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A NEW FORM OF FUZZY α-COMPACTNESS

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Abstract. A new form of α-compactness is introduced in L-topological spaces by α-open L-sets and their inequality where L is a complete de Morgan algebra. It doesn't rely on the structure of the basis lattice L. It can also be characterized by means of α-closed L-sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented and the relations between it and the other types of compactness are discussed. Countable α-compactness and the α-Lindelöf property are also researched.

Keywords: L-topology, compactness, α-compactness, countable α-compactness, α-Lindelöf property, α-irresolute map, α-continuous map

MSC 2000: 54A40, 54D35

1. Introduction

The notion of α-open sets was introduced in [13]. The concept of α-compactness for topological spaces was discussed in [12], and it was generalized to $[0,1]$-topological spaces by Thakur, Saraf and Jabalpur [18]. The definition in [18] is based on Chang's compactness which is not a good extension of ordinary compactness.

In [1], Aygün presented a new form of α-compactness which is based on Kudri's compactness [7] which is equivalent to strong compactness in [9], [19].

The concepts of SR-compactness and near SR-compactness were introduced by S. G. Li, S. Z. Bai and N. Li in terms of strongly semiopen L-sets [4], [8]. In fact, a strongly semiopen L-set is exactly an α-open set in the sense of [14]. Thus both SR-compactness and near SR-compactness are extensions of α-compactness. Moreover, the notion of SR-compactness was based on N-compactness and the notion of near

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SR-compactness was based on strong fuzzy compactness. This implies that near SR-compactness is equivalent to α-compactness in [1] when the basis lattice L is a complete distributive de Morgan algebra.

In [15], [16], a new definition of fuzzy compactness was presented in L-topological spaces by means of open L-sets and their inequality where L is a complete de Morgan algebra. This new definition doesn't depend on the structure of L. When L is completely distributive, it is equivalent to the notion of fuzzy compactness in [9], [10], [19].

In this paper, following the lines of [15], [16], we will introduce a new form of α-compactness in L-topological spaces by means of α-open L-sets and their inequality. This new form of α-compactness has many characterizations if L is completely distributive. Moreover, we will compare our α-compactness with other types of α-compactness.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a complete de Morgan algebra, X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X. We often don't distinguish a crisp subset A of X and its character function χ_A.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. An element a in L is called co-prime if a' is prime [6]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [5]. In a completely distributive de Morgan algebra L, each element b is a sup of $\{a \in L;\ a \prec b\}$. A set $\{a \in L;\ a \prec b\}$ is called the greatest minimal family of b in the sense of [9], [19], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L;\ a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$ we use the following notations from [17].

$$A_{[a]} = \{x \in X;\ A(x) \geqslant a\}, \quad A^{(a)} = \{x \in X;\ A(x) \nleqslant a\},$$
$$A_{(a)} = \{x \in X;\ a \in \beta(A(x))\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathscr{T})$, where $\mathscr{T}$ is a subfamily of L^X which contains $\chi_\emptyset$, χ_X and is closed for any suprema and finite infima. $\mathscr{T}$ is called an L-topology on X. Members of $\mathscr{T}$ are called open L-sets and their complements are called closed L-sets.

Definition 2.1 ([9], [19]). An L-space $(X, \mathscr{T})$ is called weakly induced if $\forall a \in L$, $\forall A \in \mathscr{T}$, it follows that $A^{(a)} \in [\mathscr{T}]$, where $[\mathscr{T}]$ denotes the topology formed by all crisp sets in $\mathscr{T}$.

Definition 2.2 ([9], [19]). For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X;\ A^{(a)} \in \tau,\ a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X; in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ). A topologically generated L-space is also called an induced L-space.

It is obvious that $(X, \omega_L(\tau))$ is weakly induced.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ and $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ.

Definition 2.3 ([15], [16]). Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is called (countably) compact if for every (countable) family $\mathscr{U} \subseteq \mathscr{T}$, it follows that

$$\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x) \Big) \leqslant \bigvee_{\mathscr{V} \in 2^{(\mathscr{U})}} \bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{V}} A(x) \Big).$$

Definition 2.4 ([16]). Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to have the Lindelöf property if for every family $\mathscr{U} \subseteq \mathscr{T}$, it follows that

$$\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x) \Big) \leqslant \bigvee_{\mathscr{V} \in 2^{[\mathscr{U}]}} \bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{V}} A(x) \Big).$$

Lemma 2.5 ([16]). *Let L be a complete Heyting algebra, let $f\colon X \to Y$ be a map and $f_L^{\rightarrow}\colon L^X \to L^Y$ the extension of f. Then for any family $\mathscr{P} \subseteq L^Y$, we have*

$$\bigvee_{y \in Y} \Big(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathscr{P}} B(y) \Big) = \bigvee_{x \in X} \Big(G(x) \wedge \bigwedge_{B \in \mathscr{P}} f_L^{\leftarrow}(B)(x) \Big),$$

where $f_L^{\rightarrow}\colon L^X \to L^Y$ and $f_L^{\leftarrow}\colon L^Y \to L^X$ are defined as follows:

$$f_L^{\rightarrow}(G)(y) = \bigvee_{x \in f^{-1}(y)} G(x), \quad f_L^{\leftarrow}(B) = B \circ f.$$

The notion of an α-open set was introduced by Njåstad in [13] and generalized to $[0,1]$-topological spaces by Shahana in [14]. Analogously we can generalize it to L-fuzzy setting as follows:

Definition 2.6 ([14]). An L-set G in an L-space $(X, \mathscr{T})$ is called α-open if $G \leqslant \operatorname{int}(\operatorname{cl}(\operatorname{int}(G)))$. G is called α-closed if G' is α-open.

Definition 2.7 ([18]). Let $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ be two L-spaces. A map $f\colon (X, \mathscr{T}_1) \to (Y, \mathscr{T}_2)$ is called α-continuous if $f_L^{\leftarrow}(G)$ is α-open in $(X, \mathscr{T}_1)$ for every open L-set G in $(Y, \mathscr{T}_2)$.

It can be seen that an α-continuous map was also said to be strongly semi-continuous in [14].

Definition 2.8 ([18]). Let $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ be two L-spaces. A map $f\colon (X, \mathscr{T}_1) \to (Y, \mathscr{T}_2)$ is called α-irresolute if $f_L^{\leftarrow}(G)$ is α-open in $(X, \mathscr{T}_1)$ for every α-open L-set G in $(Y, \mathscr{T}_2)$.

3. Definition and characterizations of α-compactness

Definition 3.1. Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is called (countably) α-compact if for every (countable) family $\mathscr{U}$ of α-open L-sets, it follows that

$$\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{U}} A(x)\Big) \leqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{V}} A(x)\Big).$$

Definition 3.2. Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to have the α-Lindelöf property (or be an α-Lindelöf L-set) if for every family $\mathscr{U}$ of α-open L-sets, it follows that

$$\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{U}} A(x)\Big) \leqslant \bigvee_{\mathscr{V}\in 2^{[\mathscr{U}]}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{V}} A(x)\Big).$$

Obviously we have the following theorem.

Theorem 3.3. *α-compactness implies countable α-compactness and the α-Lindelöf property. Moreover, an L-set having the α-Lindelöf property is α-compact if and only if it is countably α-compact.*

Since an open L-set is α-open, we have the following theorem.

Theorem 3.4. *α-compactness implies compactness, countable α-compactness implies countable compactness, and the α-Lindelöf property implies the Lindelöf property.*

From Definition 3.1 and Definition 3.2 we can obtain the following two theorems by simply using complements.

Theorem 3.5. *Let $(X,\mathscr{T})$ be an L-space. $G\in L^X$ is (countably) α-compact if and only if for every (countable) family $\mathscr{B}$ of α-closed L-sets, it follows that*

$$\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{B}}B(x)\Big)\geqslant\bigwedge_{\mathscr{F}\in 2^{(\mathscr{B})}}\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{F}}B(x)\Big).$$

Theorem 3.6. *Let $(X,\mathscr{T})$ be an L-space. $G\in L^X$ has the α-Lindelöf property if and only if for every family $\mathscr{B}$ of α-closed L-sets, it follows that*

$$\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{B}}B(x)\Big)\geqslant\bigwedge_{\mathscr{F}\in 2^{[\mathscr{B}]}}\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{F}}B(x)\Big).$$

In order to present characterizations of α-compactness, countable α-compactness and the α-Lindelöf property, we generalize the notions of an a-shading and an a-R-neighborhood family in [15], [16] as follows:

Definition 3.7. Let $(X,\mathscr{T})$ be an L-space, $a\in L\setminus\{1\}$ and $G\in L^X$. A family $\mathscr{A}\subseteq L^X$ is said to be

(1) an a-shading of G if for any $x\in X$, $\Big(G'(x)\vee\bigvee_{A\in\mathscr{U}}A(x)\Big)\nleqslant a$.

(2) a strong a-shading of G if $\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathscr{U}}A(x)\Big)\nleqslant a$.

(3) an a-remote family of G if for any $x\in X$, $\Big(G(x)\wedge\bigwedge_{B\in\mathscr{P}}B(x)\Big)\ngeqslant a$.

(4) a strong a-remote family of G if $\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{P}}B(x)\Big)\ngeqslant a$.

It is obvious that a strong a-shading of G is an a-shading of G, a strong a-remote family of G is an a-remote family of G, and $\mathscr{P}$ is a strong a-remote family of G if and only if $\mathscr{P}'$ is a strong a'-shading of G. Moreover, a closed a-remote family is exactly an a-remote neighborhood family and a closed strong a-remote family is exactly an a^--remote neighborhood family in the sense of [19].

Definition 3.8. Let $a\in L\setminus\{0\}$ and $G\in L^X$. A subfamily $\mathscr{A}$ of L^X is said to have a weak a-nonempty intersection in G if $\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{A\in\mathscr{A}}A(x)\Big)\geqslant a$. $\mathscr{A}$ is said to have the finite (countable) weak a-intersection property in G if every finite (countable) subfamily $\mathscr{F}$ of $\mathscr{A}$ has a weak a-nonempty intersection in G.

Definition 3.9. Let $a\in L\setminus\{0\}$ and $G\in L^X$. A subfamily $\mathscr{A}$ of L^X is said to be a weak a-filter relative to G if any finite intersection of members in $\mathscr{A}$ is weak a-nonempty in G. A subfamily $\mathscr{B}$ of L^X is said to be a weak a-filterbase relative to G if

$$\{A\in L^X;\ \text{ there exists } B\in\mathscr{B} \text{ such that } B\leqslant A\}$$

is a weak a-filter relative to G.

From Definition 3.1, Definition 3.2, Theorem 3.5 and Theorem 3.6 we immediately obtain the following two results.

Theorem 3.10. *Let* $(X, \mathscr{T})$ *be an* L*-space and* $G \in L^X$. *Then the following conditions are equivalent:*

(1) G *is (countably)* α*-compact.*

(2) *For any* $a \in L \setminus \{1\}$, *each (countable)* α*-open strong* a*-shading* $\mathscr{U}$ *of* G *has a finite subfamily which is a strong* a*-shading of* G.

(3) *For any* $a \in L \setminus \{0\}$, *each (countable)* α*-closed strong* a*-remote family* $\mathscr{P}$ *of* G *has a finite subfamily which is a strong* a*-remote family of* G.

(4) *For any* $a \in L \setminus \{0\}$, *each (countable) family of* α*-closed* L*-sets which has the finite weak* a*-intersection property in* G *has a weak* a*-nonempty intersection in* G.

(5) *For each* $a \in L \setminus \{0\}$, *every* α*-closed (countable) weak* a*-filterbase relative to* G *has a weak* a*-nonempty intersection in* G.

Theorem 3.11. *Let* $(X, \mathscr{T})$ *be an* L*-space and* $G \in L^X$. *Then the following conditions are equivalent:*

(1) G *has the* α*-Lindelöf property.*

(2) *For any* $a \in L \setminus \{1\}$, *each* α*-open strong* a*-shading* $\mathscr{U}$ *of* G *has a countable subfamily which is a strong* a*-shading of* G.

(3) *For any* $a \in L \setminus \{0\}$, *each* α*-closed strong* a*-remote family* $\mathscr{P}$ *of* G *has a countable subfamily which is a strong* a*-remote family of* G.

(4) *For any* $a \in L \setminus \{0\}$, *each family of* α*-closed* L*-sets which has the countable weak* a*-intersection property in* G *has a weak* a*-nonempty intersection in* G.

4. Properties of (countable) α-compactness

Theorem 4.1. *Let* L *be a complete Heyting algebra. If both* G *and* H *are (countably)* α*-compact, then* $G \vee H$ *is (countably)* α*-compact.*

Proof. For any (countable) family $\mathscr{P}$ of α-closed L-sets, we have by Theorem 3.5 that

$$\bigvee_{x \in X} \Big((G \vee H)(x) \wedge \bigwedge_{B \in \mathscr{P}} B(x) \Big) = \Big\{ \bigvee_{x \in X} \Big(G(x) \wedge \bigwedge_{B \in \mathscr{P}} B(x) \Big) \Big\} \vee \Big\{ \bigvee_{x \in X} \Big(H(x) \wedge \bigwedge_{B \in \mathscr{P}} B(x) \Big) \Big\}$$

$$\geqslant \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big) \right\} \vee \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big(H(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big) \right\}$$
$$= \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big((G \vee H)(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big).$$

This shows that $G \vee H$ is (countably) α-compact. □

Analogously we have the following result.

Theorem 4.2. *Let L be a complete Heyting algebra. If both G and H have the α-Lindelöf property, then so does $G \vee H$.*

Theorem 4.3. *If G is (countably) α-compact and H is α-closed, then $G \wedge H$ is (countably) α-compact.*

Proof. For any (countable) family $\mathscr{P}$ of α-closed L-sets, we have by Theorem 3.5 that

$$\bigvee_{x\in X} \Big((G \wedge H)(x) \wedge \bigwedge_{B\in\mathscr{P}} B(x) \Big)$$
$$= \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathscr{P}\cup\{H\}} B(x) \Big) \geqslant \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P}\cup\{H\})}} \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big)$$
$$= \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big) \right\} \wedge \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big(G(x) \wedge H(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big) \right\}$$
$$= \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big(G(x) \wedge H(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big) \right\}$$
$$= \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \Big((G \wedge H)(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \Big).$$

This shows that $G \wedge H$ is (countably) α-compact. □

Analogously we have the following result.

Theorem 4.4. *If G has the α-Lindelöf property and H is α-closed, then $G \wedge H$ has the α-Lindelöf property.*

Theorem 4.5. *Let L be a complete Heyting algebra and let $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ be an α-irresolute map. If G is an α-compact (or a countably α-compact, an α-Lindelöf) L-set in $(X,\mathscr{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y,\mathscr{T}_2)$.*

P r o o f. We only prove that the theorem is true for α-compactness. Suppose that $\mathscr{P}$ is a family of α-closed L-sets in $(Y,\mathscr{T}_2)$. Then by Lemma 2.5 and α-compactness of G we have that

$$\bigvee_{y\in Y}\Big(f_L^{\rightarrow}(G)(y)\wedge\bigwedge_{B\in\mathscr{P}}B(y)\Big)=\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{P}}f_L^{\leftarrow}(B)(x)\Big)$$
$$\geqslant\bigwedge_{\mathscr{F}\in2^{(\mathscr{P})}}\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{F}}f_L^{\leftarrow}(B)(x)\Big)=\bigwedge_{\mathscr{F}\in2^{(\mathscr{P})}}\bigvee_{y\in Y}\Big(f_L^{\rightarrow}(G)(y)\wedge\bigwedge_{B\in\mathscr{F}}B(y)\Big).$$

Therefore $f_L^{\rightarrow}(G)$ is α-compact. □

Analogously we have the following result.

Theorem 4.6. *Let L be a complete Heyting algebra and let $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ be an α-continuous map. If G is an α-compact (a countably α-compact, an α-Lindelöf) L-set in $(X,\mathscr{T}_1)$, then $f_L^{\rightarrow}(G)$ is a compact (countably compact, Lindelöf) L-set in $(Y,\mathscr{T}_2)$.*

Definition 4.7. Let $(X,\mathscr{T}_1)$ and $(Y,\mathscr{T}_2)$ be two L-spaces. A map $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ is called strongly α-irresolute if $f_L^{\leftarrow}(G)$ is open in $(X,\mathscr{T}_1)$ for every α-open L-set G in $(Y,\mathscr{T}_2)$.

It is obvious that a strongly α-irresolute map is α-irresolute and continuous.

Analogously we have the following result.

Theorem 4.8. *Let L be a complete Heyting algebra and let $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ be a strongly α-irresolute map. If G is a compact (countably compact, Lindelöf) L-set in $(X,\mathscr{T}_1)$, then $f_L^{\rightarrow}(G)$ is an α-compact (a countably α-compact, an α-Lindelöf) L-set in $(Y,\mathscr{T}_2)$.*

5. Further characterizations of α-compactness and goodness

In this section we assume that L is a completely distributive de Morgan algebra.

Now we generalize the notions of a β_a-open cover and a Q_a-open cover [16] as follows:

Definition 5.1. Let $(X, \mathscr{T})$ be an L-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathscr{U} \subseteq L^X$ is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x))$. $\mathscr{U}$ is called a strong β_a-cover of G if $a \in \beta\Big(\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x)\Big)\Big)$.

Definition 5.2. Let $(X, \mathscr{T})$ be an L-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathscr{U} \subseteq L^X$ is called a Q_a-cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x) \geqslant a$.

It is obvious that a strong β_a-cover of G is a β_a-cover of G, and a β_a-cover of G is a Q_a-cover of G.

Analogously to the proof of Theorem 2.9 in [16] we can obtain the following theorem.

Theorem 5.3. *Let $(X, \mathscr{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.*

(1) *G is α-compact.*

(2) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$), each α-closed strong a-remote family of G has a finite subfamily which is an a-remote (a strong a-remote) family of G.*

(3) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any α-closed strong a-remote family $\mathscr{P}$ of G, there exist a finite subfamily $\mathscr{F}$ of $\mathscr{P}$ and $b \in \beta(a)$ (or $b \in \beta^*(a)$) such that $\mathscr{F}$ is a (strong) b-remote family of G.*

(4) *For any $a \in L \setminus \{1\}$ (or $a \in P(L)$), each α-open strong a-shading of G has a finite subfamily which is an a-shading (a strong a-shading) of G.*

(5) *For any $a \in L \setminus \{1\}$ (or $a \in P(L)$) and any α-open strong a-shading $\mathscr{U}$ of G, there exist a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ and $b \in \alpha(a)$ (or $b \in \alpha^*(a)$) such that $\mathscr{V}$ is a (strong) b-shading of G.*

(6) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$), each α-open strong β_a-cover of G has a finite subfamily which is a (strong) β_a-cover of G.*

(7) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any α-open strong β_a-cover $\mathscr{U}$ of G, there exist a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ and $b \in L$ (or $b \in M(L)$) with $a \in \beta(b)$ such that $\mathscr{V}$ is a (strong) β_b-cover of G.*

(8) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$, each α-open Q_a-cover of G has a finite subfamily which is a Q_b-cover of G.*

(9) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$ (or $b \in \beta^*(a)$), each α-open Q_a-cover of G has a finite subfamily which is a (strong) β_b-cover of G.*

Analogously we also can present characterizations of countable α-compactness and the α-Lindelöf property.

Now we consider the goodness of α-compactness.

For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \qquad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases}$$

Theorem 5.4 ([17]). *For an L-set $A \in L^X$, the following facts are true.*
(1) $A = \bigvee_{a \in L} (a \wedge A_{(a)}) = \bigvee_{a \in L} (a \wedge A_{[a]})$.
(2) $A = \bigwedge_{a \in L} (a \vee A^{(a)}) = \bigwedge_{a \in L} (a \vee A^{[a]})$.

Theorem 5.5 [17]. *Let $(X, \omega_L(\tau))$ be the L-space topologically generated by (X, τ) and $A \in L^X$. Then the following facts hold.*
(1) $\mathrm{cl}(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^-) = \bigvee_{a \in L} (a \wedge (A_{[a]})^-)$;
(2) $\mathrm{cl}(A)_{(a)} \subset (A_{(a)})^- \subset (A_{[a]})^- \subset \mathrm{cl}(A)_{[a]}$;
(3) $\mathrm{cl}(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^-) = \bigwedge_{a \in L} (a \vee (A^{[a]})^-)$;
(4) $\mathrm{cl}(A)^{(a)} \subset (A^{(a)})^- \subset (A^{[a]})^- \subset \mathrm{cl}(A)^{[a]}$;
(5) $\mathrm{int}(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^\circ) = \bigvee_{a \in L} (a \wedge (A_{[a]})^\circ)$;
(6) $\mathrm{int}(A)_{(a)} \subset (A_{(a)})^\circ \subset (A_{[a]})^\circ \subset \mathrm{int}(A)_{[a]}$;
(7) $\mathrm{int}(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^\circ) = \bigwedge_{a \in L} (a \vee (A^{[a]})^\circ)$;
(8) $\mathrm{int}(A)^{(a)} \subset (A^{(a)})^\circ \subset (A^{[a]})^\circ \subset \mathrm{int}(A)^{[a]}$, *where $(A_{(a)})^-$ and $(A_{(a)})^\circ$ denote respectively the closure and the interior of $A_{(a)}$ in (X, τ) and so on,* $\mathrm{cl}(A)$ *and* $\mathrm{int}(A)$ *denote respectively the closure and the interior of A in $(X, \omega_L(\tau))$.*

Lemma 5.6. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). If A is an α-open set in (X, τ), then χ_A is an α-open L-set in $(X, \omega_L(\tau))$. If B is an α-open L-set in $(X, \omega_L(\tau))$, then $B_{(a)}$ is an α-open set in (X, τ) for every $a \in L$.*

Proof. If A is an α-open set in (X, τ), then $A \subseteq ((A^\circ)^-)^\circ$. Thus we have

$$\chi_A \leqslant \chi_{((A^\circ)^-)^\circ} = \mathrm{int}(\chi_{(A^\circ)^-}) = \mathrm{int}(\mathrm{cl}(\chi_{A^\circ})) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(\chi_A))).$$

This shows that χ_A is α-open in $(X, \omega_L(\tau))$.

If B is an α-open L-set in $(X, \omega_L(\tau))$, then $B \leqslant \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))$. From Theorem 5.5 we have

$$B_{(a)} \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))_{(a)} \subseteq (\mathrm{cl}(\mathrm{int}(B))_{(a)})^\circ \subseteq ((\mathrm{int}(B)_{(a)})^-)^\circ \subseteq (((B_{(a)})^\circ)^-)^\circ.$$

This shows that $B_{(a)}$ is an α-open set in (X, τ). $\square$

The next two theorems show that α-compactness, countable α-compactness and the α-Lindelöf property are good extensions.

Theorem 5.7. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ is (countably) α-compact if and only if (X, τ) is (countably) α-compact.*

Proof. (Necessity) Let $\mathscr{A}$ be an α-open cover (a countable α-open cover) of (X, τ). Then $\{\chi_A;\ A \in \mathscr{A}\}$ is a family of α-open L-sets in $(X, \omega_L(\tau))$ with $\bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{U}} \chi_A(x)\Big) = 1$. From (countable) α-compactness of $(X, \omega_L(\tau))$ we know that

$$1 \geqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{V}} \chi_A(x)\Big) \geqslant \bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{U}} \chi_A(x)\Big) = 1.$$

This implies that there exists $\mathscr{V} \in 2^{(\mathscr{U})}$ such that $\bigwedge_{x\in X}(\bigvee_{A\in\mathscr{V}} \chi_A(x)) = 1$. Hence $\mathscr{V}$ is a cover of (X, τ). Therefore (X, τ) is (countably) α-compact.

(Sufficiency) Let $\mathscr{U}$ be a (countable) family of α-open L-sets in $(X, \omega_L(\tau))$ and let $\bigwedge_{x\in X}(\bigvee_{B\in\mathscr{U}} B(x)) = a$. If $a = 0$, then we obviously have

$$\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{U}} B(x)\Big) \leqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{V}} B(x)\Big).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$ we have

$$b \in \beta\Big(\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{U}} B(x)\Big)\Big) \subseteq \bigcap_{x\in X} \beta\Big(\bigvee_{B\in\mathscr{U}} B(x)\Big) = \bigcap_{x\in X}\bigcup_{B\in\mathscr{U}} \beta\left(B(x)\right).$$

By Lemma 5.6 this implies that $\{B_{(b)};\ B \in \mathscr{U}\}$ is an α-open cover of (X, τ). From (countable) α-compactness of (X, τ) we know that there exists $\mathscr{V} \in 2^{(\mathscr{U})}$ such that $\{B_{(b)};\ B \in \mathscr{V}\}$ is a cover of (X, τ). Hence $b \leqslant \bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}} B(x)\Big)$. Further we have

$$b \leqslant \bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}} B(x)\Big) \leqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}} B(x)\Big).$$

This implies that

$$\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{U}} B(x)\Big) = a = \bigvee\{b;\ b \in \beta(a)\} \leqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}} B(x)\Big).$$

Therefore $(X, \omega_L(\tau))$ is (countably) α-compact. □

Analogously we have the following result.

Theorem 5.8. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ has the α-Lindelöf property if and only if (X, τ) has the α-Lindelöf property.*

6. The relations of α-compactness and other types of compactness

In this section we assume that L is again completely distributive.

Based on Kudri's compactness in [7], Aygün presented a definition of α-compactness in [1]. Since Kudri's compactness is equivalent to strong compactness in [9], [19], we shall also refer to Aygün's α-compactness as α-strong compactness. The following is its equivalent form.

Definition 6.1. Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to be α-strongly compact if for any $a \in P(L)$, each α-open a-shading of G has a finite subfamily which is an a-shading of G.

In [4], [8], Bai and Li et al. introduced the notions of SR-compactness and near SR-compactness by means of strongly semiopen L-sets. In fact, a strongly semiopen L-set is equivalent to an α-open L-set. This implies that both SR-compactness and near SR-compactness are extensions of α-compactness in general topology. Their equivalent forms can be stated as follows:

Definition 6.2 ([4]). Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to be SR-compact (we shall call it α-N-compact) if for each $a \in M(L)$, every α-closed a-remote family of G has a finite subfamily which is a strong a-remote family of G.

Definition 6.3 ([8]). Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to be near SR-compact if for each $a \in M(L)$, every α-closed a-remote family of G has a finite subfamily which is an a-remote family of G.

It is obvious that Definition 6.3 is equivalent to Definition 6.1.

From Theorem 5.3 we easily obtain the following result.

Theorem 6.4. *For an L-set G in an L-space, the following implications hold.*

α-N-compactness	$\Rightarrow$	*α-strong compactness*	$\Rightarrow$	*α-compactness*
$\Downarrow$		$\Downarrow$		$\Downarrow$
N-compactness	$\Rightarrow$	*strong compactness*	$\Rightarrow$	*compactness*

Notice that none of the above implications is invertible. We only present a counterexample which is α-compact but not α-strongly compact. The other examples can be found in [8], [9], [19] and in general topology.

Example 6.5. Take $Y=\mathbb{N}$. For all $n\in Y$, define $B_n\in[0,1]^Y$ as follows:

$$B_n(y)=\begin{cases}(n+1)^{-1}, & y=n;\\ 0, & y\neq n.\end{cases}$$

Let $\mathscr{T}$ be the $[0,1]$-topology generated by the subbase $\mathscr{B}=\{B_n;\ n\in Y\}$. Obviously $\{B_n;\ n\in Y\}$ is an open 0-shading of χ_Y, but $\{B_n;\ n\in Y\}$ has no finite subfamily which is an open 0-shading of χ_Y. Therefore $(Y,\mathscr{T})$ is not strongly compact, of course it is not α-strongly compact either.

Now we prove that $(Y,\mathscr{T})$ is α-compact. It is easy to check that if A is an α-open L-set in $(Y,\mathscr{T})$ and $A\neq\chi_Y$, then $A\leqslant\bigvee_{n\in Y}B_n$.

For each $a\in[0,1)$, suppose that $\mathscr{U}$ is an α-open strong a-shading of χ_Y. If $\chi_Y\in\mathscr{U}$, then $\{\chi_Y\}$ is a strong a-shading of χ_Y. Now we suppose that $\chi_Y\notin\mathscr{U}$. Then $\mathscr{U}$ is not a strong a-shading of χ_Y since $\bigwedge_{y\in Y}\Big(\bigvee_{A\in\mathscr{U}}A(y)\Big)\leqslant\bigwedge_{y\in Y}\Big(\bigvee_{n\in Y}B_n(y)\Big)=0$.

This shows that $(Y,\mathscr{T})$ is α-compact.

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