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THE FORCING DIMENSION OF A GRAPH

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Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the (metric) representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a resolving set for G if distinct vertices of G have distinct representations. A resolving set of minimum cardinality is a basis for G and the number of vertices in a basis is its (metric) dimension $\dim(G)$. For a basis W of G, a subset S of W is called a forcing subset of W if W is the unique basis containing S. The forcing number $f_G(W, \dim)$ of W in G is the minimum cardinality of a forcing subset for W, while the forcing dimension $f(G, \dim)$ of G is the smallest forcing number among all bases of G. The forcing dimensions of some well-known graphs are determined. It is shown that for all integers a, b with $0 \leqslant a \leqslant b$ and $b \geqslant 1$, there exists a nontrivial connected graph G with $f(G) = a$ and $\dim(G) = b$ if and only if $\{a, b\} \neq \{0, 1\}$.

Keywords: resolving set, basis, dimension, forcing dimension

MSC 2000: 05C12

1. Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

as the (*metric*) *representation of* v *with respect to* W. The set W is called a *resolving set* for G if distinct vertices have distinct representations. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G.

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The (*metric*) *dimension* $\dim(G)$ is the number of vertices in a basis for G. For example, the graph G of Figure 1 has the basis $W = \{u, z\}$ and so $\dim(G) = 2$. The representations for the vertices of G with respect to W are $r(u|W) = (0,1)$, $r(v|W) = (2,1)$, $r(x|W) = (1,2)$, $r(y|W) = (1,1)$, $r(z|W) = (1,0)$.

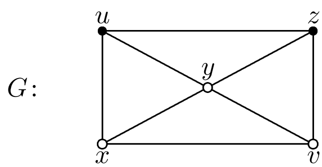

Figure 1. A graph G with $\dim(G) = 2$

The example just presented also illustrates an important point. When determining whether a given set W of vertices of a graph G is a resolving set for G, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of G whose distance from w is 0. The following lemma will be used on several occasions. The proof of this lemma is routine and is therefore omitted.

Lemma 1.1. *Let G be a nontrivial connected graph. For $u, v \in V(G)$, if $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u, v\}$, then u and v belong to every resolving set of G.*

The inspiration for these concepts stems from chemistry. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1, 2]. The dimension of directed graphs has been studied in [5, 6].

The concepts of resolving set and minimum resolving set have previously appeared in the literature. In [14] and later in [15], Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its location number. Independently, Harary and Melter [11] investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted.

For a basis W of G, a subset S of W with the property that W is the unique basis containing S is called a *forcing subset* of W. The *forcing number* $f_G(W, \dim)$ of W in G is the minimum cardinality of a forcing subset for W, while the *forcing dimension* $f(G, \dim)$ of G is the smallest forcing number among all bases of G. Since the parameter dimension is understood in this context, we write $f_G(W)$ for $f_G(W, \dim)$

and $f(G)$ for $f(G, \dim)$. Hence if G is a graph with $f(G) = a$ and $\dim(G) = b$, then $0 \leqslant a \leqslant b$ and there exists a basis W of cardinality b containing a forcing subset of cardinality a. Forcing concepts have been studied for a various of subjects in graph theory, including such diverse parameters as the chromatic number [9], the graph reconstruction number [12], and geodetic concepts in graphs [3, 7, 8]. Also, many invariants arising from the study of forcing in graph theory offer abundant new subjects for new and applicable research. A survey of graphical forcing parameters is discussed in [10].

To illustrate these concepts, we consider the graph G of Figure 2. The graph G has dimension 2 and so $f(G) \leqslant 2$. Let $W = \{x, z\}$ and $W' = \{v, z\}$. Since $r(s|W) = (2,1)$, $r(t|W) = (1,2)$, $r(u|W) = (1,3)$, $r(v|W) = (2,2)$, and $r(y|W) = (1,1)$, it follows that W is a basis of G. Also, since $r(s|W') = (1,1)$, $r(t|W') = (1,2)$, $r(u|W') = (1,3)$, $r(x|W') = (2,2)$, and $r(y|W') = (3,1)$, the set W' is a basis of G. Hence $1 \leqslant f(G) \leqslant 2$ by Lemma 1.2. Next we show that $f_G(W) = 1$ and $f_G(W') = 2$. Let $S_1 = \{x, s\}$, $S_2 = \{x, t\}$, $S_3 = \{x, u\}$, $S_4 = \{x, v\}$, and $S_5 = \{x, y\}$. Observe that $r(u|S_1) = r(y|S_1) = (1,2)$, $r(s|S_2) = r(v|S_1) = (2,1)$, $r(t|S_3) = r(y|S_3) = (1,2)$, $r(t|S_4) = r(u|S_4) = (1,1)$, and $r(u|S_5) = r(t|S_5) = (1,2)$. Hence W is the unique basis containing x and so $f_G(W) = 1$. Certainly, W' is not the unique basis containing z since $z \in W$. Moreover, $W'' = \{v, s\}$ is a basis in G containing v and so W' is not the unique basis containing v. Hence W' is not the unique basis containing any of its proper subset and so $f_G(W') = 2$. Now the forcing dimension $f(G)$ of G is the smallest forcing number among all bases of G an so $f(G) = 1$.

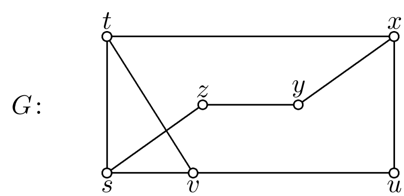

Figure 2. A graph G with $\dim(G) = 2$ and $f(G) = 1$

It is immediate that $f(G) = 0$ if and only if G has a unique basis. If G has no unique basis but contains a vertex belonging to only one basis, then $f(G) = 1$. Moreover, if for every basis W of G and every proper subset S of W, the set W is not the unique basis containing S, then $f(G) = \dim(G)$. We summarize these observations below.

Lemma 1.2. *For a graph G, the forcing dimension $f(G) = 0$ if and only if G has a unique basis, $f(G) = 1$ if and only if G has at least two distinct bases but some* *vertex of G belongs to exactly one basis, and $f(G) = \dim(G)$ if and only if no basis of G is the unique basis containing any of its proper subsets.*

2. Forcing dimensions of certain graphs

The following three theorems (see [2], [11], [14], [15]) give the dimensions of some well-known classes of graphs. In this section, we determine the forcing dimensions of these graphs.

Theorem A. *Let G be a connected graph of order $n \geqslant 2$.*
(a) *Then $\dim(G) = 1$ if and only if $G = P_n$.*
(b) *Then $\dim(G) = n - 1$ if and only if $G = K_n$.*
(c) *For $n \geqslant 3$, $\dim(C_n) = 2$.*
(d) *For $n \geqslant 4$, $\dim(G) = n - 2$ if and only if $G = K_{r,s}$ $(r, s \geqslant 1)$, $G = K_r + \overline{K_s}$ $(r \geqslant 1,\ s \geqslant 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$.*

A vertex of degree at least 3 in a tree T is called a *major vertex*. An end-vertex u of T is said to be a *terminal vertex of a major vertex* v of T if $d(u, v) < d(u, w)$ for every other major vertex w of T. The *terminal degree* $\operatorname{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of T and let $\operatorname{ex}(T)$ denote the number of exterior major vertices of T.

Theorem B. *If T is a tree that is not a path, then $\dim(G) = \sigma(T) - \operatorname{ex}(T)$.*

Theorem C. *Let T be a tree of order $n \geqslant 3$ that is not a path having p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i,1}, u_{i,2}, \ldots, u_{i,k_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path $(1 \leqslant j \leqslant k_i)$. Suppose that W is a set of vertices of T. Then W is a basis of T if and only if W contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p)$ with exactly one exception for each i with $1 \leqslant i \leqslant p$ and $k_i \geqslant 2$, and W contains no other vertices of T.*

Proposition 2.1. *Let G be a nontrivial connected graph. If G is a complete graph, cycle, or tree, then $f(G) = \dim(G)$.*

P r o o f. First assume that G is the complete graph K_n of order $n \geqslant 2$. Since every set W of $n - 1$ vertices in K_n is a basis of K_n, it follows that W is not the unique basis containing any of its proper subset. By Lemma 1.2, $f(K_n) = \dim(K_n)$. Next assume that G is a cycle C_n of order $n \geqslant 4$. If n is odd, then every pair of vertices forms a basis of C_n. If n is even, then every pair u, v of vertices with $d(u, v) \neq n/2$ forms a basis of C_n. So in either cases, there is no basis of C_n that is the unique basis containing any of its proper subset. Again, it then follows from Lemma 1.2 that $f(C_n) = \dim(C_n)$.

Now let T be a tree. First assume that T is the path P_n of order $n \geqslant 2$. Since each end-vertex of P_n forms a basis for P_n, it follows that $f(P_n) \geqslant 1 = \dim(P_n)$ by Lemma 1.2. Hence $f(P_n) = \dim(P_n) = 1$. Next assume that T is a tree of order $n \geqslant 4$ that is not a path and T has p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i,1}, u_{i,2}, \ldots, u_{i,k_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path $(1 \leqslant j \leqslant k_i)$. Let W be a basis of G. It then follows from Theorem C that W contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p)$ with exactly one exception for each i with $1 \leqslant i \leqslant p$ and $k_i \geqslant 2$, and W contains no other vertices of G. Let S be a proper subset of W and let $x \in W - S$. Then there exist i, j with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant k_i$ such that x is a vertex from the path $P_{ij} - v_i$, say x is a vertex from $P_{11} - v_1$. Since $x \in W$, it follows that $\mathrm{ter}(v_1) = k_1 \geqslant 2$. Assume, without loss of generality, that for each j with $1 \leqslant j \leqslant k_1 - 1$, there is a vertex x_j from $P_{1j} - v_1$ that belongs to W and there is no vertex of $P_{1,k_1} - v_1$ that belongs to W. So $x_1 = x$. Let x_{k_1} be a vertex of the path $P_{1,k_1} - v_1$. Then $W' = (W - \{x_1\}) \cup \{x_{k_1}\}$ is a basis of T by Theorem C. Since W' contains S and $W' \neq W$, it follows that W is not the unique basis containing S. Therefore, $f(T) = \dim(T)$ by Lemma 1.2. $\square$

Proposition 2.2. *Let G be a connected graph of order $n \geqslant 2$ with $\dim(G) = n-2$. If $G = K_{r,s}$ $(r, s \geqslant 1)$ or $G = K_r + \overline{K_s}$ $(r \geqslant 1, s \geqslant 2)$, then $f(G) = \dim(G)$. If $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$, then $f(G) = \dim(G) - 1$.*

P r o o f. By Theorem A, if $\dim(G) = n-2$, then $G = K_{r,s}$ $(r, s \geqslant 1)$, $G = K_r + \overline{K_s}$ $(r \geqslant 1, s \geqslant 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$. First let $G = K_{r,s}$ whose the partite sets are $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$. Then by Lemma 1.1 every basis W of G has the form $W = W_1 \cup W_2$, where $W_i \subseteq V_i$ $(1 = 1, 2)$ with $|W_1| = r-1$ and $|W_2| = s-1$. Assume, without loss of generality, that $W = V(G) - \{u_r, v_s\}$. Let S be a proper subset of W. Then $S = S_1 \cup S_2$, where $S_i \subseteq W_i$ $(1 = 1, 2)$ and $|S_1| \leqslant r-2$ or $|S_2| \leqslant s-2$, say $|S_1| \leqslant r-2$. Thus there is $u_i \in W$, where $1 \leqslant i \leqslant r-1$, such that $u_i \notin S_1$. Then $W' = (W - \{u_i\}) \cup \{u_r\}$ is a basis of G containing S. Since $W' \neq W$, it follows that W is not the unique basis containing S. Therefore, $f(G) = \dim G$. If $G = K_r + \overline{K_s}$, let $V_1 = V(K_r) = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = V(\overline{K_s}) = \{v_1, v_2, \ldots, v_s\}$. Since every basis W of G has the form $W = W_1 \cup W_2$, where $W_i \subseteq V_i$ $(1 = 1, 2)$ with $|W_1| = r-1$ and $|W_2| = s-1$, a similar argument shows that $f(G) = \dim G$.

Now let $G = K_r + (K_1 \cup K_s)$. Assume that $V_1 = V(K_r) = \{u_1, u_2, \ldots, u_r\}$, $V_2 = V(K_s) = \{v_1, v_2, \ldots, v_s\}$, and $V(K_1) = \{x\}$. Then by Lemma 1.1 it can be verified that every basis of G has the form $W = W_1 \cup W_2 \cup \{x\}$, where $W_i \subseteq V_i$ $(i = 1, 2)$ and $|W_1| = r - 1$ and $|W_2| = s - 1$. Since the vertex x belongs to every basis, $f(G) \leqslant |W| - 1 = \dim(G) - 1$. On the other hand, let W be a basis of G, say $W = V(G) - \{u_r, v_s\}$, and let S be a subset of W with $|S| \leqslant |W| - 2$. Then there is a vertex $y \in W - S$ such that $y \neq x$. We may assume that $y \in V_1$. Then $W' = (W - \{y\}) \cup \{u_r\}$ is a basis of G containing S. So W is not the unique basis containing S. Thus $f(G) \geqslant |W| - 1 = \dim(G) - 1$. Therefore, $f(G) = \dim(G) - 1$. □

3. Graphs with prescribed dimensions and forcing dimensions

We have already noted that if G is a graph with $f(G) = a$ and $\dim(G) = b$, then $0 \leqslant a \leqslant b$ and $b \geqslant 1$. We now determine which pairs a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 1$ are realizable as the forcing dimension and dimension of some nontrivial connected graph. In order to do this, we state the following result obtained in [1].

Theorem D. *For $k \geqslant 2$, there exists a connected graph of dimension k with a unique basis.*

Theorem 3.1. *For all integers a, b with $0 \leqslant a \leqslant b$ and $b \geqslant 1$, there exists a nontrivial connected graph G with $f(G) = a$ and $\dim(G) = b$ if and only if $\{a, b\} \neq \{0, 1\}$.*

Proof. By Theorem A, the path P_n of order $n \geqslant 2$ is the only nontrivial connected graph of order n with dimension 1. However, $f(P_n) = 1$ for all $n \geqslant 2$ by Proposition 2.1. Hence there is no nontrivial connected graph G with $f(G) = 0$ and $\dim(G) = 1$.

We now verify the converse. Let $a = 0$ and $b \geqslant 2$. By Theorem D there is a connected graph G of dimension b with a unique basis. Thus $f(G) = 0$ by Lemma 1.2 and $\dim(G) = b$. Hence the result is true for $a = 0$ and $b \geqslant 2$. So we may assume that $a > 0$. First assume that $b = a$. When $b = a = 1$, each path P_n $(n \geqslant 2)$ has the desired property. When $b = a = 2$, the star $K_{1,3}$ has the desired property. When $b = a \geqslant 3$, then the complete graph K_{a+1} has the desired property. So we now assume that $a < b$. We consider two cases.

Case 1. $b = a + 1$. Let G be the graph obtained from the 4-cycle u_1, u_2, u_3, u_4, u_1 by adding a new edge u_2u_4 and then joining b new vertices $v_1, v_2, \ldots, v_b$ to u_2 and u_3. The graph G is shown in Figure 3. First note every basis of G contains at least

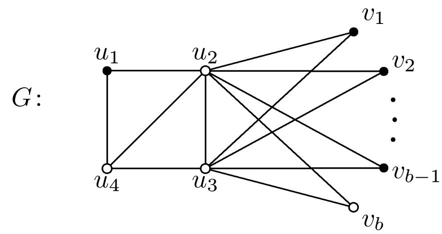

Figure 3. A graph G with $\dim(G) = b$ and $f(G) = b - 1$

$b-1$ vertices from $\{v_1, v_2, \ldots, v_b\}$ by Lemma 1.1. However, it can be verified that if W is a basis of G, then W contains exactly $b-1$ vertices from $\{v_1, v_2, \ldots, v_b\}$ and the vertex u_1. Hence $\dim(G) = b$. Next we show that $f(G) = b-1$. Let W be a basis of G, say $W = \{u_1, v_1, v_2, \ldots, v_{b-1}\}$. Since u_1 belongs to every basis of G, it follows that W is the unique basis containing the subset $\{v_1, v_2, \ldots, v_{b-1}\}$, which implies that $f_G(W) \leqslant b-1$. On the other hand, if S is a subset of W with $|S| \leqslant b-2$, then, without loss of generality, we assume that $v_{b-1} \notin S$. Then $W' = (W - \{v_{b-1}\}) - \{v_b\}$ is a basis of G containing S. Thus W is not the unique basis containing S and so $f_G(W) \geqslant b-1$. Hence $f_G(W) = b-1$ for every basis W of G and so $f(G) = b-1 = a$.

Case 2. $b \geqslant a+2$. Let $r = b - a$. Then $2 \leqslant r \leqslant b-1$. First we construct a graph H of order $r + 2^r$ with $V(H) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_{2^r-1}\}$ and the ordered set $V = \{v_{r-1}, v_{r-2}, \ldots, v_0\}$ are disjoint. The induced subgraph $\langle U \rangle$ of H is complete, while V is independent. It remains to define the adjacencies between V and U. Let each integer j $(0 \leqslant j \leqslant 2^r - 1)$ be expressed in its base 2 (binary) representation. Thus, each such j can be expressed as a sequence of r coordinates, that is, an r-vector, where the rightmost coordinate represents the value (either 0 or 1) in the 2^0 position, the coordinate to its immediate left is the value in the 2^1 position, etc. For integers i and j, with $0 \leqslant i \leqslant r-1$ and $0 \leqslant j \leqslant 2^r - 1$, we join v_i and u_j if and only if the value in the 2^i position in the binary representation of j is 1. The structure of H is based on one given in the proof of Theorem D (see [1]), where it was shown that H has dimension r and V is the unique basis of H. Now the graph G is obtained from H by adding the a new vertices $x_1, x_2, \ldots, x_a$ such that each x_i $(1 \leqslant i \leqslant a)$ has the same neighborhood as u_0 in V and the induced subgraph $\langle U \cup \{x_1, x_2, \ldots, x_a\} \rangle$ is complete.

We first show that $\dim G = b$. Let $T = \{u_0, x_1, x_2, \ldots, x_a\}$. Note that if $t_1, t_2 \in T$ and $v \in V(G)$, then $d(t_1, v) = d(t_2, v)$. Hence every resolving set of G must contain at least a vertices from T by Lemma 1.1. Let $W = V \cup \{x_1, x_2, \ldots, x_a\}$. We show that W is a resolving set of G. It suffices to show that the metric representations of vertices in U are distinct. Observe that the first r coordinates of the metric representation for each u_j $(0 \leqslant j \leqslant 2^r - 1)$ can be expressed as $r(u_j|V)$. Since V

is the basis of H, the metric representations $r(u_j|V)$ $(0 \leqslant j \leqslant 2^r - 1)$ of u_j with respect to V are distinct. In fact, $r(u_j|V) = (2 - a_{r-1}, 2 - a_{r-2}, \ldots, 2 - a_0)$, where a_m $(0 \leqslant m \leqslant r - 1)$ is the value in the 2^m position of the binary representation of j. Since the binary representations $a_{r-1}a_{r-2}\ldots a_1a_0$ are distinct for the vertices of U, their metric representations $(2 - a_{r-1}, 2 - a_{r-2}, \ldots, 2 - a_0)$ (with respect to V) are distinct. This implies that the metric representations $r(u_j|W)$ are distinct as well. Hence W is a resolving set of G and so $\dim G \leqslant |W| = (b - a) + a = b$. Next we show that $\dim G \geqslant b$. Assume, to the contrary, that $\dim(G) \leqslant b - 1$. Let S be a basis of G with $|S| = \dim(G)$. Let $S = S' \cup X$, where $X \subseteq T$ and $S' \subseteq V(G) - T$. Then $|X| \geqslant a$ by Lemma 1.1. Let $S^* = S' \cup \{u_0\}$. Hence $|S^*| = |S| - |X| + 1 \leqslant (b - 1) - a + 1 = b - a$. Since V is the unique basis of H and $u_0 \notin V$, it follows that S^* is not a basis of H. Thus there exist $z, z' \in V(H) - \{u_0\}$ such that $r(z|S^*) = r(z'|S^*)$ and so $d(z, u_0) = d(z', u_0)$. Thus $d(z, x_i) = d(z', x_i)$ for all i. This implies that $r(z|S) = r(z'|S)$ and so S is not a basis, which is a contradiction. Therefore, $\dim(G) \leqslant b$ and so $\dim(G) = b$.

In order to determine $f(G)$, we first show that V belongs to every basis of G. Assume, to the contrary, there exists a basis W of G such that $V \nsubseteq W$. If $T \subseteq W$, then $W' = (W - T) \cup \{u_0\} \neq V$ and so W' is not a basis of H. Thus there exist $z, z' \in V(H) - \{u_0\}$ such that $r(z|W') = r(z'|W')$. This implies that $r(z|W) = r(z'|W)$ and so W is not a basis, a contradiction. Hence W contains exactly a vertices from T. Assume, without loss of generality, that $W = S \cup X$, where $X = T - \{u_0\}$ and $S \subseteq V(H) - T$. A similar argument to the one employed in the proof of Theorem D [1] shows that there exist two vertices z and z' in $U = V(H) - V$ such that $r(z|S) = r(z'|S)$. Since the distance between every two vertices in $U \cup T$ is 1, it follows that $r(z|W) = r(z'|W)$. This contradicts the fact that W is a basis. Therefore, V belongs to every basis W of G.

We are now prepared to show that $f(G) = a$. Let W be a basis of G. Since V must belong to W, it follows that W is the unique basis containing $W - V$. Thus $f_G(W) \leqslant |W - V| = b - (b - a) = a$. This is true for every basis W of G and so $f(G) \leqslant a$. On the other hand, let W be a basis and S be a subset of W with $|S| \leqslant a - 1$. Without loss of generality, assume that $W = V \cup X$ with $X = \{x_1, x_2, \ldots, x_a\}$. Since $|S| \leqslant a - 1$, there exists $x \in W \cap X$ such that $x \notin S$. Then $W' = (W - \{x\}) \cup \{u_0\}$ is a basis of G that contains S. Hence W is not the unique basis containing S and so $f_G(W) \geqslant |S| + 1 = a$. Again, this is true for every basis W in G and so $f(G) \leqslant a$. Therefore, $f(G) = a$ and $\dim(G) = b$, as desired. □

4. Open problem

While the forcing dimension $f(G)$ of a graph G is the minimum forcing number among all bases of G, we define the *upper forcing dimension* $f^+(G)$ as the maximum forcing number among all bases of G. Hence

$$0 \leqslant f(G) \leqslant f^+(G) \leqslant \dim(G).$$

If a graph G has a unique basis, then $f(G) = f^+(G) = 0$. Also, there are numerous examples of graphs G, such as complete graphs and trees, with $f(G) = f^+(G) = \dim(G)$. On the other hand, as we have seen, the graph G of Figure 1 contains two bases with distinct forcing numbers and so $f(G) = 1$ and $f^+(G) = 2$. Hence $f(G) < f^+(G)$. We close with the following open problem.

Problem 4.1. For which pairs a, b of integers with $0 \leqslant a \leqslant b$, does there exist a nontrivial connected graph G with $f(G) = a$ and $f^+(G) = b$?

References

[1] *P. Buczkowski, G. Chartrand, C. Poisson, P. Zhang*: On k-dimensional graphs and their bases. Submitted.

[2] *G. Chartrand, L. Eroh, M. Johnson*: Resolvability in graphs and the metric dimension of a graph. To appear in Appl. Discrete Math.

[3] *G. Chartrand, F. Harary, P. Zhang*: On the geodetic number of a graph. To appear in Networks.

[4] *G. Chartrand, C. Poisson, P. Zhang*: Resolvability and the upper dimension of graphs. To appear in Int. J. Comput. Math. Appl.

[5] *G. Chartrand, M. Raines, P. Zhang*: The directed distance dimension of oriented graphs. Math. Bohem. *125* (2000), 155–168.

[6] *G. Chartrand, M. Raines, P. Zhang*: On the dimension of oriented graphs. To appear in Utilitas Math.

[7] *G. Chartrand, P. Zhang*: The geodetic number of an oriented graph. European J. Combin. *21* (2000), 181–189.

[8] *G. Chartrand, P. Zhang*: The forcing geodetic number of a graph. Discuss. Math. Graph Theory *19* (1999), 45–58.

[9] *C. Ellis, F. Harary*: The chromatic forcing number of a graph. To appear.

[10] *F. Harary*: A survey of forcing parameters in graph theory. Preprint.

[11] *F. Harary, R. A. Melter*: On the metric dimension of a graph. Ars Combin. *2* (1976), 191–195.

[12] *F. Harary, M. Plantholt*: The graph reconstruction number. J. Graph Theory *9* (1985), 451–454.

[13] *C. Poisson, P. Zhang*: The dimension of unicyclic graphs. Submitted. To appear in J. Combin. Math. Combin. Comput.

[14] *P. J. Slater*: Leaves of trees. Congress. Numer. *14* (1975), 549–559.

[15] *P. J. Slater*: Dominating and reference sets in graphs. J. Math. Phys. Sci. *22* (1988), 445–455.

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