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PSEUDO BL -ALGEBRAS AND $DR\ell$ -MONOIDS

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Abstract. It is shown that pseudo BL -algebras are categorically equivalent to certain bounded $DR\ell$ -monoids. Using this result, we obtain some properties of pseudo BL -algebras, in particular, we can characterize congruence kernels by means of normal filters. Further, we deal with representable pseudo BL -algebras and, in conclusion, we prove that they form a variety.

Keywords: pseudo BL -algebra, $DR\ell$ -monoid, filter, polar, representable pseudo BL -algebra

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1. CONNECTIONS BETWEEN PSEUDO BL -ALGEBRAS AND $DR\ell$ -MONOIDS

Recently, pseudo BL -algebras were introduced by A. Di Nola, G. Georgescu and A. Iorgulescu in [3] as a noncommutative extension of Hájek's BL -algebras (see [6]).

An algebra $\mathfrak{A} = (A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is called a *pseudo BL -algebra* iff $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a monoid and the following conditions are satisfied for all $x, y, z \in A$:

- (1) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,
- (2) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$,
- (3) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

By [3, Corollary 3.29], pseudo BL -algebras satisfying the identity

$$(x \rightsquigarrow 0) \rightarrow 0 = (x \rightarrow 0) \rightsquigarrow 0 = x$$

are the duals of pseudo MV -algebras.

In the same way, (noncommutative) $DR\ell$ -monoids extend Swamy's $DR\ell$ -semi-groups which were introduced in [12] as a common generalization of abelian ℓ -groups and Brouwerian algebras.

An algebra $\mathfrak{A} = (A, +, 0, \vee, \wedge, \rightarrow, \leftarrow)$ of type $\langle 2, 0, 2, 2, 2, 2 \rangle$ is a *dually residuated lattice ordered monoid*, or simply a *DR ℓ -monoid*, iff

- (1) $(A, +, 0, \vee, \wedge)$ is an ℓ -monoid, that is, $(A, +, 0)$ is a monoid, (A, \vee, \wedge) is a lattice and, for any $x, y, s, t \in A$, the following distributive laws are satisfied:

$$\begin{aligned} s + (x \vee y) + t &= (s + x + t) \vee (s + y + t), \\ s + (x \wedge y) + t &= (s + x + t) \wedge (s + y + t); \end{aligned}$$

- (2) for any $x, y \in A$, $x \rightarrow y$ is the least $s \in A$ such that $s + y \geq x$, and $x \leftarrow y$ is the least $t \in A$ such that $y + t \geq x$;
- (3) \mathfrak{A} fulfils the identities

$$\begin{aligned} ((x \rightarrow y) \vee 0) + y &\leq x \vee y, \quad y + ((x \leftarrow y) \vee 0) \leq x \vee y, \\ x \rightarrow x &\geq 0, \quad x \leftarrow x \geq 0. \end{aligned}$$

Note that the inequalities $x \rightarrow x \geq 0$ and $x \leftarrow x \geq 0$ can be omitted, and the condition (2) is equivalent to the system of identities (see [10])

$$\begin{aligned} (x \rightarrow y) + y &\geq x, \quad y + (x \leftarrow y) \geq x, \\ x \rightarrow y &\leq (x \vee z) \rightarrow y, \quad x \leftarrow y \leq (x \vee z) \leftarrow y, \\ (x + y) \rightarrow y &\leq x, \quad (y + x) \leftarrow y \leq x. \end{aligned}$$

In [11], mutual relationships between *BL*-algebras and bounded representable commutative *DR ℓ -monoids* are described.

Theorem 1.1. *Let $\mathfrak{A} = (A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo *BL*-algebra. If we set*

$$\begin{aligned} x + y &:= x \odot y, \quad x \vee_d y := x \wedge y, \quad x \wedge_d y := x \vee y, \\ x \rightarrow y &:= y \rightarrow x, \quad x \leftarrow y := y \rightsquigarrow x, \quad 0_d := 1, \quad 1_d := 0 \end{aligned}$$

for any $x, y \in A$, then $\mathfrak{A}_d = (A, +, 0_d, \vee_d, \wedge_d, \rightarrow, \leftarrow)$ is a bounded *DR ℓ -monoid* with the greatest element 1_d . In addition, this *DR ℓ -monoid* satisfies the identities

$$(*) \quad \begin{aligned} (x \rightarrow y) \wedge_d (y \rightarrow x) &= 0_d, \\ (x \leftarrow y) \wedge_d (y \leftarrow x) &= 0_d. \end{aligned}$$

Proof. Since $(A, \odot, 1, \vee, \wedge)$ is an ℓ -monoid, by [3, Propositions 3.3, 3.9], so is $(A, +, 0_d, \vee_d, \wedge_d)$. The rest follows directly by the definitions. Note that if a *DR ℓ -monoid* \mathfrak{A}_d contains the greatest element 1_d then 0_d is its least element, by [8, Theorem 1.2.3]. \square

In view of Theorem 1.1, it is easily seen that in the definition of a pseudo BL -algebra, the condition (1) can be equivalently replaced by the following identities:

$$\begin{aligned}(x \rightarrow y) \odot x &\leq y, \quad x \odot (x \rightsquigarrow y) \leq y, \\ x \rightarrow y &\geq x \rightarrow (y \wedge z), \quad x \rightsquigarrow y \geq x \rightsquigarrow (y \wedge z), \\ y \rightarrow (x \odot y) &\geq x, \quad y \rightsquigarrow (y \odot x) \geq x.\end{aligned}$$

Consequently, pseudo BL -algebras form a variety of algebras of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$. This variety is arithmetical; in accordance with [8, Theorem 3.1.1], the Pixley term of the variety of pseudo BL -algebras can be taken as follows:

$$p(x, y, z) = ((x \rightsquigarrow y) \rightarrow z) \wedge ((z \rightsquigarrow y) \rightarrow x) \wedge (x \vee z).$$

Theorem 1.2. *Let $\mathfrak{A} = (A, +, 0, \vee, \wedge, \rightarrow, \leftarrow)$ be a $DR\ell$ -monoid with the greatest element 1. For any $x, y \in A$ set*

$$\begin{aligned}x \odot y &:= x + y, \quad x \vee_d y := x \wedge y, \quad x \wedge_d y := x \vee y, \\ x \rightarrow y &:= y \leftarrow x, \quad x \rightsquigarrow y := y \leftarrow x, \quad 0_d := 1, \quad 1_d := 0.\end{aligned}$$

Then $\mathfrak{A}_d = (A, \vee_d, \wedge_d, \odot, \rightarrow, \rightsquigarrow, 0_d, 1_d)$ is a pseudo BL -algebra if and only if \mathfrak{A} satisfies ().*

Proof. In any $DR\ell$ -monoid we have

$$x \vee y = ((y \leftarrow x) \vee 0) + x = x + ((y \leftarrow x) \vee 0).$$

Since \mathfrak{A} is bounded, that is, $0 \leq x \leq 1$ for any $x \in A$, it follows that

$$x \wedge_d y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y).$$

The rest is obvious. □

Let \mathcal{PBL} be the category of pseudo BL -algebras, that is, the category whose objects are pseudo BL -algebras and morphisms are homomorphisms of pseudo BL -algebras. Let $\mathcal{DR}\mathcal{L}_{1(*)}$ be the category of bounded $DR\ell$ -monoids satisfying (*). Its morphisms are homomorphisms of $DR\ell$ -monoids which preserve also 1, thus in the sequel, bounded $DR\ell$ -monoids are regarded as algebras $(A, +, 0, \vee, \wedge, \rightarrow, \leftarrow, 1)$ of type $\langle 2, 0, 2, 2, 2, 2, 0 \rangle$.

Theorem 1.3. *The categories \mathcal{PBL} and $\mathcal{DR}\mathcal{L}_{1(*)}$ are equivalent.*

Proof. Theorems 1.1 and 1.2 enable us to define a functor $\mathcal{F}: \mathcal{PBL} \rightarrow \mathcal{DR}\mathcal{L}_{1(*)}$ as follows: (i) $\mathcal{F}(\mathfrak{A}) = \mathfrak{A}_d$ for any pseudo BL -algebra \mathfrak{A} , and (ii) $\mathcal{F}(h) = h$ for any pseudo BL -homomorphism h . It is easy to see that \mathcal{F} is really a categorical equivalence. □

2. FILTERS

According to [3], a subset F of a pseudo BL -algebra \mathfrak{A} with the following properties is said to be a *filter* of \mathfrak{A} :

- (F1) $1 \in F$;
- (F2) $\forall x, y \in F; x \odot y \in F$;
- (F3) $\forall x \in F \forall y \in A; x \leq y \implies y \in F$.

For any subset $M \subseteq A$, the intersection of all filters containing M is called a *filter generated by M* and denoted by $[M]$. It is clear that

$$[M] = \{x \in A; x \geq a_1 \odot \dots \odot a_n \text{ for some } a_1, \dots, a_n \in M \text{ and } n \geq 1\},$$

and if we write briefly $[a]$ for $[\{a\}]$ then

$$[a] = \{x \in A; x \geq a^n \text{ for some } n \geq 1\}.$$

In Section 1, we have already proved that $DR\ell$ -monoids include the duals of pseudo BL -algebras. It is obvious that $F \subseteq A$ is a filter of a pseudo BL -algebra \mathfrak{A} iff it is an ideal of the induced bounded $DR\ell$ -monoid \mathfrak{A}_d , that is,

- (I1) $0_d \in F$;
- (I2) $\forall x, y \in F; x + y \in F$;
- (I3) $\forall x \in F \forall y \in A; x \geq_d y \implies y \in F$.

Ideals of noncommutative $DR\ell$ -monoids were studied in [9]. Considering the above facts, we immediately obtain the following results.

Proposition 2.1. *The set of all filters of any pseudo BL -algebra \mathfrak{A} , ordered by set inclusion, is an algebraic Brouwerian lattice. For any filters F, G of \mathfrak{A} , the relative pseudocomplement of F with respect to G is given by*

$$F * G = \{a \in A; a \vee x \in G \text{ for all } x \in F\}.$$

Let \mathfrak{A} be a pseudo BL -algebra and $X \subseteq A$. The set

$$X^\perp = \{a \in A; a \vee x = 1 \text{ for any } x \in X\}$$

is called the *polar* of X . For any $x \in A$ we write x^\perp instead of $\{x\}^\perp$.

A subset X of A is a *polar in \mathfrak{A}* iff $X = Y^\perp$ for some $Y \subseteq A$.

Proposition 2.2 [3, Propositions 4.38, 4.39]. *For all subsets X, Y of a pseudo BL -algebra \mathfrak{A} , (i) X^\perp is a filter of \mathfrak{A} , (ii) $X \subseteq X^{\perp\perp}$, (iii) $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$, (iv) $X^\perp = X^{\perp\perp\perp}$.*

Proposition 2.3. For any subset X of a pseudo BL -algebra \mathfrak{A} , X is a polar in \mathfrak{A} iff $X = X^{\perp\perp}$.

Proof. Let $X = Y^\perp$; then $X^{\perp\perp} = Y^{\perp\perp\perp} = Y^\perp = X$. □

By Proposition 2.1, the pseudocomplement of a filter F is

$$F^* = \{a \in A; a \vee x = 1 \text{ for any } x \in F\}.$$

Moreover, it is clear that $F^\perp = F^*$ whenever F is a filter, and conversely, any polar is the pseudocomplement of some filter; in fact, $X = (X^\perp)^*$. Thus the polars in any pseudo BL -algebra are precisely the pseudocomplements in the lattice of its filters. Therefore, by the Glivenko-Frink Theorem, we directly obtain

Theorem 2.4. The set of all polars in any pseudo BL -algebra, ordered by set inclusion, is a complete Boolean algebra.

A filter F of a pseudo BL -algebra \mathfrak{A} is said to be *normal* iff it satisfies the following condition for each $x, y \in A$:

$$x \rightarrow y \in F \iff x \rightsquigarrow y \in F.$$

Proposition 2.5. For any filter F , the following conditions are equivalent:

- (i) F is normal;
- (ii) $x \odot F = F \odot x$ for each $x \in A$.

Proposition 2.6. If F and G are normal filters of \mathfrak{A} then

$$F \vee G = \{x \in A; x \geq a \odot b \text{ for some } a \in F, b \in G\}.$$

In addition, $F \vee G$ is a normal filter. Consequently, normal filters of any pseudo BL -algebra form a complete sublattice of the lattice of all its filters.

Theorem 2.7. In any pseudo BL -algebra, there is a one-to-one correspondence between the normal filters and the congruence relations. In fact, F corresponds to $\Theta(F)$ defined by

$$\langle x, y \rangle \in \Theta(F) = \Theta_1(F) \iff (x \rightarrow y) \wedge (y \rightarrow x) \in F,$$

or equivalently,

$$\langle x, y \rangle \in \Theta(F) = \Theta_2(F) \iff (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F.$$

As proved in [3], and in general for noncommutative $DR\ell$ -monoids in [9], if F is not a normal filter then the binary relations defined in the previous theorem, $\Theta_1(F)$ and $\Theta_2(F)$, are two distinct congruence relations on the distributive lattice $\mathfrak{L}(\mathfrak{A}) = (A, \vee, \wedge, 0, 1)$. In the quotient lattices $\mathfrak{L}(\mathfrak{A})/\Theta_1(F)$ and $\mathfrak{L}(\mathfrak{A})/\Theta_2(F)$ we have

$$(2.1) \quad [x]\Theta_1(F) \leq [y]\Theta_1(F) \iff x \rightarrow y \in F$$

and

$$(2.2) \quad [x]\Theta_2(F) \leq [y]\Theta_2(F) \iff x \rightsquigarrow y \in F,$$

respectively.

Let \mathfrak{A} be a pseudo BL -algebra. A filter F of \mathfrak{A} is said to be *prime* if it is a finitely meet-irreducible element in the lattice of filters of \mathfrak{A} .

By [3, Theorem 4.28], for any filter F of a pseudo BL -algebra \mathfrak{A} and for each ideal I of the lattice $\mathfrak{L}(\mathfrak{A})$, if $F \cap I = \emptyset$ then there exists a prime filter P of \mathfrak{A} with $F \subseteq P$ and $P \cap I = \emptyset$. Consequently, every proper filter is the intersection of all prime filters including it. In particular, the intersection of all prime filters is equal to $\{1\}$.

Theorem 2.8. *For any filter F of a pseudo BL -algebra \mathfrak{A} , the following conditions are equivalent:*

- (i) F is prime;
- (ii) for all filters G, H of \mathfrak{A} , $G \cap H \subseteq F$ implies $G \subseteq F$ or $H \subseteq F$;
- (iii) for any $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$;
- (iv) for any $x, y \in A$, $x \vee y = 1$ implies $x \in F$ or $y \in F$;
- (v) for any $x, y \in A$, $x \rightarrow y \in F$ or $y \rightarrow x \in F$;
- (vi) for any $x, y \in A$, $x \rightsquigarrow y \in F$ or $y \rightsquigarrow x \in F$;
- (vii) $\mathfrak{L}(\mathfrak{A})/\Theta_1(F)$ is totally ordered;
- (viii) $\mathfrak{L}(\mathfrak{A})/\Theta_2(F)$ is totally ordered;
- (ix) the set of all filters including F is totally ordered under set inclusion.

Remark. The equivalence of (iii), (v), (vi), (vii) and (viii) is due to [3, Proposition 4.25].

Proof. (i) \Rightarrow (ii): Using the distributivity of the lattice of filters, $G \cap H \subseteq F$ implies $F = F \vee (G \cap H) = (F \vee G) \cap (F \vee H)$, whence $F = F \vee G$ or $F = F \vee H$, that is, $F \supseteq G$ or $F \supseteq H$.

(ii) \Rightarrow (iii): Obviously, $x \vee y \in F$ yields $[x] \cap [y] = [x \vee y] \subseteq F$. Hence, by (ii), $[x] \subseteq F$ or $[y] \subseteq F$ and thus $x \in F$ or $y \in F$.

(iii) \Rightarrow (iv): This is evident since $1 \in F$.

(iv) \Rightarrow (v) and (iv) \Rightarrow (vi): By the definition of a pseudo BL -algebra,

$$(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1,$$

which implies the assertion by (iv).

(v) \Rightarrow (vii) and (vi) \Rightarrow (viii): This is obvious from (2.1) and (2.2), respectively.

(vii) \Rightarrow (ix): If $F \subseteq G, H$ and neither $G \subseteq H$ nor $H \subseteq G$ then there exist $a, b \in A$ with $a \in G \setminus H$ and $b \in H \setminus G$. For instance, let $a \rightarrow b \in F$. Then $b \geq a \wedge b = (a \rightarrow b) \odot a \in G$, whence $b \in G$; a contradiction. Similarly (viii) \Rightarrow (ix).

(ix) \Rightarrow (i): $F = G \cap H$ entails $F = G$ or $F = H$, because either $G \subseteq H$ or $H \subseteq G$. \square

3. REPRESENTABLE PSEUDO BL -ALGEBRAS

Proposition 3.1. *If P is a minimal prime filter of a pseudo BL -algebra \mathfrak{A} then $A \setminus P$ is a maximal ideal of the lattice $\mathfrak{L}(\mathfrak{A})$.*

Proof. By Zorn's Lemma, there is a maximal ideal I of $\mathfrak{L}(\mathfrak{A})$ with $A \setminus P \subseteq I$. (Since P is also a prime filter of $\mathfrak{L}(\mathfrak{A})$, it follows that $A \setminus P$ is a prime ideal of $\mathfrak{L}(\mathfrak{A})$ which is included in some maximal (prime) ideal.) We will show that $I = A \setminus P$. Denote $Q = \bigcup \{a^\perp; a \in I\}$. We claim that $P = Q$.

If $x \in a^\perp$ for some $a \in I$, then $x \vee a = 1$ and $x \notin I$. Indeed, if $x \in I$ then $x \vee a \neq 1$ since $x \vee a = 1$ would mean $I = A$. Thus $x \in A \setminus I \subseteq A \setminus (A \setminus P) = P$, whence $a^\perp \subseteq A \setminus I \subseteq P$ and consequently, $Q \subseteq A \setminus I \subseteq P$.

We shall now prove that Q is a prime filter of \mathfrak{A} . (F1): Since any principal polar a^\perp contains 1, so does Q . (F2): If $x, y \in Q$, that is, $x \in a^\perp, y \in b^\perp$ for some $a, b \in I$, then $a \vee b \in I$ and

$$(x \odot y) \vee a \vee b \geq (x \vee a \vee b) \odot (y \vee a \vee b) = 1 \odot 1 = 1.$$

Therefore $x \odot y \in (a \vee b)^\perp \subseteq Q$. (F3): It is obvious since a^\perp is a filter of \mathfrak{A} for each $a \in I$.

To prove that Q is prime, suppose $x \vee y = 1$ and $x \notin Q$, that is, $x \vee a \neq 1$ for all $a \in I$. If $x \notin I$ then the ideal in the lattice $\mathfrak{L}(\mathfrak{A})$ generated by $I \cup \{x\}$, $(I \cup \{x\}]$, is proper, i.e., $A \setminus P \subseteq I \subset (I \cup \{x\}] \neq A$, since $(I \cup \{x\}] = A$ would entail $1 \leq x \vee a$ for some $a \in I$; a contradiction. Hence $x \in I$ and thus $y \in x^\perp \subseteq Q$, proving that Q is prime.

However, P is a minimal prime filter of \mathfrak{A} ; thus $Q \subseteq A \setminus I \subseteq P$ yields $Q = A \setminus I = P$ as claimed. Therefore $I = A \setminus P$. \square

Corollary 3.2. *If P is a minimal prime filter then*

$$P = \bigcup \{a^\perp; a \notin P\}.$$

Proof. By the proof of the previous proposition, $P = \bigcup \{a^\perp; a \in I\}$, where $I = A \setminus P$. \square

A pseudo BL -algebra is said to be *representable* if it is a subdirect product of linearly ordered pseudo BL -algebras.

By Theorems 2.7 and 2.8, subdirect representations by totally ordered pseudo BL -algebras are associated with families of normal prime filters whose intersections are precisely $\{1\}$. Therefore it is obvious that every BL -algebra is representable (see also [11]). In contrast, for pseudo BL -algebras, this assertion fails.

The following results generalize the similar properties of pseudo MV -algebras, [4, Theorem 2.20], [1, Theorem 5.9], and [2, Theorem 6.11].

Theorem 3.3. *For any pseudo BL -algebra \mathfrak{A} , the following statements are equivalent.*

- (i) \mathfrak{A} is representable.
- (ii) There exists a family $\{P_i\}_{i \in I}$ of normal prime filters of \mathfrak{A} such that

$$\bigcap_{i \in I} P_i = \{1\}.$$

- (iii) Any polar of \mathfrak{A} is a normal filter of \mathfrak{A} .
- (iv) Any principal polar is a normal filter.
- (v) Any minimal prime filter is normal.

Proof. As argued above, the equivalence of (i) and (ii) is clear.

(i) \Rightarrow (iii): Suppose that \mathfrak{A} is a subdirect product of linearly ordered pseudo BL -algebras $\{\mathfrak{A}_i\}_{i \in I}$. Observe that

$$(3.1) \quad x \vee y = 1 \text{ iff } \{i \in I; x_i \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} = \emptyset$$

for all $x, y \in A$, since \mathfrak{A}_i are totally ordered.

Let now P be a polar in \mathfrak{A} , i.e. $P = P^{\perp\perp}$. Let $x \in A, a \in P$ and $y \in P^\perp$. Then $x \odot a \leq x$ implies $x \odot a = (x \odot a) \wedge x = (x \rightarrow (x \odot a)) \odot x$. Further, $\{i \in I; x_i \rightarrow (x_i \odot a_i) \neq 1_i\} \subseteq \{i \in I; a_i \neq 1_i\}$. Indeed, if $a_i = 1_i$ then $x_i \rightarrow (x_i \odot a_i) = x_i \rightarrow (x_i \odot 1_i) = x_i \rightarrow x_i = 1_i$. Hence we obtain

$$\{i \in I; x_i \rightarrow (x_i \odot a_i) \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} \subseteq \{i \in I; a_i \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} = \emptyset$$

by (3.1), since $a \in P$ and $y \in P^\perp$. Therefore $(x \rightarrow (x \odot a)) \vee y = 1$, and thus $x \rightarrow (x \odot a) \in P^{\perp\perp} = P$. Hence $x \odot a = (x \rightarrow (x \odot a)) \odot x \in P \odot x$, proving $x \odot P \subseteq P \odot x$.

(iii) \Rightarrow (iv): Obvious.

(iv) \Rightarrow (v): By Corollary 3.2, $P = \bigcup\{a^\perp; a \notin P\}$ for any minimal prime filter P . If $x \rightarrow y \in P$ then there is $a \notin P$ with $x \rightarrow y \in a^\perp$ which is a normal filter, and hence $x \rightsquigarrow y \in a^\perp \subseteq P$. Summarizing, $x \rightarrow y \in P$ iff $x \rightsquigarrow y \in P$.

(v) \Rightarrow (i): Since any prime filter contains a minimal prime filter and the intersection of all prime filters of \mathfrak{A} is obviously $\{1\}$, so does the intersection of minimal prime filters. Thus, by (ii), \mathfrak{A} is representable. \square

Theorem 3.4. *A pseudo BL-algebra is representable if and only if it satisfies the identities*

$$(3.2) \quad (y \rightarrow x) \vee (z \rightsquigarrow ((x \rightarrow y) \odot z)) = 1,$$

$$(3.3) \quad (y \rightsquigarrow x) \vee (z \rightarrow (z \odot (x \rightsquigarrow y))) = 1.$$

Consequently, the class of representable pseudo BL-algebras is a variety.

Proof. In any linearly ordered pseudo BL-algebra \mathfrak{A} , either $y \rightarrow x = 1$ or $x \rightarrow y = 1$ (and also $y \rightsquigarrow x = 1$ or $x \rightsquigarrow y = 1$), and so it is easy to verify that \mathfrak{A} fulfils (3.2) and (3.3). Therefore the part “only if” is obvious.

Conversely, suppose that (3.2) and (3.3) are satisfied by \mathfrak{A} . In view of Theorem 3.3 (iv), it suffices to prove that any principal polar x^\perp is a normal filter of \mathfrak{A} .

Let $y \in x^\perp$, that is, $y \vee x = 1$. Observe that in this case

$$x = 1 \rightarrow x = (y \vee x) \rightarrow x = (y \rightarrow x) \wedge (x \rightarrow x) = (y \rightarrow x) \wedge 1 = y \rightarrow x$$

and similarly $y = x \rightarrow y$. Hence, by (3.2),

$$x \vee (z \rightsquigarrow (y \odot z)) = (y \rightarrow x) \vee (z \rightsquigarrow ((x \rightarrow y) \odot z)) = 1,$$

thus $z \rightsquigarrow (y \odot z) \in x^\perp$. Further, $y \odot z \leq z$ implies $y \odot z = (y \odot z) \wedge z = z \odot (z \rightsquigarrow (y \odot z)) \in z \odot x^\perp$, which shows $x^\perp \odot z \subseteq z \odot x^\perp$. The other inclusion follows similarly by (3.3). \square

References

- [1] *A. Dvurečenskij*: On pseudo MV -algebras. *Soft Computing* 5 (2001), 347–354.
- [2] *A. Dvurečenskij*: States on pseudo MV -algebras. *Studia Logica* 68 (2001), 301–327.
- [3] *A. Di Nola, G. Georgescu, A. Iorgulescu*: Pseudo BL -algebras: Part I. Preprint.
- [4] *G. Georgescu, A. Iorgulescu*: Pseudo MV -algebras. *Mult. Val. Logic* 6 (2001), 95–135.
- [5] *G. Grätzer*: General Lattice Theory. Birkhäuser, Berlin, 1998.
- [6] *P. Hájek*: Basic fuzzy logic and BL -algebras. *Soft Computing* 2 (1998), 124–128.
- [7] *P. Hájek*: Metamathematics of Fuzzy Logic. Kluwer, Amsterdam, 1998.
- [8] *T. Kovář*: A general theory of dually residuated lattice ordered monoids. Ph.D. thesis, Palacký Univ., Olomouc, 1996.
- [9] *J. Kühr*: Ideals of noncommutative $DR\ell$ -monoids. Manuscript.
- [10] *J. Rachůnek*: A non-commutative generalization of MV -algebras. *Czechoslovak Math. J.* 52 (2002), 255–273.
- [11] *J. Rachůnek*: A duality between algebras of basic logic and bounded representable $DR\ell$ -monoids. *Math. Bohem.* 126 (2001), 561–569.
- [12] *K. L. N. Swamy*: Dually residuated lattice ordered semigroups. *Math. Ann.* 159 (1965), 105–114.

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