cap BV([0,b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution $U * V$ as multiplication and $\eta \operatorname{var}_{[0,b]}(U)$ as the norm.*

Using Banach algebra techniques we come to the following result.

**28. Proposition.** *If $A \in C([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$, $A(0) = 0$ and if there is a $c \in (0,b]$ such that*

$$(\mathcal{B}) \operatorname*{var}_{[0,c]}(A) < 1, \tag{37}$$

*then there exists a unique $R \in C([0,b], L(X)) \cap (\mathcal{B})BV([0,b], L(X))$ with $R(0) = 0$ such that*

$$R(t) - \int_0^t \mathrm{d}[A(s)]R(t-s) = A(t), \ t \in [0,b] \tag{38}$$

*and*

$$R(t) - \int_0^t \mathrm{d}[R(s)]A(t-s) = A(t), \ t \in [0,b]. \tag{39}$$

P r o o f. By Lemma 24, by (29) and (37) we have

$$\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(A) \leqslant \eta(\mathcal{B})\operatorname*{var}_{[0,c]}(A) + \mathrm{e}^{-\eta c}\eta(\mathcal{B})\operatorname*{var}_{[c,b]}(A)$$
$$\leqslant (\mathcal{B})\operatorname*{var}_{[0,c]}(A) + \mathrm{e}^{-\eta c}(\mathcal{B})\operatorname*{var}_{[c,b]}(A) < 1 + \mathrm{e}^{-\eta c}(\mathcal{B})\operatorname*{var}_{[0,b]}(A)$$

and this yields that taking $\eta > 0$ sufficiently large we get

$$\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(A) < 1. \tag{40}$$

Let us now define $A_0(t) = A(t)$ and $A_{n+1}(t) = (A * A_n)(t)$, $t \in [0,b]$ and put

$$R(t) = \sum_{n=0}^{\infty} A_n(t). \tag{41}$$

By (32) from Proposition 22 we get the inequalities

$$\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(A_n) \leqslant (\eta(\mathcal{B})\operatorname*{var}_{[0,b]}(A))^n, \ \ n \in \mathbb{N}.$$

Since (40) holds, this inequality implies the convergence of the series (41) in the space $(\mathcal{B})BV([0,b],L(X))$ and by Proposition 13 also the continuity of its sum $R(t)$, i.e. $R \in C([0,b],L(X)) \cap (\mathcal{B})BV([0,b],L(X))$ and clearly $R(0) = 0$.

By the definitions we have

$$\left(\left(\sum_{n=0}^{N} A_n\right) * A\right)(t) = \left(A * \left(\sum_{n=0}^{N} A_n\right)\right)(t) = \sum_{n=1}^{N+1} A_n(t) = \sum_{n=0}^{N+1} A_n(t) - A(t)$$

for every $N \in \mathbb{N}$, and passing to the limit for $N \to \infty$ we obtain (38) and (39).

Concerning the uniqueness let us assume that

$$Q \in C([0,b],L(X)) \cap (\mathcal{B})BV([0,b],L(X))$$

satisfies also (38) and (39). Then

$$Q - A * Q = A \text{ and } \ R - R * A = A.$$

Using the associativity of convolution products we get

$$R = A + R * A = A + R * (Q - A * Q) = A + R * Q - R * A * Q$$
$$= A + (R - R * A) * Q = A + A * Q = Q$$

and the uniqueness is proved. □

**29. Corollary.** *Assume that* $A\colon [0,\infty) \to L(X)$, $A(0) = 0$. *If*

$$A \in C([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$$

*and if there is a* $c \in (0, b]$ *such that*

$$(\mathcal{B}) \operatorname*{var}_{[0,c]}(A) < 1$$

*then there exists a unique* $R\colon [0,\infty) \to L(X)$,

$$R \in C([0,\infty), L(X)) \cap BV_{\mathrm{loc}}([0,\infty), L(X))$$

*with* $R(0) = 0$ *such that* (38) *and* (39) *hold for every* $b > 0$.

$R \in C([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$ given in Corollary 29 is called the *resolvent* of $A \in C([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$.

**30. Theorem.** *Assume that* $A\colon [0,\infty) \to L(X)$,

$$A \in C([0,\infty), L(X)) \cap (\mathcal{B})BV_{\mathrm{loc}}([0,\infty), L(X))$$

*and that there is a* $c \in (0, b]$ *such that*

$$(\mathcal{B}) \operatorname*{var}_{[0,c]}(A) < 1.$$

*Then for every* $F \in G([0,\infty), L(X))$ *and* $f \in G([0,\infty), X)$ *there exist unique solutions* $X\colon [0,\infty) \to L(X)$ *and* $x\colon [0,\infty) \to X$ *of the abstract renewal equations*

$$X(t) = F(t) + \int_0^t \mathrm{d}[A(s)]X(t-s) \tag{42}$$

*and*

$$x(t) = f(t) + \int_0^t \mathrm{d}[A(s)]x(t-s), \tag{43}$$

*respectively, and the relations*

$$X(t) = F(t) + \int_0^t \mathrm{d}[R(s)]F(t-s), \tag{44}$$

$$x(t) = f(t) + \int_0^t \mathrm{d}[R(s)]f(t-s) \tag{45}$$

*hold for $t > 0$ where $R$ is the resolvent of $A$.*

P r o o f. Whithout any loss of generality we may assume that $A(0) = 0$. Indeed, if $A(0) \neq 0$ then $B(t) = A(t) - A(0)$ can be used because clearly $\int_0^t \mathrm{d}[A(s)]x(t-s) = \int_0^t \mathrm{d}[B(s)]x(t-s)$ provided one of these integrals exists.

The expression on the right hand side of (44) is well defined and it reads $X(t) = F(t) + (R * F)(t)$.

Hence using (38) and the associativity established in Lemma 14 we obtain

$$\begin{aligned} A * X(t) = A * F(t) + (A * (R * F))(t) &= ((A + A * R) * F)(t) \\ &= (R * F)(t) = X(t) - F(t) \end{aligned}$$

and this shows that by (44) a solution of (42) is given.

Concerning the unicity assume that two solutions $X$ and $Y$ of (42) are given. For $Z(t) = X(t) - Y(t)$ we have

$$Z(t) = \int_0^t \mathrm{d}[A(s)]Z(t-s),\ t \geqslant 0.$$

Hence by Proposition 6 we get

$$\|Z(t)\|_{L(X)} = \left\| \int_0^t \mathrm{d}[A(s)]Z(t-s) \right\|_{L(X)} \leqslant (\mathcal{B}) \operatorname*{var}_{[0,t]}(A) \cdot \sup_{s\in[0,t]} \|Z(s)\|_{L(X)}.$$

If $Z(t) \neq 0$ for $t \in [0,c]$ then by the assumption $(\mathcal{B})\operatorname{var}_{[0,c]}(A) < 1$ we obtain

$$\|Z(t)\|_{L(X)} \leqslant (\mathcal{B}) \operatorname*{var}_{[0,c]}(A) \cdot \sup_{s\in[0,c]} \|Z(s)\|_{L(X)} < \sup_{s\in[0,c]} \|Z(s)\|_{L(X)}$$

and this implies that $Z(t) = 0$ on $t \in [0,c]$.

For $t \in [c, 2c]$ we have

$$Z(t) = \int_0^t \mathrm{d}[A(s)]Z(t-s) = \int_0^{t-c} \mathrm{d}[A(s)]Z(t-s)$$

and in the same way as above we get

$$\begin{aligned} \|Z(t)\|_{L(X)} &\leqslant (\mathcal{B}) \operatorname*{var}_{[0,t-c]}(A) \cdot \sup_{s\in[c,2c]} \|Z(s)\|_{L(X)} \\ &\leqslant (\mathcal{B}) \operatorname*{var}_{[0,c]}(A) \cdot \sup_{s\in[0,c]} \|Z(s)\|_{L(X)} < \sup_{s\in[c,2c]} \|Z(s)\|_{L(X)} \end{aligned}$$

and this yields $Z(t) = 0$ on $t \in [c, 2c]$. In this way we can proceed step by step to show that $Z(t) = 0$ for all $t \in [0, \infty)$.

An analogous result for (43) can be shown similarly. □

Our approach to convolution equations of the type (42) or (43) has been based on Theorem 26 and on the well known Banach algebra techniques. Let us turn our attention to the special case of equations (42) and (43) when $A \in C([0,\infty), L(X)) \cap BV_{\rm loc}([0,\infty), L(X))$.

Using Theorem 27 all considerations from Proposition 28, Corollary 29 and Theorem 30 can be repeated without changes provided the assumption (37) requiring $(\mathcal{B})\,\mathrm{var}_{[0,c]}(A) < 1$ for some $c > 0$ is replaced by the assumption

$$\mathop{\mathrm{var}}_{[0,c]}(A) < 1$$

which has to be satisfied for some $c > 0$.

Since in the case $A \in C([0,\infty), L(X)) \cap BV_{\rm loc}([0,\infty), L(X))$ the function

$$V(r) = \mathop{\mathrm{var}}_{[0,r]}(A), \ \ r \geqslant 0$$

is continuous and $V(0) = 0$ we can see immediately that there is a $c > 0$ such that

$$\mathop{\mathrm{var}}_{[0,c]}(A) < 1$$

holds.

Using this we arrive immediately at the following result.

**31. Theorem.** *Assume that* $A\colon [0,\infty) \to L(X)$,

$$A \in C([0,\infty), L(X)) \cap BV_{\rm loc}([0,\infty), L(X)).$$

*Then for every* $F \in G([0,\infty), L(X))$ *and* $f \in G([0,\infty), X)$ *there exist unique solutions* $X\colon [0,\infty) \to L(X)$ *and* $x\colon [0,\infty) \to X$ *of the abstract renewal equations* (42) *and* (43) *respectively, and the relations*

$$X(t) = F(t) + \int_0^t \mathrm{d}[R(s)]F(t-s),$$
$$x(t) = f(t) + \int_0^t \mathrm{d}[R(s)]f(t-s)$$

*hold for* $t > 0$ *where* $R$ *is the resolvent of* $A$.

*References*

[1] *O. Diekmann, M. Gyllenberg, H. R. Thieme*: Perturbing semigroups by solving Stieltjes renewal equations. Differ. Integral Equ. *6* (1993), 155–181.

[2] *O. Diekmann, M. Gyllenberg, H. R. Thieme*: Perturbing evolutionary systems by step responses on cumulative outputs. Differ. Integral Equ. *8* (1995), 1205–1244.

[3] *N. Dunford, J. T. Schwartz*: Linear Operators I. Interscience Publishers, New York, London, 1958.

[4] *Ch. S. Hönig*: Volterra-Stieltjes Integral Equations. North-Holland Publ. Comp., Amsterdam, 1975.

[5] *J. Kurzweil*: Nichtabsolut konvergente Integrale. BSB B. G. Teubner, Leipzig, 1980.

[6] *Š. Schwabik*: Generalized Ordinary Differential Equations. World Scientific, Singapore, 1992.

[7] *Š. Schwabik, M. Tvrdý, O. Vejvoda*: Differential and Integral Equations. Academia & Reidel, Praha & Dordrecht, 1979.

[8] *Š. Schwabik*: Abstract Perron-Stieltjes integral. Math. Bohem. *121* (1996), 425–447.

[9] *Š. Schwabik*: Linear Stieltjes integral equations in Banach spaces. Math. Bohem. *124* (1999), 433–457.

[10] *Š. Schwabik*: Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. Math. Bohem. *125* (2000), 431–454.

[11] *Š. Schwabik*: A note on integration by parts for abstract Perron-Stieltjes integrals. Math. Bohem. *126* (2001), 613–626.

*Author's address*: *Štefan Schwabik*, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `schwabik@math.cas.cz`.