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# MULTIPLIERS FOR GENERALIZED RIEMANN INTEGRALS IN THE REAL LINE

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*Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday*

*Abstract.* We use an elementary method to prove that each $BV$ function is a multiplier for the $C$-integral.

*Keywords*: multiplier, $C$-integral, $BV$ function

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## 1. Introduction

It is well known that if $f$ is Henstock-Kurzweil integrable on a compact interval $[a, b] \subset \mathbb{R}$ and $g$ is of bounded variation there, then $fg$ is Henstock-Kurzweil integrable on $[a, b]$ and the integration by parts formula holds; see, for example, [7, Theorem 12.21]. Here $g$ is known as a multiplier for the Henstock-Kurzweil integral. In [2] Bongiorno used the above mentioned result to prove that each $BV$ function is a multiplier for the $C$-integral. See [2, Theorem 4.2] for details. In this paper, we will use elementary properties of the $C$-integral to obtain a new proof of [2, Theorem 4.2]. As a result, we also obtain an alternative proof of the well-known results that each $BV$ function is a multiplier for both the McShane and Henstock-Kurzweil integrals.

## 2. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$. A set $Z \subset \mathbb{R}$ is said to be $\mu_1$-*negligible* whenever $\mu_1(Z) = 0$, where $\mu_1$ is the one-dimensional Lebesgue measure. Given two subsets $X$, $Y$ of $\mathbb{R}$, we say that $X$ and $Y$ are non-overlapping if their intersection is $\mu_1$-negligible. A function is always real-valued. When no confusion is possible we do not distinguish between a function defined on a set $Z$ and its restriction to a set $W \subset Z$.

An interval in $\mathbb{R}$ is always a compact non-degenerate interval in $\mathbb{R}$. The family of all non-degenerate subintervals of $[a,b]$, where $-\infty < a < b < \infty$, is denoted by $\mathcal{I}_1$. For any given $I \in \mathcal{I}_1$, we write $\mu_1(I)$ as $|I|$.

A *partition* $P$ is a finite collection $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$, where $I_1,\dots,I_p$ are pairwise non-overlapping intervals in $\mathcal{I}_1$, and $\xi_i \in [a,b]$ for each $i = 1,\dots,p$. Given $Z \subseteq [a,b]$, a positive function $\delta$ on $Z$ is called a *gauge* on $Z$. A partition $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$ is said to be:

(i) a partition *of* $Z$ if $\bigcup\limits_{i=1}^{p} I_i = Z$;

(ii) a subpartition *of* $Z$ if $\bigcup\limits_{i=1}^{p} I_i \subseteq Z$;

(iii) $\delta$-*fine* if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1,\dots,p$;

(iv) McShane if for each $i = 1,\dots,p$, $\xi_i$ need not be in $I_i$.

**Lemma 2.1** [8, Lemma 6.2.6]**.** *Given a gauge $\delta$ on $[a,b]$, $\delta$-fine partitions of $[a,b]$ exist.*

**Definition 2.2** ([3])**.** A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is said to be $C$-*integrable on* $[a,b]$ if there exists $A \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a gauge $\delta$ on $[a,b]$ such that

$$\Bigl|\sum_{i=1}^{p} f(\xi_i)|I_i| - A\Bigr| < \varepsilon$$

for each $\delta$-fine McShane partition $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$ of the interval $[a,b]$ such that $\sum\limits_{i=1}^{p} \operatorname{dist}(\xi_i, I_i) < 1/\varepsilon$. Here $A$ is called the $C$-*integral of* $f$ *over* $[a,b]$, and we write $A$ as $\int_a^b f(x)\,\mathrm{d}x$ or $\int_{[a,b]} f(x)\,\mathrm{d}x$.

The $C$-integral is the minimal integral which includes Lebesgue integrable functions and derivatives. See [3, Main Theorem] for details. The following properties of the $C$-integral can be found in [1], [2], [3], [6].

Remark 2.3. (a) The $C$-integral is linear; the class of $C$-integrable functions on $[a,b]$ is a linear space.

(b) $C$-integrability on an interval $I$ implies $C$-integrability on each subinterval of $I$.

**Lemma 2.4** (Saks-Henstock). *Let $f$ be $C$-integrable on $[a,b]$. Then for each $\varepsilon > 0$ there exists a gauge $\delta$ on $[a,b]$ such that*

$$\sum_{i=1}^{p} \left| f(\xi_i)|I_i| - \int_{I_i} f(x)\,\mathrm{d}x \right| < \varepsilon \tag{1}$$

*for each $\delta$-fine McShane subpartition $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$ of $[a,b]$ such that $\sum\limits_{i=1}^{p} \operatorname{dist}(\xi_i, I_i) < 1/\varepsilon$.*

## 3. Multipliers for the $C$-integral

Let $\chi_X$ denote the characteristic function of a set $X$. The following lemma is an easy consequence of [9, 4.32 Theorem].

**Lemma 3.1.** *If $a \leqslant u < v \leqslant b$, $g \in BV[a,b]$ and $g(a) = 0$, then*

$$\int_a^b \chi_{[u,v]}(x) g(x)\,\mathrm{d}x = \int_a^b \left( \int_x^b \chi_{[u,v]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x).$$

As an easy application of Lemma 3.1, we have the following crucial theorem for this paper.

**Theorem 3.2.** *Let $f$ be $C$-integrable on $[a,b]$. If $g \in BV[a,b]$ and $g(a) = 0$, then the inequality*

$$\begin{aligned} &\left| \sum_{i=1}^{p} \left\{ f(\xi_i)g(\xi_i)(v_i - u_i) - \int_a^b \left( \int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right| \\ &\qquad \leqslant \sum_{i=1}^{p} |f(\xi_i)| \int_{u_i}^{v_i} |g(\xi_i) - g(t)|\,\mathrm{d}t \\ &\qquad\quad + \sup_{x\in[a,b]} \left| \int_x^b \sum_{i=1}^{p} \{ f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t)\}\,\mathrm{d}t \right| \operatorname{Var}(g,[a,b]) \end{aligned}$$

*holds for each subpartition $\{([u_1,v_1],\xi_1),\dots,([u_p,v_p],\xi_p)\}$ of $[a,b]$.*

Proof. Let $\{([u_1,v_1],\xi_1),\dots,([u_p,v_p],\xi_p)\}$ be a subpartition of $[a,b]$. In view of Lemma 3.1, we see that

$$\left| \sum_{i=1}^{p} \left\{ f(\xi_i)g(\xi_i)(v_i - u_i) - \int_a^b \left( \int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right|$$

$$\leqslant \sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)(v_i - u_i) - \int_{u_i}^{v_i} g(t)\,\mathrm{d}t \right|$$
$$+ \left| \sum_{i=1}^{p} \left\{ f(\xi_i) \int_{u_i}^{v_i} g(t)\,\mathrm{d}t - \int_a^b \left( \int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right|$$
$$\leqslant \sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)(v_i - u_i) - \int_{u_i}^{v_i} g(t)\,\mathrm{d}t \right|$$
$$+ \left| \sum_{i=1}^{p} \left\{ f(\xi_i) \int_a^b \left( \int_x^b \chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right.\right.$$
$$\left.\left. - \int_a^b \left( \int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right|.$$

Since

$$\sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)(v_i - u_i) - \int_{u_i}^{v_i} g(t)\,\mathrm{d}t \right| \leqslant \sum_{i=1}^{p} |f(\xi_i)| \int_{u_i}^{v_i} |g(\xi_i) - g(t)|\,\mathrm{d}t$$

and

$$\left| \int_a^b \left( \int_x^b \sum_{i=1}^{p} \{ f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t) \}\,\mathrm{d}t \right) \mathrm{d}g(x) \right|$$
$$\leqslant \sup_{x\in[a,b]} \left| \int_x^b \sum_{i=1}^{p} \{ f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t) \}\,\mathrm{d}t \right| \operatorname{Var}(g,[a,b]),$$

the theorem is proved.

We can now give an elementary proof of the following result.

**Theorem 3.3** [2, Theorem 4.2]**.** *Each BV function is a multiplier for the C-integral.*

P r o o f. We may assume that $g(a) = 0$ and $\operatorname{Var}(g,[a,b]) < 1$. According to the Saks-Henstock Lemma for the $C$-integral, given $\varepsilon > 0$ there exists a gauge $\delta_1$ on $[a,b]$ such that

$$\sum_{i=1}^{q} \left| f(\zeta_i)(t_i - s_i) - \int_{s_i}^{t_i} f(x)\,\mathrm{d}x \right| < \frac{\varepsilon}{3} \tag{2}$$

for each $\delta_1$-fine McShane subpartition $\{([s_1,t_1],\zeta_1),\ldots,([s_q,t_q],\zeta_q)\}$ of $[a,b]$ such that

$$\sum_{i=1}^{q} \operatorname{dist}(\zeta_i,[s_i,t_i]) < \frac{3}{\varepsilon}.$$

Observe that if $s < r < t$, then $(r,t] = (s,t] - (s,r]$. Then it follows from our choice of $\delta_1$ that for each $x \in [a,b]$, the inequality

$$\Bigl|\sum_{i=1}^{q}\Bigl\{f(\zeta_i)\mu_1([x,b]\cap[s_i,t_i]) - \int_a^b f(t)\chi_{[x,b]\cap[s_i,t_i]}(t)\,\mathrm{d}t\Bigr\}\Bigr| < \frac{2\varepsilon}{3}$$

holds for each $\delta_1$-fine McShane subpartition $\{([s_1,t_1],\zeta_1),\dots,([s_q,t_q],\zeta_q)\}$ of $[a,b]$ such that

$$\sum_{i=1}^{q}\operatorname{dist}(\zeta_i,[s_i,t_i]) < \frac{3}{\varepsilon}.$$

As $f$ is real-valued and $g$ is of bounded variation on $[a,b]$, it is not difficult to select a gauge $\delta_2$ on $[a,b]$ such that

$$\sum_{j=1}^{r}|f(z_j)|\int_{\alpha_i}^{\beta_i}|g(z_j)-g(t)|\,\mathrm{d}t < \frac{\varepsilon}{3}$$

for each $\delta_2$-fine McShane subpartition $\{([\alpha_1,\beta_1],z_1),\dots,([\alpha_r,\beta_r],z_r)\}$ of $[a,b]$.

Define a gauge $\delta$ on $[a,b]$ by $\delta(x) = \min\{\delta_1(x),\delta_2(x)\}$. For each $\delta$-fine McShane partition $\{([u_1,v_1],\xi_1),\dots,([u_p,v_p],\xi_p)\}$ of $[a,b]$ satisfying

$$\sum_{i=1}^{p}\operatorname{dist}(\xi_i,[u_i,v_i]) < \frac{1}{\varepsilon},$$

we infer from Theorem 3.2 and the above estimates that

$$\begin{aligned}
&\Bigl|\sum_{i=1}^{p} f(\xi_i)g(\xi_i)(v_i-u_i) - \int_a^b\Bigl(\int_x^b f(t)\,\mathrm{d}t\Bigr)\mathrm{d}g(x)\Bigr|\\
&\quad= \Bigl|\sum_{i=1}^{p}\Bigl\{f(\xi_i)g(\xi_i)(v_i-u_i) - \int_a^b\Bigl(\int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t\Bigr)\mathrm{d}g(x)\Bigr\}\Bigr|\\
&\quad\leqslant \sum_{i=1}^{p}|f(\xi_i)|\int_{u_i}^{v_i}|g(\xi_i)-g(t)|\,\mathrm{d}t\\
&\qquad+\sup_{x\in[a,b]}\Bigl|\int_x^b\sum_{i=1}^{p}\Bigl\{f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t)\Bigr\}\mathrm{d}t\Bigr|\operatorname{Var}(g,[a,b]) < \varepsilon,
\end{aligned}$$

thereby completing the proof of the theorem.

By modifying the proof of the above theorem, we obtain the following well-known theorem.

**Theorem 3.4.** *Each BV function is a multiplier for each of the generalized Riemann integrals:*

(i) the McShane integral;
(ii) the classical Henstock-Kurzweil integral;
(iii) the $\widetilde{C}$-integral in [5];
(iv) the improper Lebesgue integral in [4].

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