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ORDER CONVERGENCE OF VECTOR MEASURES ON
TOPOLOGICAL SPACES

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Abstract. Let X be a completely regular Hausdorff space, E a boundedly complete vector lattice, $C_b(X)$ the space of all, bounded, real-valued continuous functions on X , \mathcal{F} the algebra generated by the zero-sets of X , and $\mu: C_b(X) \rightarrow E$ a positive linear map. First we give a new proof that μ extends to a unique, finitely additive measure $\mu: \mathcal{F} \rightarrow E^+$ such that μ is inner regular by zero-sets and outer regular by cozero sets. Then some order-convergence theorems about nets of E^+ -valued finitely additive measures on \mathcal{F} are proved, which extend some known results. Also, under certain conditions, the well-known Alexandrov's theorem about the convergent sequences of σ -additive measures is extended to the case of order convergence.

Keywords: order convergence, tight and τ -smooth lattice-valued vector measures, measure representation of positive linear operators, Alexandrov's theorem

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1. INTRODUCTION AND NOTATION

All vector spaces are taken over reals. E , in this paper, is always assumed to be a boundedly complete vector lattice (and so, necessarily Archimedean) ([??], [??], [??]). If E is a locally convex space and E' its topological dual, then $\langle \cdot, \cdot \rangle: E \times E' \rightarrow \mathbb{R}$ will stand for the bilinear mapping $\langle x, f \rangle = f(x)$. For a completely regular Hausdorff space X , $\mathcal{B}(X)$ and $\mathcal{B}_1(X)$ are the classes of Borel and Baire subsets of X , $C(X)$ and $C_b(X)$ are the spaces of all real-valued, and real-valued and bounded, continuous functions on X and \tilde{X} is the Stone-Ćech compactification of X respectively. For an $f \in C_b(X)$, \tilde{f} is its unique continuous extension to \tilde{X} . The sets $\{f^{-1}(0): f \in C_b(X)\}$ are called the zero-sets of X and their complements the cozero sets of X .

For a compact Hausdorff space X and a boundedly complete vector lattice G , let $\mu: \mathcal{B}(X) \rightarrow G^+$ be a countably additive (countable additivity in the order convergence

of G) Borel measure; then μ is said to be quasi-regular if for any open $V \subset X$, $\mu(V) = \sup\{\mu(C): C \text{ compact, } C \subset V\}$. Integration with respect to these measures is taken in the sense of ([??], [??]). There is a 1-1 correspondence between these quasi-regular, positive, G -valued Borel measures on X and positive linear mappings $\mu: C(X) \rightarrow G$ ([??], [??], [??]); $M_{(o)}^+(X, G)$ will denote the set of all these measures.

Now suppose that X is a completely regular Hausdorff space. A positive countably additive Borel measure $\mu: \mathcal{B}(X) \rightarrow G^+$ is said to be tight if for any open $V \subset X$, $\mu(V) = \sup\{\mu(C): C \text{ compact, } C \subset V\}$ ([??], p. 207). This measure gives a positive linear mapping $\tilde{\mu}: C(\tilde{X}) \rightarrow G$, $\tilde{\mu}(f) = \mu(f|_X)$ and the corresponding quasi-regular, positive, G -valued Borel measure on \tilde{X} is given by $\tilde{\mu}(B) = \mu(X \cap B)$ for every Borel $B \subset \tilde{X}$. $M_{(o,t)}^+(X, G)$ will denote the set of all tight measures.

If $\mu: \mathcal{B}(X) \rightarrow G^+$ is a countably additive Borel measure, then μ is said to be τ -smooth if for any increasing net $\{U_\alpha\}$ of open subsets of X , $\mu(\bigcup U_\alpha) = \sup \mu(U_\alpha)$ (some properties of these measures are given in [??], p. 207). Any such measure gives a positive linear mapping $\tilde{\mu}: C(\tilde{X}) \rightarrow G$, $\tilde{\mu}(f) = \mu(f|_X)$ and the corresponding quasi-regular, positive, G -valued Borel measure on \tilde{X} is given by $\tilde{\mu}(B) = \mu(X \cap B)$ for every Borel set $B \subset \tilde{X}$. $M_{(o,\tau)}^+(X, G)$ will denote the set of all τ -smooth measures. If $\mu: \mathcal{B}_1(X) \rightarrow G^+$ is a countably additive Baire measure then, as in the case of τ -smooth measure, we get $\tilde{\mu}: C(\tilde{X}) \rightarrow G$, $\tilde{\mu}(f) = \mu(f|_X)$; $M_{(o,\sigma)}^+(X, G)$ will denote the set of all these Baire measures.

In ([??], [??]) some interesting results are derived about the weak order convergence of nets of positive lattice-valued measures. In this paper we extend some of those results to a more general setting. Before doing that we need Alexandrov's Theorem.

2. ALEXANDROV'S THEOREM

Suppose X is a completely regular Hausdorff space and \mathcal{F} is the algebra generated by the zero-sets of X . Assume that $\mu: C_b(X) \rightarrow \mathbb{R}$ is a positive linear mapping. By well-known Alexandrov's Theorem, there exists a unique finitely additive measure ν , $\nu: \mathcal{F} \rightarrow \mathbb{R}^+$, such that

(i) ν is inner regular by zero-sets and outer regular by cozero sets; (ii) $\int f d\nu = \mu(f)$ for all $f \in C_b(X)$ ([??], Theorem 5, p. 165; [??]) (Note that $C_b(X)$ is contained in the uniform closure of \mathcal{F} -simple functions on X in the space of all bounded functions on X and so each $f \in C_b(X)$ is ν -integrable; ν is also generally denoted by μ .) This theorem has been extended to the vector case (e.g. [??], p. 353). The proof is quite sophisticated. We give a quite different proof based on the regularity properties of the corresponding quasi-regular Borel measure on \tilde{X} ; this also provides a relation

between this finitely additive measure and the corresponding quasi-regular Borel measure on \tilde{X} . We start with a lemma.

Lemma 1. *If Z_1 and Z_2 are zero-sets in X then $\overline{(Z_1 \cap Z_2)} = \overline{Z_1} \cap \overline{Z_2}$ (for a subset $A \subset X$, \bar{A} denotes the closure of A in \tilde{X}).*

Proof. Suppose this is not true. Take a point $a \in \overline{Z_1} \cap \overline{Z_2} \setminus \overline{(Z_1 \cap Z_2)}$ (note that $Z_1 \cap Z_2$ can be empty). Take an $f \in C_b(X)$, $0 \leq f \leq 1$, such that $\tilde{f}(a) = 1$ and $f = 0$ on $(Z_1 \cap Z_2)$. For $i = 1, 2$, take $h_i \in C_b(X)$ such that $0 \leq h_i \leq 1$ and $Z_i = h_i^{-1}(0)$. Define $f_i(x) = f(x)h_i(x)/(h_1(x) + h_2(x))$ for $x \notin (Z_1 \cap Z_2)$ and 0 otherwise. These functions are continuous and $f = f_1 + f_2$. Thus $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. Since $f_i = 0$ on Z_i , $\tilde{f}_i = 0$ on $\overline{Z_i}$ and so $\tilde{f}_1 + \tilde{f}_2 = 0$ on $\overline{Z_1} \cap \overline{Z_2}$. This means $\tilde{f}(a) = 0$, a contradiction. \square

Now we come to the main theorem.

Theorem 2 ([??], p. 353). *Suppose X is a completely regular Hausdorff space, E is a boundedly order-complete vector lattice and $\mu: C_b(X) \rightarrow E$ is a positive linear mapping. Then there exist a unique finitely additive measure $\nu: \mathcal{F} \rightarrow E^+$ such that, in terms of order convergence,*

- (i) ν is inner regular by zero-sets and outer regular by cozero sets;
- (ii) $\int f d\nu = \mu(f)$ for all $f \in C_b(X)$;
- (iii) For any zero-set $Z \subset X$ we have $\nu(Z) = \tilde{\mu}(\overline{Z})$, \overline{Z} being the closure of Z in \tilde{X} .

Proof. There is no loss of generality if we assume that $E = C(S)$, S being a Stonian space. The given mapping gives a positive linear mapping $\tilde{\mu}: C(\tilde{X}) \rightarrow E$; by ([??], [??]) we get an E -valued, positive quasi-regular Borel measure $\tilde{\mu}: \mathcal{B}(\tilde{X}) \rightarrow E^+$. If A is a subset of X or \tilde{X} , \bar{A} will denote the closure of A in \tilde{X} . We prove this theorem in several steps.

I. Let $\overline{\mathcal{Z}} = \{\bar{A}: A \text{ a zero-set in } X\}$. Then for every $Q \in \overline{\mathcal{Z}}$, $\inf\{\tilde{\mu}((\tilde{X} \setminus Q) \setminus W): W \in \overline{\mathcal{Z}}\} = 0$.

Proof. Using the quasi-regularity of $\tilde{\mu}$, take an increasing net $\{C_\alpha\}$ of compact subsets of $(\tilde{X} \setminus Q)$ such that $\inf(\tilde{\mu}((\tilde{X} \setminus Q) \setminus C_\alpha)) = 0$. Fix α and take a $g \in C(\tilde{X})$, $0 \leq g \leq 1$, such that $g = 1$ on C_α and $g = 0$ outside $\tilde{X} \setminus Q$. Let $V = \{x \in \tilde{X}: g(x) > \frac{1}{2}\}$, $Z = \{x \in \tilde{X}: g(x) \geq \frac{1}{3}\}$. We have $Z \supset (\overline{Z \cap X}) \supset (\overline{V \cap X}) \supset V \supset C_\alpha$ (note that X is dense in \tilde{X}). Now $Z \cap X$ is a zero-set in X and taking $W = \overline{(Z \cap X)}$, we have $C_\alpha \subset W \subset (\tilde{X} \setminus Q)$. Since $\overline{\mathcal{Z}}$ is closed under finite unions, the result follows.

II. Let \mathcal{A} be the algebra in \tilde{X} generated by $\overline{\mathcal{Z}}$ and denote by \mathcal{A}_0 the elements of \mathcal{A} which have the property that these elements and their complements are inner regular by the elements of $\overline{\mathcal{Z}}$. Then $\mathcal{A}_0 = \mathcal{A}$.

Proof. We use I and Lemma 1 to prove it. By I, $\mathcal{A}_0 \supset \bar{\mathcal{Z}}$. By definition, \mathcal{A}_0 is closed under complements. Using Lemma 1, it is a routine verification that if A and B are in \mathcal{A}_0 then $A \cup B$ and $A \cap B$ are also in \mathcal{A}_0 . This proves the result.

III. Let \mathcal{F} be the algebra in X generated by zero-sets in X . Then it is a simple verification that $\mathcal{A} \cap X \supset \mathcal{F}$. Also, if $A \in \mathcal{A}$ and $A \cap X = \emptyset$, then $\tilde{\mu}(A) = 0$. To prove this, take any $\bar{Z} \in \bar{\mathcal{Z}}$, Z being a zero-set in X , such that $\bar{Z} \subset A$. This means Z is empty and so $\tilde{\mu}(A) = 0$. Now we can define a $\nu: \mathcal{F} \rightarrow E$, $\nu(B) = \tilde{\mu}(A)$, A being any element in \mathcal{A} with $B = A \cap X$; it is a trivial verification that ν is well-defined, finitely additive and it is inner regular by zero-sets in X and outer regular by positive-sets in X . We also have $\nu(Z) = \tilde{\mu}(\bar{Z})$ for any zero-set $Z \subset X$.

IV. For any $f \in C_b(X)$, $\mu(f) = \int f d\nu$.

Proof. Let \mathcal{M} be the vector space of all \mathcal{F} -simple functions on X . With the norm topology $\|\cdot\|$ on $C(S)$, the mapping $\tilde{\mu}: \mathcal{M} \rightarrow C(S)$, $f \rightarrow \int f d\mu$ is positive and continuous and $C_b(X)$ lies in the norm completion of \mathcal{M} ; this implies that every $f \in C_b(X)$ is ν -integrable. Put $\mu(1) = e \in C(S)$.

Take an $f \in C_b(X)$, $0 \leq f \leq 1$, and fix a large positive integer k . For i , $1 \leq i \leq k$, let $Z_i = f^{-1}[i/k, 1]$ and $W_i = \tilde{f}^{-1}[i/k, 1]$. On X we get $k^{-1} \sum_{i=1}^k \chi_{Z_i} \leq f \leq k^{-1} + k^{-1} \sum_{i=1}^k \chi_{Z_i}$. From this we get $0 \leq \nu(f) - k^{-1} \sum_{i=1}^k \nu(Z_i) \leq k^{-1}e$. On \tilde{X} we get $\tilde{f} \geq k^{-1} \sum_{i=1}^k \chi_{W_i} \geq k^{-1} \sum_{i=1}^k \chi_{\bar{Z}_i}$. Define $h: \tilde{X} \rightarrow \mathbb{R}^+$, $h(\tilde{x}) = k^{-1} + k^{-1} \sum_{i=1}^k \chi_{\bar{Z}_i}$. Then h is usc (upper semi-continuous). Take an $\tilde{x} \in \tilde{X}$ and a net $\{x_\alpha\} \subset X$ such that $x_\alpha \rightarrow \tilde{x}$. Now $\tilde{f}(\tilde{x}) = \lim f(x_\alpha) \leq \overline{\lim} h(x_\alpha) \leq h(\tilde{x})$ (note that h is usc). Thus $k^{-1} \sum_{i=1}^k \chi_{\bar{Z}_i} \leq \tilde{f} \leq k^{-1} + k^{-1} \sum_{i=1}^k \chi_{\bar{Z}_i}$. Integrating relative to $\tilde{\mu}$, we have $0 \leq \mu(f) - k^{-1} \sum_{i=1}^k \nu(Z_i) \leq k^{-1}e$ (note $\tilde{\mu}(\bar{Z}_i) = \nu(Z_i)$). Combining these results, we have $|\mu(f) - \nu(f)| \leq k^{-1}e$. Taking the limit over k , we get the result.

V. Uniqueness.

Proof. For $i = 1, 2$, let $\nu_i: \mathcal{F} \rightarrow E^+$ be two finitely additive regular (inner regular by zero-sets and outer regular by positive-sets in X) measures such that $\int f d\nu_1 = \int f d\nu_2$ for all $f \in C_b(X)$. Fix a zero-set $Z \subset X$ and take a decreasing net $\{U_\alpha\}$ of cozero sets in X such that $\nu_i(U_\alpha \setminus Z) \downarrow 0$ for $i = 1, 2$. For each α , take an $f_\alpha \in C_b(X)$ with $0 \leq f_\alpha \leq 1$, $f_\alpha = 1$ on Z , and $f_\alpha = 0$ outside U_α . For $i = 1, 2$, $\nu_i(U_\alpha) \geq \nu_i(f_\alpha) \geq \nu_i(Z)$. From this we get, since $\nu_1(f_\alpha) = \nu_2(f_\alpha)$, $\nu_1(U_\alpha) - \nu_2(Z) \geq 0 \geq \nu_1(Z) - \nu_2(U_\alpha)$. Taking limits we get $\nu_1(Z) = \nu_2(Z)$. By regularity, we have $\nu_1 = \nu_2$. This proves the result. \square

We denote by $M_{(o)}^+(X, E)$ the set of all finitely additive $\mu: \mathcal{F} \rightarrow E^+$ which are inner regular by zero-sets; they are just the positive linear operators $\mu: C_b(X) \rightarrow E^+$.

3. ORDER CONVERGENCE OF MEASURES

In this section we consider the order convergence of these measures. A net $\{\mu_\alpha\} \subset M_{(o)}^+(X, E)$ is said to order-converge weakly to a $\mu \in M_{(o)}^+(X, E)$ if $\mu_\alpha(f) \rightarrow \mu(f)$ in order-convergence for each $f \in C_b(X)$; this is equivalent to $\tilde{\mu}_\alpha(f) \rightarrow \tilde{\mu}(f)$ in order-convergence for each $f \in C(\tilde{X})$.

Theorem 3. *Suppose X is a Hausdorff completely regular space, E is a boundedly order-complete vector-lattice, $\{\mu_\alpha\}$ is a uniformly order-bounded net in $M_{(o)}^+(X, E)$ and $\mu \in M_{(o)}^+(X, E)$. Then, with order convergence, the following statements are equivalent:*

- (i) $\mu_\alpha \rightarrow \mu$, pointwise on $C_b(X)$;
- (ii) $\overline{\lim}_\alpha \mu_\alpha(Z) \leq \mu(Z)$ for every zero-set Z and $\mu_\alpha(X) \rightarrow \mu(X)$;
- (iii) $\underline{\lim}_\alpha \mu_\alpha(U) \geq \mu(U)$ for every positive-set U and $\mu_\alpha(X) \rightarrow \mu(X)$;

If μ is τ -smooth, then each of the above statements is also equivalent to

- (iv) $\mu_\alpha \rightarrow \mu$ pointwise on $C_{ub}(X)$, where $C_{ub}(X)$ is the set of all uniformly continuous functions on X relative to a uniformity \mathcal{U} on X which gives the original topology on X (if the uniformity \mathcal{U} comes from a single metric, then it is enough to assume that μ is σ -smooth).

Proof. The positive linear mappings $\mu: C_b(X) \rightarrow E$ and $\mu_\alpha: C_b(X) \rightarrow E$ give the positive linear mappings $\tilde{\mu}: C(\tilde{X}) \rightarrow E$ and $\tilde{\mu}_\alpha: C(\tilde{X}) \rightarrow E$. Since the net $\{\mu_\alpha\}$ is a uniformly order-bounded, we can assume that $\mu_\alpha(1) \leq p$ for all α , for some $p \in E$ ($p > 0$).

(ii) and (iii) are easily seen to be equivalent.

(i) implies (ii). Fix a zero-set $Z \subset X$ and let \bar{Z} be its closure in \tilde{X} . Take a decreasing net $\{\tilde{f}_\gamma\} \subset C(\tilde{X})$, $0 \leq \tilde{f}_\gamma \leq 1$ for every γ such that $\tilde{f}_\gamma \downarrow \chi_{\bar{Z}}$. This means that, for some $\eta_\gamma \downarrow 0$ in E we have $\mu(Z) = \tilde{\mu}(\bar{Z}) = \tilde{\mu}(\tilde{f}_\gamma) - \eta_\gamma \geq \tilde{\mu}(\tilde{f}_\gamma) - 2\eta_\gamma = \lim_\alpha \tilde{\mu}_\alpha(\tilde{f}_\gamma) - 2\eta_\gamma \geq \overline{\lim}_\alpha \mu_\alpha(Z) - 2\eta_\gamma$. Taking the limit over γ , we get the result.

(ii) implies (i). Take an $f \in C_b(X)$, $0 \leq f \leq 1$, and fix a large positive integer k . For i , $1 \leq i \leq k$, put $Z_i = f^{-1}[i/k, 1]$. We get $\sum_{i=1}^k \chi_{Z_i} \leq f \leq \sum_{i=1}^k \chi_{Z_i} + k^{-1}$. From this we get $\mu_\alpha(f) \leq \sum_{i=1}^k \mu_\alpha(Z_i) + k^{-1}p$. This means $\overline{\lim}_\alpha (\mu_\alpha(f)) \leq \overline{\lim}_\alpha \left(\sum_{i=1}^k \mu_\alpha(Z_i) \right) + k^{-1}p$. Using (ii), this gives $\overline{\lim}_\alpha (\mu_\alpha(f)) \leq \left(\sum_{i=1}^k \mu(Z_i) \right) + k^{-1}p$. From this it follows

that $\overline{\lim}_\alpha(\mu_\alpha(f)) \leq \mu(f) + k^{-1}p$. Taking the limit as $k \rightarrow \infty$, we get $\overline{\lim}_\alpha(\mu_\alpha(f)) \leq \mu(f)$. The same result holds for $1 - f$ also (note that $\mu_\alpha(X) \rightarrow \mu(X)$). Combining these two results, we get the desired implication.

(i) implies (iv) trivially.

(iv) implies (ii). Fix a zero-set $Z \subset X$ and take a decreasing net $\{f_\gamma\} \subset C_{ub}(X)$ such that $f_\gamma \downarrow \chi_Z$ (if the uniformity comes from a single metric then the net $\{f_\gamma\}$ can be taken to be a sequence). Since μ is τ -smooth, $\mu(Z) = \lim_\gamma \mu(f_\gamma)$ (in case the uniformity is metrizable, it is enough to assume μ to be σ -smooth). The rest of the proof is identical with that given above in ((i) implies (ii)). \square

Remark 4. This generalizes ([??], Theorem 7, p. 4).

Suppose X is a uniform space. An $H \subset C_{ub}(X)$ is called ueb if it is uniformly bounded and uniformly equicontinuous. Now we have the following theorem:

Theorem 5. *Suppose X is a topological space having a uniformity \mathcal{U} which gives the same topology on X , E is a boundedly order-complete vector-lattice and $\{\mu_\alpha\}$ is a uniformly order-bounded net in $M_o^+(X, E)$. Suppose there is a $\mu \in M_{(o,t)}^+(X, E)$ such that $\mu_\alpha \rightarrow \mu$ pointwise on $C_{ub}(X)$ and H is a ueb set in $C_{ub}(X)$. Then $\mu_\alpha \rightarrow \mu$ uniformly on H .*

Proof. Because $\{\mu_\alpha\}$ is uniformly order-bounded, we can take $E = C(S)$ for some Stonian compact Hausdorff space S and we can also assume that $\mu_\alpha(1) \leq e$ for every α , e being the unit function in $C(S)$. Also assume H to be absolutely convex and pointwise compact and $\|f\| \leq 1$ for all $f \in H$. Take a compact $K \subset X$. By the Arzelà-Ascoli theorem, $H|_K$ is norm compact in $C(K)$. Further $d(x, y) = \sup_{f \in H} |f(x) - f(y)|$ is a uniformly continuous pseudometric on X . Fix $c > 0$. Define $h: X \rightarrow \mathbb{R}$, $h(x) = d(x, K)$; then $h \in C_{ub}(X)$. This means $V = \{x: h(x) < c\}$ is a positive set, it is open in X , $V \supset K$, and for an $x \in V$ there is a $y \in K$ such that $d(x, y) < c$. By the Arzelà-Ascoli theorem, there is a finite subset $\{f_i: 1 \leq i \leq n\} \subset H$ such that $H = \bigcup_{i=1}^n H_i$ where $H_i = \{f \in H: \|f - f_i\|_K < c\}$. Now take an $x \in V$ and $f \in H_i$. There is a $y \in K$ such that $d(x, y) < c$. We get $|f(x) - f_i(x)| \leq |f(x) - f(y)| + |f(y) - f_i(y)| + |f_i(y) - f_i(x)| \leq 3c$. So $|f - f_i| \leq 3c$ on V . From the given hypothesis, $\mu_\alpha \rightarrow \mu$ uniformly on finite subsets of $C_{ub}(X)$. Thus there exists a net $\{\eta_\alpha\} \subset E$ such that $\eta_\alpha \downarrow 0$ and $|\mu_\alpha(f_i) - \mu(f_i)| \leq \eta_\alpha$ for $1 \leq i \leq n$. Fix i and take an $f \in H_i$. We have $|\int f d\mu_\alpha - \int f d\mu| \leq |\int (f - f_i) d\mu_\alpha - \int (f - f_i) d\mu| + |\int f_i d\mu_\alpha - \int f_i d\mu| \leq |\int_V (f - f_i) d\mu_\alpha| + |\int_{X \setminus V} (f - f_i) d\mu_\alpha| + \int_K (f - f_i) d\mu + \int_{X \setminus K} (f - f_i) d\mu + \eta_\alpha \leq 3ce + 2\mu_\alpha(X \setminus V) + 3ce + 2\mu(X \setminus K) + \eta_\alpha$. Since this is true for each i , $1 \leq i \leq n$, the above result holds for every $f \in H$. So we get

$\sup_{f \in H} |\int f d\mu_\alpha - \int f d\mu| \leq 6ce + 2\mu_\alpha(X \setminus V) + 2\mu(X \setminus K) + \eta_\alpha$. Taking limit superior, we get $\overline{\lim}_\alpha (\sup_{f \in H} |\int f d\mu_\alpha - \int f d\mu|) \leq 2\mu(X \setminus V) + 2\mu(X \setminus K) + 6ce$. Letting $c \downarrow 0$, we get $\overline{\lim}_\alpha (\sup_{f \in H} |\int f d\mu_\alpha - \int f d\mu|) \leq 4\mu(X \setminus K)$. Since $\mu \in M_{(o,t)}^+(X, E)$, the result follows. \square

Corollary 6. *Suppose X is a Hausdorff completely regular space, E is a boundedly order-complete vector-lattice and $\{\mu_\alpha\}$ is a uniformly order-bounded net in $M_o^+(X, E)$. Suppose there is a $\mu \in M_{(o,t)}^+(X, E)$ such that $\mu_\alpha \rightarrow \mu$ pointwise on $C_b(X)$ and H is a uniformly bounded and pointwise equicontinuous subset of $C_b(X)$. Then $\mu_\alpha \rightarrow \mu$ uniformly on H .*

Proof. Consider X to be a uniform space with uniformity determined by all continuous pseudo-metrics on X . In this uniformity, H is a ueb set and so the result follows from Theorem 5. \square

4. ALEXANDROV'S THEOREM FOR A σ -ADDITIVE CASE

In this case we take E to be a boundedly complete vector lattice and, E^* and E_n^* to be its order dual and order continuous dual. E_n^* is a band in E^* and we assume that E_n^* separates the points of E . Take a sequence $\{\mu_n\} \subset M_{(o,\sigma)}^+(X, E)$ and assume that, in order convergence, $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. If $E = \mathbb{R}$, the well-known Alexandrov's theorem says that $\mu \in M_\sigma^+(X)$ ([??], p.195); in ([??], Theorem 2, p.73), this result is extended to the case when E is a topological vector space. In the next theorem we extend the result to the case when E is a boundedly complete vector lattice.

Theorem 7. *Suppose X is a Hausdorff completely regular space, E is a boundedly order-complete vector lattice and E_n^* its order dual. Assume that E is weakly σ -distributive ([??]) and E_n^* separates the points of E . Let $\{\mu_n\} \subset M_{(o,\sigma)}^+(X, E)$ be a sequence such that, in order convergence, $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. Then the positive $\mu: C_b(X) \rightarrow E$ is generated by the E^+ -valued Baire measure on X .*

Proof. E_n^* is a band in E^* and so the order intervals of E_n^* are $\sigma(E_n^*, E)$ -compact and convex. Now the topology on E of uniform convergence on the order intervals of E_n^* is a locally convex topology for which lattice operations are continuous and so, in this topology, the positive cone E_+ of E is closed and convex. Since this topology is compatible with the duality $\langle E, E_n^* \rangle$, E_+ is also closed in $\sigma(E, E_n^*)$. Now

we consider E to be a locally convex space with the topology $\sigma(E, E_n^*)$. By given hypothesis, $\mu_n: C_b(X) \rightarrow E$ are countably additive measures (note that E_n^* is the order continuous dual of E) and $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. By ([??], Theorem 2, p. 73), if $g_m \downarrow 0$ in $C_b(X)$, then $\mu_n(g_m) \rightarrow 0$ uniformly in n . So we get $\mu(g_m) \rightarrow 0$ in E . We claim that in order convergence in E , $\mu(g_m) \rightarrow 0$. This will be proved if we prove that $\inf_m \mu(g_m) = 0$ (note that $\mu(g_m) \downarrow$). Let $\inf_m \mu(g_m) = a > 0$. Take a positive element $f \in E_n^*$ such that $f(a) > 0$ (note that E_n^* separates the points of E). This implies that $\lim \langle f, \mu(g_m) \rangle = f(a) > 0$. This contradicts $\mu(g_m) \rightarrow 0$ in $(E, \sigma(E, E_n^*))$ and so the claim is proved. We get a positive linear mapping $\tilde{\mu}: C(\tilde{X}) \rightarrow E$, $\tilde{\mu}(f) = \mu(f|_X)$. For any zero-set $Z \subset \tilde{X} \setminus X$, take a sequence $\{g_m\} \subset C(\tilde{X})$ and $g_m \downarrow \chi_Z$. This means $(g_m)|_X \downarrow 0$. By ([??]), $\tilde{\mu}$ can be considered a Baire measure on \tilde{X} and so $\tilde{\mu}(Z) = \lim \tilde{\mu}(g_m) = \mu((g_m)|_X) = 0$. Since E is weakly σ -distributive, $\tilde{\mu}$ is a regular Baire measure and so for any Baire set $B \subset \tilde{X} \setminus X$, $\tilde{\mu}(B) = 0$. It is a simple verification that the class of Baire subsets of X is equal to the class of Baire subsets of \tilde{X} intersected with X . Now for any Baire subset B_0 of X , take a Baire subset B of \tilde{X} such that $B_0 = B \cap X$; define $\mu(B_0) = \tilde{\mu}(B)$. It is a simple verification the μ is well-defined and $\mu \in M_{(\sigma, \sigma)}^+(X, E)$ ([??]). This proves the theorem. \square

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References

- [1] *Aliprantis, C. D., Burkinshaw, O.*: Positive Operators. Academic Press, 1985. zbl
- [2] *Diestel, J., Uhl, J. J.*: Vector Measures. Amer. Math. Soc. Surveys 15, Amer. Math. Soc., 1977. zbl
- [3] *Kawabe, Jun*: The Portmanteau theorem for Dedekind complete Riesz space-valued measures. Nonlinear Analysis and Convex Analysis. Yokohama Publ., Yokohama, 2004, pp. 149–158. zbl
- [4] *Kawabe, Jun*: Uniformity for weak order convergence of Riesz space-valued measures. Bull. Austral. Math. Soc. 71 (2005), 265–274. zbl
- [5] *Khurana, Surjit Singh*: Lattice-valued Borel measures. Rocky Mt. J. Math. 6 (1976), 377–382. zbl
- [6] *Khurana, Surjit Singh*: Lattice-valued Borel measures II. Trans. Amer. Math. Soc. 235 (1978), 205–211. zbl
- [7] *Khurana, Surjit Singh*: Topologies on spaces of continuous vector-valued functions. Trans Amer. Math. Soc. 241 (1978), 195–211. zbl
- [8] *Khurana, Surjit Singh*: A vector form of Alexanderov’s theorem. Math. Nachr. 135 (1988), 73–77. zbl
- [9] *Lipectki, Z.*: Riesz representation theorems for positive operators. Math. Nachr. 131 (1987), 351–356. zbl
- [10] *Schaeffer, H. H.*: Topological Vector Spaces. Springer, 1980. zbl
- [11] *Schaeffer, H. H.*: Banach Lattices and Positive Operators. Springer, 1974. zbl

- [12] *Wheeler, R. F.*: Survey of Baire measures and strict topologies. *Expo. Math.* 2 (1983), 97–190. zbl
- [13] *Varadarajan, V. S.*: Measures on topological spaces. *Amer. Math. Soc. Transl.* 48 (1965), 161–220. zbl
- [14] *Wright, J. D. M.*: Stone-algebra-valued measures and integrals. *Proc. Lond. Math. Soc.* 19 (1969), 107–122. zbl
- [15] *Wright, J. D. M.*: Vector lattice measures on locally compact spaces. *Math. Z.* 120 (1971), 193–203. zbl
- [16] *Wright, J. D. M.*: The measure extension problem for vector lattices. *Ann. Inst. Fourier (Grenoble)* 21 (1971), 65–85. zbl
- [17] *Wright, J. D. M.*: Measures with values in partially ordered vector spaces. *Proc. Lond. Math. Soc.* 25 (1972), 675–688. zbl
- [18] *Wright, J. D. M.*: An algebraic characterization of vector lattices with Borel regularity property. *J. Lond. Math. Soc.* 7 (1973), 277–285. zbl

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