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Mathematica Bohemica, Vol. 131 (2006), No. 1, 1–9

Persistent URL: http://dml.cz/dmlcz/134078

131 (2006) MATHEMATICA BOHEMICA No. 1, 1–9

THE DUAL OF THE SPACE OF FUNCTIONS OF BOUNDED VARIATION

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(Received May 31, 2005)

Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. In the paper, we show that the space of functions of bounded variation and the space of regulated functions are, in some sense, the dual space of each other, involving the Henstock-Kurzweil-Stieltjes integral.

Keywords: bounded variation, two-norm space, dual space, linear functional, Henstock integral, Stieltjes integral, regulated function

MSC 2000: 26A39, 26A42, 26A45, 46B26, 46E99

1. Introduction

The classical Riesz representation theorem is well known [5]. It states that a continuous linear functional on the space $C[a,b]$ of continuous functions on $[a,b]$ can be represented in terms of the Riemann-Stieltjes integral $\int_a^b f\,\mathrm{d}g$ for $f\in C[a,b]$ where g is a function of bounded variation on $[a,b]$. In other words, the Riesz representation theorem showed that the dual of $C[a,b]$ is the space BV of functions of bounded variation or more precisely a subspace of BV. However we fail to say in whatever sense that the space $C[a,b]$ is the dual of BV.

In 1966, Hildebrandt [9] has characterized continuous linear functionals on the space BV regarding BV as a two-norm space. By a two-norm space we mean that the space is endorsed with two-norm topology or two-norm convergence. More precisely, a sequence is convergent in the two-norm sense if the sequence is bounded in one given norm and convergent in another given norm. Further Hildebrandt used the left Cauchy integral [8, p. 87] among others. It is not explicit how his representation theorem is connected with the Riesz theorem. In this paper, we reformulate Hildebrandt's results in terms of the Henstock-Kurzweil-Stieltjes integral, and establish a duality in which a part of it is the Riesz representation theorem. More precisely, let BVN be the subspace of BV such that the condition $f(x) = \frac{1}{2}[f(x+) + f(x-)]$ holds for $x \in (a,b)$ with $f(a) = f(a+)$ and $f(b) = f(b-)$ whenever $f \in$ BVN. A *regulated function* [2] on $[a,b]$ is a function whose one-sided limits exist at every point of $[a,b]$. Let RF denote the space of regulated functions on $[a,b]$, and RFN the subspace of RF satisfying the condition as in BVN. Regarding BVN as a two-norm space and RFN a normed space, then the dual of BVN is RFN and vice versa. Note that $C[a,b]$ is a subspace of RFN and its dual is BVN. The fact that the dual of $C[a,b]$ is BVN can now embedded into the duality of RFN and BVN.

Let $V(g;[a,b])$ or $V(g)$ denote the *total variation* of g on $[a,b]$. A function g is *of bounded variation* on $[a,b]$ or $g \in$ BV if $V(g)$ is finite. A sequence $\{g_n\}$ of functions on $[a,b]$ is said to be *two-norm convergent* to g in BV if $V(g_k) \leqslant M$ for all k and g_k is uniformly convergent to g on $[a,b]$. The space BV is now a *two-norm space* with the two-norm structure provided by the two-norm convergence in BV. Further, RF is a *normed space* with norm given by $\|f\| = \sup\{|f(x)| : x \in [a,b]\}$.

2. The Henstock-Stieltjes integral

We define the Henstock-Kurzweil-Stieltjes integral [6], [7] and prove a convergence theorem. A real-valued function f is said to be *Henstock-Kurzweil-Stieltjes integrable* to A with respect to g on $[a,b]$ or (f,g) *Henstock-Kurzweil-Stieltjes integrable* on $[a,b]$, if for every $\varepsilon > 0$, there exists a function $\delta(\xi) > 0$ for $\xi \in [a,b]$ such that for any division D of $[a,b]$ given by $a = x_0 < x_1 < \ldots < x_n = b$, with $\xi_1, \xi_2, \ldots, \xi_n$ satisfying $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \ldots, n$, we have

$$\left| \sum_{i=1}^{n} f(\xi_i)[g(x_i) - g(x_{i-1})] - A \right| < \varepsilon.$$

For brevity, we write $D = \{(\xi, [u,v])\}$ in which $(\xi, [u,v])$ denotes a typical point-interval pair $(\xi_i, [x_{i-1}, x_i])$ in D, and also we write the Riemann sum in the form $(D)\sum f(\xi)[g(v) - g(u)]$. Here D is said to be *δ-fine* if the above condition holds [6], [7]. That is, (f,g) is *Henstock-Kurzweil-Stieltjes* integrable to A on $[a,b]$ if for every $\varepsilon > 0$, there is a positive function δ such that for any δ-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$, we have

$$(*) \qquad |s(f,g,D) - A| < \varepsilon$$

where $s(f,g,D) = (D)\sum f(\xi)\,[g(v) - g(u)]$. For simplicity, we write $A = \int_a^b f\,\mathrm{d}g$. A division D_1 is said to be *finer* than a division D_2 if every interval of D_1 is a subinterval of some interval of D_2. If $g(x) = x$, then the integral reduces to the *Henstock-Kurzweil integral.* If δ is a constant, then it reduces to the *Riemann-Stieltjes integral.* Moreover, if $g(x) = x$ and δ is a constant, then it reduces to the *Riemann integral* [10]. We remark that the Henstock-Kurzweil-Stieltjes integral of (f,g) does allow both functions f and g to be discontinuous on the same side at a point.

Let f be a real-valued function defined on $[a,b]$. A family $\mathcal{F}$ of Henstock-Kurzweil-Stieltjes integrable functions (f,g_k), $k = 1,2,\ldots$, is said to be *equi-integrable* on $[a,b]$, if the above inequality $(*)$ holds with g and A replaced by g_k and the integral A_k of g_k, and δ being independent of k [7, p. 104]. It is clear that if $f \in \mathrm{RF}$ and $g \in \mathrm{BV}$, then $\int_a^b f\,\mathrm{d}g$ exists, for reference see [3].

Lemma 1. *Suppose $f \in \mathrm{RF}$. If $g_k \in \mathrm{BV}$ and $V(g_k) \leqslant M$ for every k, then $\{(f,g_k) : k = 1,2,\ldots\}$ is equi-integrable on $[a,b]$.*

Proof. Since $f \in \mathrm{RF}$, it is known [1] that for every $\varepsilon > 0$, there exists a division D_0: $a = x_0 < x_1 < \ldots < x_n = b$ such that $|f(\xi) - f(\eta)| < \varepsilon$ for $\xi, \eta \in (x_{i-1}, x_i)$, $i = 1,2,\ldots,n$. Fix i and put $g_k^*(x) = g_k(x)$ for every $x \in (x_{i-1}, x_i)$ and $g_k^*(x_{i-1}) = g_k(x_{i-1}+)$, $g_k^*(x_i) = g_k(x_i-)$. Then we have

$$\int_{x_{i-1}}^{x_i} f\,\mathrm{d}g_k = \int_{x_{i-1}}^{x_i} f\,\mathrm{d}g_k^* + f(x_{i-1})\,[g_k(x_{i-1}+) - g_k(x_{i-1})] + f(x_i)\,[g_k(x_i) - g_k(x_i-)].$$

Define a function $\delta(\xi) > 0$ for every $\xi \in [a,b]$ such that every δ-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$ is finer than D_0. Take any δ-fine division D. Then we can write $D = D_1 \cup D_2 \cup \ldots \cup D_n$ where D_i is a δ-fine division of $[x_{i-1}, x_i]$. Thus we have

$$\left| s(f, g_k, D_i) - \int_{x_{i-1}}^{x_i} f\,\mathrm{d}g_k \right| = \left| s(f, g_k^*, D_i) - \int_{x_{i-1}}^{x_i} f\,\mathrm{d}g_k^* \right| \leqslant \varepsilon V(g_k; [x_{i-1}, x_i]).$$

Hence for every δ-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$,

$$\left| s(f, g_k, D) - \int_a^b f\,\mathrm{d}g_k \right| \leqslant \sum_{i=1}^n \left| s(f, g, D_i) - \int_{x_{i-1}}^{x_i} f\,\mathrm{d}g_k \right| \leqslant \varepsilon V(g_k; [a,b]) \leqslant \varepsilon M$$

for every k. That is, $\{(f,g_k) : k = 1,2,\ldots\}$ is equi-integrable on $[a,b]$. $\square$

Theorem 1. *Suppose* $g_n \in \mathrm{BV}$ *for* $n = 1, 2, \ldots$, *and* $f \in \mathrm{RF}$. *If* g_n *is two-norm convergent to* g *in* BV, *then* $\int_a^b f \, \mathrm{d}g$ *exists and* $\lim_{n\to\infty} \int_a^b f \, \mathrm{d}g_n = \int_a^b f \, \mathrm{d}g$.

Proof. It is clear that $g \in \mathrm{BV}$ and $\int_a^b f \, \mathrm{d}g$ exists. Further, by Lemma 1, $\{(f, g_n) : n = 1, 2, \ldots\}$ is equi-integrable on $[a, b]$. Then for every $\varepsilon > 0$, there exists a function $\delta(\xi)$ for $\xi \in [a, b]$ such that for every δ-fine division $D = \{(\xi, [u, v])\}$ of $[a, b]$ and for every n, we have $|s(f, g, D) - \int_a^b f \, \mathrm{d}g| < \varepsilon$ and $|s(f, g_n, D) - \int_a^b f \, \mathrm{d}g_n| < \varepsilon$. Since g_n is uniformly convergent to g on $[a, b]$, we can prove that there exists D_0 such that $|s(f, g_n, D_0) - s(f, g, D_0)| < \varepsilon$ for large n. Therefore we have

$$\left| \int_a^b f \, \mathrm{d}g_n - \int_a^b f \, \mathrm{d}g \right| \leqslant \left| \int_a^b f \, \mathrm{d}g_n - s(f, g_n, D_0) \right| + |s(f, g_n, D_0) - s(f, g, D_0)| + \left| \int_a^b f \, \mathrm{d}g - s(f, g, D_0) \right| < 3\varepsilon$$

for large n. □

3. Two-norm continuous linear functionals

A functional F defined on BV or RF is *linear* if $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$ for $f, g \in \mathrm{BV}$ or RF, and real α, β. A functional F defined on BV is said to be *two-norm continuous* if $F(g_n) \to F(g)$ as $n \to \infty$ whenever g_n is two-norm convergent to g in BV. Theorem 1 shows that if $f \in \mathrm{RF}$, then $F(g) = \int_a^b f \, \mathrm{d}g$ is two-norm continuous in BV. A functional F defined on RF is *continuous* if $F(f_n) \to F(f)$ as $n \to \infty$ whenever $\|f_n - f\| \to 0$ as $n \to \infty$. In this section, we characterize the dual or the two-norm dual of the space BV, that is, the space of all two-norm continuous linear functionals on BV, in terms of the Henstock-Kurzweil-Stieltjes integral. We define *characteristic functions* γ_t and δ_t for a fixed $t \in [a, b]$. When $t \in (a, b)$, define $\gamma_t(x) = 0$ for every $x \in [a, t)$, $\frac{1}{2}$ when $x = t$ and 1 for every $x \in (t, b]$. When $t = a$, define $\gamma_a(x) = 0$ when $x = a$ and 1 for every $x \in (a, b]$. When $t = b$, define $\gamma_b(x) = 0$ for every $x \in [a, b)$ and 1 when $x = b$. We also define $\delta_t(x) = 1$ when $x = t$ and 0 otherwise. We now give a series of lemmas leading to the main theorem of this paper.

Lemma 2. *If* F *is a two-norm continuous linear functional defined on* BV, *then* $f \in \mathrm{RFN}$ *where* $f(t) = F(\gamma_t)$ *for every* $t \in [a, b]$.

Proof. Suppose $f \notin \mathrm{RF}$. Then there exists a point x such that $f(x+)$ or $f(x-)$ does not exist. Suppose $f(x+)$ does not exist. Then there exist $\eta > 0$ and a sequence of pairwise disjoint intervals $[x_j', x_j'']$ on the same right side of x and approaching

x such that $|f(x''_j) - f(x'_j)| > \eta$ for every j. Define $g_n = \frac{1}{n}\sum\limits_{j=1}^{n} \varepsilon_j(\gamma_{x''_j} - \gamma_{x'_j})$ where $\varepsilon_j = \operatorname{sgn}(f(x''_j) - f(x'_j))$. It is clearly seen that g_n is uniformly convergent to zero on $[a,b]$ and $V(g_n) \leqslant 2$ but

$$F(g_n) = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j(F(\gamma_{x''_j}) - F(\gamma_{x'_j})) = \frac{1}{n}\sum_{j=1}^{n} |f(x''_j) - f(x'_j)| > \eta,$$

for every n. That is, F is not a two-norm continuous linear functional on BV. Therefore $f(x+)$ and $f(x-)$ must exist for every x. Hence f is a regulated function. In view of the definition of γ_t, indeed $f \in$ RFN. $\square$

Lemma 3. *Suppose F is a two-norm continuous linear functional defined on* BV*. If $f(t) = F(\gamma_t)$ for every $t \in [a,b]$, then $f \in$* RFN*, the integral $\int_a^b f\,\mathrm{d}g$ exists and $F(g) = \int_a^b f\,\mathrm{d}g$ for $g \in$* BVN*.*

P r o o f. By Lemma 2, $f \in$ RFN and consequently $\int_a^b f\,\mathrm{d}g$ exists. In view of Theorem 1, it is sufficient to prove Lemma 3 for the case when g is a step function in BVN. Let t_i, $i = 1,\ldots,n-1$, be the discontinuity points of g. Consider the case when $a = t_0 < t_1 < \ldots < t_n = b$. Then

$$\int_a^b f\,\mathrm{d}g = \sum_{i=1}^{n-1} f(t_i)\,[g(t_i+) - g(t_i-)] = F\bigg(\sum_{i=1}^{n-1} \gamma_{t_i}\cdot[g(t_i+) - g(t_i-)]\bigg).$$

Here we note that

$$\sum_{i=1}^{n-1} \gamma_{t_i}\cdot[g(t_i+) - g(t_i-)] = g$$

and hence $\int_a^b f\,\mathrm{d}g = F(g)$ for the step function $g \in$ BVN and consequently for any $g \in$ BVN. $\square$

The linear space $c_0\,[a,b]$ [5] is the space of all functions f defined on $[a,b]$ such that the set $\{t \in [a,b] :\ |f(t)| \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$.

Lemma 4. *If F is a two-norm continuous linear functional defined on* BV*, then $f \in c_0\,[a,b]$ where $f(t) = F(\delta_t)$ for every $t \in [a,b]$.*

P r o o f. Suppose $f \notin c_0\,[a,b]$. Then there exists $\varepsilon > 0$ such that $|f(t_i)| \geqslant \varepsilon$ for a sequence $\{t_i\}_{i\geqslant 1} \subset [a,b]$. Define $g_n = \frac{1}{n}\sum\limits_{i=1}^{n} \varepsilon_i\delta_{t_i}$, where $\varepsilon_i = \operatorname{sgn} f(t_i)$. Then we obtain $V(g_n) \leqslant 2$ and g_n is uniformly convergent to 0 on $[a,b]$ but

$$F(g_n) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i F(\delta_{t_i}) = \frac{1}{n}\sum_{i=1}^{n} |f(t_i)| \geqslant \varepsilon \text{ for every } n.$$

That is, F is not a two-norm continuous linear functional on BV. Therefore $f \in c_0[a,b]$. □

We define the *normalized function* g^* of g on $[a,b]$ to be $g^*(x) = \frac{1}{2}[g(x+)+g(x-)]$ for $x \in (a,b)$, and $g^*(a) = g(a+)$, $g^*(b) = g(b-)$. If $g \in$ BV, then $g^* \in$ BVN.

The main result of this paper is the following theorem.

Theorem 2. *A functional F defined on* BV *is linear and two-norm continuous if and only if there exist functions $f_1 \in$ RFN and $f_2 \in c_0[a,b]$ such that*

$$F(g) = \int_a^b f_1 \, \mathrm{d}g^* + \sum_{i=1}^{\infty} [g(t_i) - g^*(t_i)] f_2(t_i)$$

for every $g \in$ BV, where t_i, $i = 1, 2, \ldots$, are the discontinuity points of g, and g^* the normalized function of g.

Proof. ⇒: Put $f_1(t) = F(\gamma_t)$ for every $t \in [a,b]$. By Lemma 2, $f_1 \in$ RFN. Let $g \in$ BV and g^* the normalized function of g. Then

$$F(g) = F(g^*) + F(g - g^*).$$

Next, put $f_2(t) = F(\delta_t)$ for every $t \in [a,b]$. By Lemma 4, $f_2 \in c_0[a,b]$. Let t_i, $i = 1, 2, \ldots$, be the discontinuity points of g. Then for any n

$$\begin{aligned} F(g - g^*) &= F\Bigg(\sum_{i=1}^{\infty} [g(t_i) - g^*(t_i)] \, \delta_{t_i} \Bigg) \\ &= \sum_{i=1}^{n} [g(t_i) - g^*(t_i)] f_2(t_i) + F\Bigg(\sum_{i=n+1}^{\infty} [g(t_i) - g^*(t_i)] \, \delta_{t_i} \Bigg). \end{aligned}$$

Since $\{f_2(t_i)\}$ is a null sequence, that is, a sequence converging to 0, and $\sum\limits_{i=1}^{\infty} [g(t_i) - g^*(t_i)]$ absolutely convergent, the series $\sum\limits_{i=1}^{\infty} [g(t_i) - g^*(t_i)] f_2(t_i)$ converges.

Further, $\sum\limits_{i=n+1}^{\infty} [g(t_i) - g^*(t_i)] \, \delta_{t_i}$ is two-norm convergent to zero in BV as $n \to \infty$. Then, by the two-norm continuity of F, we have

$$\lim_{n \to \infty} F\Bigg(\sum_{i=n+1}^{\infty} [g(t_i) - g^*(t_i)] \delta_{t_i} \Bigg) = 0.$$

Together with Lemma 3, we obtain the representation of F.

$\Leftarrow$: Let

$$F_1\,(g) = \int_a^b f_1\, \mathrm{d}g^*.$$

Take $g_n \in \mathrm{BV}$ such that g_n is two-norm convergent to g in BV. So does g_n^*. Since $f_1 \in$ RFN, by applying Lemma 1 and Theorem 1, $\{(f, g_k):\ k = 1, 2, \ldots\}$ is equi-integrable on $[a, b]$, $\int_a^b f_1\, \mathrm{d}g^*$ exists and $\lim\limits_{n\to\infty} \int_a^b f_1\, \mathrm{d}g_n^* = \int_a^b f_1\, \mathrm{d}g^*$. Hence $\lim\limits_{n\to\infty} F_1(g_n) = F_1(g)$. That is, $F_1(g)$ defines a two-norm continuous linear functional on BV.

Let

$$F_2\,(g) = \sum_{i=1}^{\infty} [g\,(t_i) - g^*\,(t_i)] f_2(t_i).$$

Since $f_2 \in c_0\,[a, b]$, for every $\varepsilon > 0$, the set $X = \{x \in [a, b]\colon\ |f_2(x)| \geqslant \varepsilon\}$ is finite. Consequently, we have $|f_2(x)| < \varepsilon$ for every $x \notin X$. Since g_n is two-norm convergent to g in BV, for every $\varepsilon > 0$, there exists N such that for every $n \geqslant N$ and for every $\xi \in [a, b]$, we have

$$|g_n(\xi) - g(\xi)| < \frac{\varepsilon}{n\,(X)}$$

where $n\,(X)$ denotes the number of the points in X. We may assume that $|f_2(x)| \leqslant M_1$ for every $x \in [a, b]$ and also we have $V(g_n) \leqslant M_2$ for every n and $V(g) \leqslant M_2$. Therefore for every $n \geqslant N$, we have

$$\sum_{t\in X} (|g_n(t) - g(t)| + |g_n^*(t) - g^*(t)|)|f_2(t)| \leqslant \sum_{t\in X} 2\frac{\varepsilon}{n(X)} M_1$$

and

$$\sum_{t\notin X} (|g_n(t) - g_n^*(t)| + |g(t) - g^*(t)|)|f_2(t)| \leqslant \varepsilon[V(g_n; [a, b]) + V(g; [a, b])].$$

Therefore

$$\left| \sum_{t\in(a,b)} ([g_n(t) - g_n^*(t)] - [g(t) - g^*(t)]) f_2(t) \right| \leqslant 2\varepsilon M_1 + \varepsilon(2M_2)$$

and hence $\lim\limits_{n\to\infty} F_2(g_n) = F_2(g)$. That is, $F_2(g)$ defines a two-norm continuous linear functional on BV. Put $F(g) = F_1(g) + F_2(g)$. Hence the proof is complete. □

As a special case, we obtain the following corollary.

Corollary 1. *A functional F defined on* BVN *is linear and two-norm continuous if and only if there exists a function $f \in$* RFN *such that*

$$F(g) = \int_a^b f \, \mathrm{d}g \quad \textit{for } g \in \mathrm{BVN}\,.$$

That is, the dual of the space BVN is the space RFN . Conversely, the dual of RFN is BVN . This has been proved in [4] and is stated below. The proof is similar to that of the Riesz theorem. A functional F on RFN is *continuous* if it is continuous with respect to the norm in RFN .

Theorem 3 [4]**.** *A functional F defined on* RFN *is linear and continuous if and only if there exist a function $g \in$* BV *and a real number d such that*

$$F\,(f) = df(a) + \int_a^b g \, \mathrm{d}f \quad \textit{for } f \in \mathrm{RFN}\,.$$

Therefore the spaces BVN and RFN are the duals of each other. A representation theorem for continuous linear functionals on RF of the form like Theorem 2 is also available [3], [12]. We shall not elaborate here.

Finally, the following Riesz representation theorem is a consequence of Theorem 3, integration by parts, and the Hahn-Banach theorem, since $C\,[a,b]$ is a closed subspace of RFN .

Corollary 2 (Riesz representation theorem)**.** *A functional F defined on $C\,[a,b]$ is linear and continuous if and only if there exists a function $g \in$* BVN *such that*

$$F(f) = \int_a^b f \, \mathrm{d}g \quad \textit{for } f \in C\,[a,b]\,.$$

Representation theorem for continuous linear functionals on RF was first given by Schwabik [13]. For another relevant reference, see [14].

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