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OSCILLATORY AND NONOSCILLATORY BEHAVIOUR OF SOLUTIONS OF DIFFERENCE EQUATIONS OF THE THIRD ORDER

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Abstract. In this paper, sufficient conditions are obtained for oscillation of all solutions of third order difference equations of the form

$$y_{n+3} + r_n y_{n+2} + q_n y_{n+1} + p_n y_n = 0, \quad n \geqslant 0.$$

These results are generalization of the results concerning difference equations with constant coefficients

$$y_{n+3} + r y_{n+2} + q y_{n+1} + p y_n = 0, \quad n \geqslant 0.$$

Oscillation, nonoscillation and disconjugacy of a certain class of linear third order difference equations are discussed with help of a class of linear second order difference equations.

Keywords: third order difference equation, oscillation, nonoscillation, disconjugacy, generalized zero

MSC 2000: 39A10, 39A12

1. Introduction

In [4], [5] an attempt was made to generalize the results concerning oscillation/nonoscillation of solutions of third order difference equations with constant coefficients of the form

$$y_{n+3} + r y_{n+2} + q y_{n+1} + p y_n = 0, \quad n \geqslant 0, \quad p \neq 0, \tag{1}$$

to third order difference equations with variable coefficients of the form

$$y_{n+3} + r_n y_{n+2} + q_n y_{n+1} + p_n y_n = 0, \quad n \geqslant 0, \tag{2}$$

where $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ are sequences of real numbers such that $p_n \neq 0$. In this paper we obtain some new results in this direction. A comparison is made of the present results with the results in [4], [6] in order to emphasize the significance of all the results. Equation (2) may be put in the form

$$\Delta(\alpha_n \Delta^2 y_n) + \beta_n \Delta y_{n+1} + \gamma_n y_{n+1} = 0, \tag{3}$$

where

$$\alpha_n = \frac{(-1)^n \alpha_0}{\prod\limits_{i=0}^{n-1} p_i}, \quad \beta_n = \frac{(-1)^n \alpha_0 (p_n - r_n - 2)}{\prod\limits_{i=0}^{n} p_i}, \quad \gamma_n = \frac{(-1)^{n+1} \alpha_0 (p_n + q_n + r_n + 1)}{\prod\limits_{i=0}^{n} p_i},$$

with $\alpha_0 \neq 0$. However, in the present situation it is convenient to handle (2) rather than (3). In Section 2 some results concerning Eq. (1) are presented as a motivating factor. Then these results are generalized to hold for Eq. (2) in Section 3. Further, known results for second order difference equations are used to predict the behaviour of solutions of a class of third order difference equations.

By a solution of Eq. (1)/(2) on $[0, \infty) = \{0, 1, 2, \ldots\}$ we mean a sequence $\{y_n\}$ of real numbers which satisfies (1)/(2) for $n \geqslant 0$. If y_0, y_1, y_2 are given, then y_n for $n \geqslant 3$ can be obtained from (1)/(2). A solution $\{y_n\}$ of Eq. (1)/(2) on $[0, \infty)$ is said to be nontrivial if for every integer $m > 0$ there exists an integer $n > m$ such that $y_n \neq 0$. In this work, by a solution of (1)/(2) we understand a nontrivial solution. A solution $\{y_n\}$ of Eq. (1)/(2) is said to be nonoscillatory if there exists an integer $M > 0$ such that either $y_n > 0$ or < 0 for all $n \geqslant M$; otherwise, $\{y_n\}$ is said to be oscillatory. Equation (1)/(2) is said to be oscillatory if all of its solutions are oscillatory.

2. Oscillatory behaviour of solutions of eq. (1)

In this section we discuss the oscillatory behaviour of solutions of Eq. (1) under different sign conditions on the coefficients. If y_0, y_1 and y_2 are known, then Eq. (1) can be solved explicitly. However, in order to study the oscillatory behaviour of its solutions, we proceed as follows:

Theorem 2.1. *Let $p > 0$ and $r < 0$.*

(i) *If $q \geqslant 0$ and $p > -\frac{4}{27} r^3$, then Eq. (1) is oscillatory.*

(ii) *If $q \leqslant 0$ and Eq. (1) is oscillatory, then $p > -\frac{4}{27} r^3$.*

Proof. (i) The characteristic equation of (1) is

$$\lambda^3 + r\lambda^2 + q\lambda + p = 0. \tag{4}$$

If $p > 0$, then (4) admits a root $\alpha < 0$. Further, $G^2 + 4H^3 > 0$ implies that (4) has two complex roots, where

$$G = p - \frac{qr}{3} + \frac{2r^3}{27} \quad \text{and} \quad H = \frac{1}{3}\Big(q - \frac{r^2}{3}\Big).$$

We may observe that $q \geqslant \frac{1}{3}r^2$ implies $G^2 + 4H^3 > 0$. Moreover, $q < \frac{1}{3}r^2$ implies that $-\frac{2}{3}pqr > \frac{8}{27}r^2q^2$ since $p > -\frac{4}{27}r^3$. Then

$$\begin{aligned} G^2 + 4H^3 &= p\Big(p + \frac{4}{27}r^3\Big) + \frac{4}{27}q^3 - \frac{2}{3}pqr - \frac{q^2r^2}{27} \\ &\geqslant p\Big(p + \frac{4}{27}r^3\Big) + \frac{4}{27}q^3 + \frac{7q^2r^2}{27} > 0. \end{aligned} \tag{5}$$

Hence in any case $G^2 + 4H^3 > 0$. Thus (4) has two complex roots. Consequently, all solutions of (1) are oscillatory. This completes the proof of the first part of the theorem.

(ii) On the contrary, let $p \leqslant -\frac{4}{27}r^3$. Then $G^2+4H^3 \leqslant 0$. Hence all roots of (4) are real. As $p > 0$, we conclude that (4) has a root $\alpha < 0$. Let β and γ be two other real roots. However, $\alpha\beta\gamma = -p$ implies that $\beta\gamma = -p/\alpha > 0$ and hence $\beta > 0$ and $\gamma > 0$ or $\beta < 0$ and $\gamma < 0$. On the other hand, $\alpha+\beta+\gamma = -r$ implies $\beta+\gamma = -r-\alpha > 0$. Thus $\beta > 0$ and $\gamma > 0$. Consequently, (1) admits two positive solutions $\{\beta^n\}$ and $\{\gamma^n\}$, a contradiction to the assumption that all solutions of (1) are oscillatory. This completes the proof of the second part of the theorem.

Corollary 2.2. *If $p > 0$ and $r < 0$, then all solutions of $y_{n+3} + ry_{n+2} + py_n = 0$ are oscillatory if and only if $p > -\frac{4}{27}r^3$.*

Theorem 2.3. *Let $p > 0$ and $q < 0$.*
(i) *If $r \geqslant 0$ and $p > 2\left(-\frac{1}{3}q\right)^{3/2}$, then (1) is oscillatory.*
(ii) *If $r \leqslant 0$ and (1) is oscillatory, then $p > 2\left(-\frac{1}{3}q\right)^{3/2}$.*

Proof. (i) It is enough to show that $G^2 + 4H^3 > 0$ because in this case (4) admits two complex roots. Further, $p > 0$ implies that (4) has a negative root. We consider two cases, viz., $2p \leqslant -\frac{1}{9}qr$ and $2p > -\frac{1}{9}qr$. Let $2p \leqslant -\frac{1}{9}qr$. Hence

$$\frac{4pr^3}{27} \geqslant \frac{4pr^2}{27} \times \frac{18p}{-q} = \frac{-8p^2r^2}{3q} > \frac{32}{81}r^2q^2,$$

since $p > 2\left(-\frac{1}{3}q\right)^{3/2}$ implies that $p^2 > -\frac{4}{27}q^3$. Consequently, from (5) we obtain

$$\begin{aligned} G^2 + 4H^3 &> p^2 + \frac{32r^2q^2}{81} + \frac{4}{27}q^3 - \frac{2}{3}pqr - \frac{q^2r^2}{27} = p^2 + \frac{29r^2q^2}{81} + \frac{4}{27}q^3 - \frac{2}{3}pqr \\ &> \frac{29r^2q^2}{81} - \frac{2}{3}pqr > 0. \end{aligned}$$

Let $2p > -\frac{1}{9}qr$. Then (5) implies

$$G^2 + 4H^3 > \frac{4pr^3}{27} - \frac{2}{3}pqr - \frac{q^2r^2}{27} \geqslant \frac{4pr^3}{27} \geqslant 0.$$

Thus the first part of the theorem is proved.

(ii) We claim that $p > 2\left(-\frac{1}{3}q\right)^{3/2}$. If not, then $p \leqslant 2\left(-\frac{1}{3}q\right)^{3/2}$. Hence $p^2 \leqslant -\frac{4}{27}q^3$. Thus (5) yields

$$G^2 + 4H^3 \leqslant \frac{4}{27}pr^3 - \frac{2}{3}pqr - \frac{1}{27}q^2r^2 \leqslant 0,$$

which implies that all solutions of (4) are real. Since $p > 0$, then (4) has a root $\alpha < 0$. If β and γ are other two real roots of (4), then $\alpha\beta\gamma = -p < 0$ implies that $\beta\gamma > 0$. But $\alpha + \beta + \gamma = -r$ implies that $\beta + \gamma > 0$ and hence $\beta > 0$ and $\gamma > 0$. Thus (1) admits two positive solutions, a contradiction which completes the proof of the second part of the theorem. □

Corollary 2.4. *If* $p > 0$ *and* $q < 0$, *then all solutions of* $y_{n+3} + qy_{n+1} + py_n = 0$ *are oscillatory if and only if* $p > 2\left(-\frac{1}{3}q\right)^{3/2}$.

3. Oscillatory behaviour of solutions of eq. (2)

In this section we obtain results similar to Theorems 2.1 and 2.3 for Eq. (2). We will show that all solutions of (2) oscillate under different sign conditions on the coefficient sequences.

Theorem 3.1. *Let* $p_n > 0$, $q_n \geqslant 0$ *and* $r_n < 0$. *If*

$$p > -\frac{4r^3}{27},$$

then (2) *is oscillatory, where* $p = \liminf_{n\to\infty} p_n$ *and* $r = \liminf_{n\to\infty} r_n > -\infty$.

P r o o f. Let $\{y_n\}$ be a nonoscillatory solution of (2). Hence $y_n > 0$ or $y_n < 0$ for $n \geqslant M$, where $M > 0$ is an integer. Without any loss of generality we can take $y_n > 0$ for $n \geqslant M$ because $\{-y_n\}$ is also a solution of (2). Setting $x_n = y_{n+1}/y_{n+2}$, $n \geqslant M$, and taking $\liminf_{n\to\infty} x_n = \mu$, we obtain from (2)

(6) $$y_{n+3} + r_ny_{n+2} + p_ny_n \leqslant 0, \quad n \geqslant M,$$

that is,

$$\frac{y_{n+3}}{y_{n+2}} + p_n\frac{y_n}{y_{n+2}} \leqslant -r_n,$$

that is,

(7)
$$x_{n+1}^{-1} \leqslant -r_n - p_n x_{n-1} x_n.$$

Hence $x_{n+1}^{-1} \leqslant -r_n$. Then $\limsup_{n\to\infty} x_{n+1}^{-1} \leqslant \limsup_{n\to\infty}(-r_n)$, that is, $\mu \geqslant -1/r > 0$. From (7) we obtain

$$\limsup_{n\to\infty} x_{n+1}^{-1} \leqslant \limsup_{n\to\infty}(-r_n - p_n x_{n-1} x_n),$$

that is,

$$(\liminf_{n\to\infty} x_{n+1})^{-1} \leqslant -\liminf_{n\to\infty}(r_n + p_n x_{n-1} x_n) \leqslant -(\liminf_{n\to\infty} r_n + \liminf_{n\to\infty}(p_n x_{n-1} x_n)),$$

that is, $\mu^{-1} \leqslant -(r + p\mu^2)$, that is,

(8)
$$p \leqslant \frac{-r\mu - 1}{\mu^3}.$$

From (6) we obtain

$$-r_n > p_n \frac{y_n}{y_{n+2}} = p_n x_{n-1} x_n.$$

Hence

$$\limsup_{n\to\infty}(-r_n) \geqslant \limsup_{n\to\infty}(p_n x_{n-1} x_n) \geqslant \liminf_{n\to\infty}(p_n x_{n-1} x_n),$$

that is,

$$-\liminf_{n\to\infty} r_n \geqslant \liminf_{n\to\infty} p_n \liminf_{n\to\infty} x_{n-1} \liminf_{n\to\infty} x_n,$$

that is, $-r \geqslant p\mu^2$, that is, $\mu < \infty$. If

$$f(\mu) = \frac{-r\mu - 1}{\mu^3},$$

then f attains its maximum at $\mu = -3/2r$ and

$$\mathrm{Max}\,\{f(\mu)\} = f\Big(-\frac{3}{2r}\Big) = -\frac{4r^3}{27}.$$

Hence (8) implies

$$p \leqslant -\frac{4r^3}{27},$$

a contradiction to the hypothesis. Thus the theorem is proved.

In [6], the following theorem is proved.

Theorem 3.2. *If* $p_n > 0$, $q_n \geqslant 0$, $r_n < 0$ *and*

$$\frac{p_{n+1}}{r_{n+1}r_{n-1}} > \frac{q_{n+1}}{r_{n+1}} + \frac{q_n}{r_{n-1}} - r_n$$

for large n, *then* (2) *is oscillatory.*

Example 1. Consider

$$y_{n+3} - 6y_{n+2} + 4y_{n+1} + 33y_n = 0, \quad n \geqslant 0. \tag{9}$$

Hence $r_n = r = -6 < 0$, $q_n = q = 4 > 0$, $p_n = p = 33 > 0$. Since $-\frac{4}{27}r^3 = 32 < p$, all solutions of (9) are oscillatory by Theorem 3.1. The characteristic equation $\lambda^3 - 6\lambda^2 + 4\lambda + 33 = 0$ of (9) has a negative root, that is, $\lambda = a < 0$ because $33 > 0$. Hence $\{a^n\}$ is an oscillatory solution of (9). We may observe that Theorem 3.2 fails to hold for (9) because

$$\frac{p_{n+1}}{r_{n+1}r_{n-1}} = \frac{33}{36} < 1 \quad \text{and} \quad \frac{q_{n+1}}{r_{n+1}} + \frac{q_n}{r_{n-1}} - r_n = -\frac{8}{6} + 6 = \frac{28}{6} > 1.$$

On the other hand, all solutions of

$$y_{n+3} - 6y_{n+2} + 20y_{n+1} + 31y_n = 0, \quad n \geqslant 0, \tag{10}$$

are oscillatory by Theorem 3.2 but Theorem 3.1 fails to hold for (10) because $p = 31 < 32 = -\frac{4}{27}r^3$. Clearly, $p > -\frac{4}{27}r^3$ is not a necessary condition for Eq. (10) to be oscillatory. However, there are equations with $p_n > 0$, $q_n \geqslant 0$ and $r_n < 0$ which admit nonoscillatory solutions for which $p \leqslant -\frac{4}{27}r^3$ is satisfied. Consider (2) with $r_n = -\frac{3}{2} - 1/n$, $p_n = \frac{1}{3} + 1/2n$, $n \geqslant 1$ and $q_n = -r_n - p_n - 1 = \frac{1}{6} + 1/2n$. Then $\liminf_{n\to\infty} r_n = -\frac{3}{2}$ and $\liminf_{n\to\infty} p_n = \frac{1}{3}$. Clearly, $p \leqslant -\frac{4}{27}r^3$ holds and

$$y_{n+3} + r_n y_{n+2} + (-r_n - p_n - 1)y_{n+1} + p_n y_n = 0$$

admits a nonoscillatory solution $y_n = c \neq 0$, a constant.

In the next theorems the sign of p_n remains the same as in Theorems 3.1 and 3.2 but q_n and r_n interchange their signs.

Theorem 3.3. *Suppose that $p_n > 0$, $q_n < 0$ and $r_n \geqslant 0$. If*

$$p > 2\left(\frac{-q}{3}\right)^{3/2},$$

then Eq. (2) *is oscillatory, where $p = \liminf_{n\to\infty} p_n > 0$ and $q = \liminf_{n\to\infty} q_n > -\infty$.*

P r o o f. On the contrary, let $\{y_n\}$ be a nonoscillatory solution of (2). Hence, without any loss of generality, we can assume that $y_n > 0$ for $n \geqslant M > 0$. From (2) we obtain

$$y_{n+3} + q_n y_{n+1} + p_n y_n \leqslant 0, \quad n \geqslant M.$$

Setting $x_n = y_n/y_{n+1}$ and assuming $\liminf_{n\to\infty} x_n = \mu$, we get

$$\frac{y_{n+3}}{y_{n+1}} + p_n \frac{y_n}{y_{n+1}} \leqslant -q_n \tag{11}$$

that is,

$$x_{n+2}^{-1} x_{n+1}^{-1} \leqslant -q_n - p_n x_n.$$

Hence

$$\limsup_{n\to\infty} (x_{n+1}x_{n+2})^{-1} \leqslant \limsup_{n\to\infty}(-q_n - p_n x_n) \leqslant -\liminf_{n\to\infty}(q_n + p_n x_n),$$

that is,

$$(\liminf_{n\to\infty}(x_{n+1}x_{n+2}))^{-1} \leqslant -\liminf_{n\to\infty} q_n - \liminf_{n\to\infty}(p_n x_n),$$

that is,

$$\frac{1}{\mu^2} \leqslant -q - p\mu.$$

We may observe that $\mu \to 0$ implies $q \to -\infty$, a contradiction. Hence $\mu > 0$. Thus

$$p \leqslant \frac{-q\mu^2 - 1}{\mu^3}. \tag{12}$$

From (11) we obtain $p_n x_n \leqslant -q_n$. Hence

$$\liminf_{n\to\infty}(p_n x_n) \leqslant \liminf_{n\to\infty}(-\,q_n) \leqslant \limsup_{n\to\infty}(-\,q_n) = -\liminf_{n\to\infty} q_n,$$

that is, $\mu \leqslant -q/p < \infty$. If we set $f(\mu) = -(q\mu^2 + 1)/\mu^3$, then it attains its maximum at $\mu = (-3/q)^{1/2}$ and hence the maximum value of f is given by $f((-3/q)^{1/2}) = \frac{2}{3\sqrt{3}}(-q)^{3/2}$. From (12) we get

$$p \leqslant \frac{2}{3\sqrt{3}}(-q)^{3/2},$$

a contradiction. Hence the theorem is proved.

Now we state some results from [4].

Theorem 3.4 (Theorem 2.7, [4]). *If* $p_n > 0$, $q_n < 0$, $r_n > 0$ *and*

$$p - \frac{qr}{3} + \frac{2r^3}{27} - \frac{2}{3\sqrt{3}}\Big(\frac{r^2}{3} - q\Big)^{3/2} > 0,$$

then (2) *is oscillatory, where* $\liminf\limits_{n\to\infty} p_n = p > 0$, $\liminf\limits_{n\to\infty} q_n = q < 0$ *and* $\liminf\limits_{n\to\infty} r_n = r > 0$.

Theorem 3.5 (Theorem 2.11, [4]). *If* $p_n \geqslant 0$, $q_n < 0$, $r_n \geqslant 0$ *and*

$$q_n > \frac{r_n p_{n+1}}{q_{n+1}} + \frac{p_n r_{n-1}}{q_{n-1}}$$

for large n, *then* (2) *is oscillatory.*

Example 2. Consider

$$y_{n+3} + (1 + (-1)^n)\, y_{n+2} - \left(3 + \frac{1}{n}\right) y_{n+1} + \left(3 + \frac{1}{n}\right) y_n = 0, \quad n \geqslant 1. \tag{13}$$

Here

$$r_n = \begin{cases} 2, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

$q_n = -\,(3 + 1/n) < 0$ and $p_n = (3 + 1/n) > 0$. Then $\liminf\limits_{n\to\infty} p_n = 3$, $\liminf\limits_{n\to\infty} q_n = -3$ and $\liminf\limits_{n\to\infty} r_n = 0$. Since all conditions of Theorem 3.3 are satisfied, (13) is oscillatory. Choosing $y_1 = 0$, $y_2 = 1$ and $y_3 = 2$, we obtain $y_4 = 4 > 0$, $y_5 = -\frac{9}{2} < 0$, $y_6 = \frac{20}{3} > 0$, $y_7 = -\frac{983}{24} < 0$ and so y_n are alternately of positive and negative values. Thus it is an oscillatory solution of (13). But Theorem 3.4 cannot be applied to (13) since $\liminf\limits_{n\to\infty} r_n = r = 0$.

Example 3. Consider

$$y_{n+3} + 3y_{n+2} - 12y_{n+1} + 18y_n = 0, \quad n \geqslant 0. \tag{14}$$

Hence $p = 18 > 16 = 2\left(-\frac{1}{3}q\right)^{3/2}$. From Theorem 2.3 or 3.3 it follows that (14) is oscillatory. However, Theorem 3.5 is not applicable to (14) because

$$\frac{r_n p_{n+1}}{q_{n+1}} + \frac{p_n r_{n-1}}{q_{n-1}} = -9 > -12 = q_n.$$

On the other hand, Theorem 3.5 can be applied to

$$y_{n+3} + 6y_{n+2} - 12y_{n+1} + 14y_n = 0$$

to conclude that all solutions of the equation are oscillatory. But Theorem 3.3 cannot be applied to this equation because

$$2\left(\frac{-q}{3}\right)^{3/2} = 16 > 14 = p.$$

4. Study of third order equations via second order equations

We begin with the following observation.

Proposition 4.1. *A real sequence* $\{y_n\}$ *is a solution of*

$$y_{n+3} + r_n y_{n+2} - (r_n + p_n + 1) y_{n+1} + p_n y_n = 0 \tag{15}$$

if and only if $\{x_n\}$, *where* $x_n = (-1)^n y_n$, *is a solution of*

$$x_{n+3} - r_n x_{n+2} - (r_n + p_n + 1) x_{n+1} - p_n x_n = 0 \tag{16}$$

where $\{p_n\}$ *and* $\{r_n\}$ *are real sequences such that* $p_n \neq 0$. *Moreover,* $\{\{y_n^{(1)}\}, \{y_n^{(2)}\}, \{y_n^{(3)}\}\}$ *is a basis of the solution space of* (15) *if and only if* $\{\{x_n^{(1)}\}, \{x_n^{(2)}\}, \{x_n^{(3)}\}\}$ *is a basis of the solution space of* (16).

The proof is straightforward and hence it is omitted.

Corollary 4.2. *If all solutions of* (15) *are nonoscillatory, then all solutions of* (16) *are oscillatory. Further, if* (16) *has a nonoscillatory solution, then* (15) *has an oscillatory solution.*

The corollary follows from Proposition 4.1. However, the converse of neither of the above two statements is true.

Example 4. The equation

$$y_{n+3} + y_{n+2} - 2y_n = 0$$

admits a positive solution $\{1\}$ and two oscillatory solutions $\{\cos n\theta\}$ and $\{\sin n\theta\}$, where $\theta = \frac{3}{4}\pi$. However, all solutions of

$$x_{n+3} - x_{n+2} + 2x_n = 0$$

are oscillatory because $\{\{(-1)^n\}, \{\cos n\theta\}, \{\sin n\theta\}\}$, where $\theta = \frac{1}{4}\pi$, is a basis of the solution space of this equation.

Corollary 4.3. *Each of Eqs.* (15) *and* (16) *possesses both oscillatory and nonoscillatory solutions simultaneously if each of these equations admits a nonoscillatory solution.*

Theorem 4.4. *If $r_n + p_n = 0$, then each of* (15) *and* (16) *admits both oscillatory and nonoscillatory solutions.*

P r o o f. Clearly, $y_n \equiv 1$ is a positive solution of (15). Further, $r_n + p_n = 0$ implies that $x_n = c \neq 0$, where c is a constant, is a nonoscillatory solution of (16). Hence the theorem follows from Corollary 4.3. □

Theorem 4.5. *If $p_n > 0$ and $r_n \geqslant 0$, then each of* (15) *and* (16) *admits both oscillatory and nonoscillatory solutions.*

P r o o f. We may notice that Eq. (16) with initial conditions $x_0 > 0$, $x_1 > 0$ and $x_2 > 0$ admits a positive solution $\{x_n\}$ because $p_n > 0$ and $r_n \geqslant 0$. Further, $y_n \equiv 1$ is a positive solution of (15). Hence the theorem is proved. □

E x a m p l e 5. (i) Each of the equations

$$y_{n+3} - y_{n+2} - y_{n+1} + y_n = 0 \quad \text{and} \quad x_{n+3} + x_{n+2} - x_{n+1} - x_n = 0$$

admits an oscillatory solution $\{(-1)^n\}$ and a positive solution $\{1\}$.

(ii) The equation $y_{n+3} + y_{n+2} - 3y_{n+1} + y_n = 0$ admits a positive solution $\{1\}$ and an oscillatory solution $\{(-1 - \sqrt{2})^n\}$. On the other hand, the equation $x_{n+3} - x_{n+2} - 3x_{n+1} - x_n = 0$ admits a positive solution $\{(1 + \sqrt{2})^n\}$ and an oscillatory solution $\{(-1)^n\}$.

Proposition 4.6. *A real sequence $\{y_n\}$ is a solution of* (15) *if and only if $\{\Delta y_n\}$ is a solution of the second order difference equation*

$$u_{n+2} + (r_n + 1)u_{n+1} - p_n u_n = 0, \tag{17}$$

where Δy_n is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$. This follows from the fact that Eq. (15) *can be written in the form*

$$y_{n+3} - y_{n+2} + (r_n + 1)(y_{n+2} - y_{n+1}) - p_n(y_{n+1} - y_n) = 0,$$

that is,

$$\Delta y_{n+2} + (r_n + 1)\Delta y_{n+1} - p_n \Delta y_n = 0.$$

Lemma 4.7. *Let $\{y_n\}$ be a sequence of real numbers. If $\{\Delta y_n\}$ is eventually of one sign, then $\{y_n\}$ is eventually of one sign.*

The proof follows easily and hence is omitted.

Theorem 4.8. *If all solutions of* (17) *are nonoscillatory, then all solutions of* (15) *are nonoscillatory.*

Proof. Let $\{y_n\}$ be a solution of (15). So $\{\Delta y_n\}$ is a solution of (17) by Proposition 4.6. Since all solutions of (17) are nonoscillatory, then Δy_n is eventually of one sign. Then, by Lemma 4.7, $\{y_n\}$ is eventually of one sign, that is, $\{y_n\}$ is nonoscillatory. This completes the proof of the theorem. □

Remark. We may notice that such a result is not possible for Eq. (16) because it always admits an oscillatory solution $\{(-1)^n\}$.

In literature there are several sufficient conditions for nonoscillation of all solutions of linear second order difference equations. The following result may be obtained from [2].

Theorem 4.9. *Let $\beta_n < 0$ and $\gamma_n > 0$ for large n.*

(i) *If $\beta_n\beta_{n+1} \geqslant 4\gamma_{n+1}$ for large n, then all solutions of*

$$x_{n+2} + \beta_n x_{n+1} + \gamma_n x_n = 0 \tag{18}$$

are nonoscillatory.

(ii) *If $-\beta_n \geqslant \max\{\gamma_n, 4\}$ for large n, then all solutions of* (18) *are nonoscillatory.*

Theorem 4.10. *Let $p_n < 0$ and $r_n < -1$ for large n.*

(i) *If $(r_n + 1)(r_{n+1} + 1) + 4p_{n+1} \geqslant 0$ for large n,*
then all solutions of (15) *are nonoscillatory and hence* (16) *is oscillatory.*

(ii) *If $-(r_n + 1) \geqslant \max\{-p_n, 4\}$ for large n,*
then all solutions of (15) *are nonoscillatory and hence* (16) *is oscillatory.*

The theorem follows from Theorems 4.8 and 4.9 and Corollary 4.2.

Definition. A sequence of real numbers $\{y_n\colon\ n \geqslant 0\}$ is said to have a generalized zero at k if one of the following conditions holds: (i) $y_k = 0$ if $k = 0$ and (ii) if $k \geqslant 1$, then either $y_k = 0$ or $y_{k-1}y_k < 0$.

Definition. Equation (17) is said to be disconjugate on $[0,\infty)$ if no nontrivial solution of (17) has two or more generalized zeros in $[0,\infty)$. Similarly, Eq. (15) is said to be disconjugate on $[0,\infty)$ if no nontrivial solution of (15) has three or more generalized zeros in $[0,\infty)$.

Theorem 4.11. *If Eq.* (17) *is disconjugate on* $[0,\infty) = \{0,1,2,\ldots\}$, *then Eq.* (15) *is disconjugate on* $[0,\infty)$.

P r o o f. On the contrary, let $\{y_n\}$, $n \geqslant 0$, be a solution of (15) which has three generalized zeros at m_1, m_2 and m_3 $(0 \leqslant m_1 < m_2 < m_3)$.

If $m_1 \geqslant 1$, then $y_{m_1} \neq 0$ or $y_{m_1} = 0$. From Proposition 4.6 it follows that $\{\Delta y_n\}$ is a solution of (17). We consider the following four possible cases:

(i) Let $y_{m_1-1} < 0$, $y_{m_1} > 0$, $y_{m_2-1} > 0$, $y_{m_2} \leqslant 0$, $y_{m_3-1} < 0$ and $y_{m_3} \geqslant 0$. Then $\Delta y_{m_1-1} = y_{m_1} - y_{m_1-1} > 0$, $\Delta y_{m_2-1} = y_{m_2} - y_{m_2-1} < 0$ and $\Delta y_{m_3-1} = y_{m_3} - y_{m_3-1} > 0$. Thus the solution $\{\Delta y_n\}$ of (17) has two generalized zeros in $[m_1, m_3-1] = \{m_1, m_1+1, \ldots, m_3-2, m_3-1\}$.

(ii) Let $y_{m_1-1} > 0$, $y_{m_1} < 0$, $y_{m_2-1} < 0$, $y_{m_2} \geqslant 0$, $y_{m_3-1} > 0$ and $y_{m_3} \leqslant 0$. Hence $\Delta y_{m_1-1} < 0$, $\Delta y_{m_2-1} > 0$ and $\Delta y_{m_3-1} < 0$. Thus $\{\Delta y_n\}$ has two generalized zeros in $[m_1, m_3-1]$.

(iii) Let $y_{m_1} = 0$, $y_{m_1+1} < 0$, $y_{m_2-1} < 0$, $y_{m_2} \geqslant 0$, $y_{m_3-1} > 0$ and $y_{m_3} \leqslant 0$. Then $\Delta y_{m_1} < 0$, $\Delta y_{m_2-1} > 0$ and $\Delta y_{m_3-1} < 0$. Hence $\{\Delta y_n\}$ has two generalized zeros in $[m_1+1, \quad m_3-1]$.

(iv) Let $y_{m_1} = 0$, $y_{m_1+1} > 0$, $y_{m_2-1} > 0$, $y_{m_2} \leqslant 0$, $y_{m_3-1} < 0$ and $y_{m_3} \geqslant 0$. Then $\Delta y_{m_1} > 0$, $\Delta y_{m_2-1} < 0$ and $\Delta y_{m_3-1} > 0$. Thus $\{\Delta y_n\}$ has two generalized zeros in $[m_1+1,\ m_3-1]$.

If $m_1 = 0$, then $y_{m_1} = 0$. We arrive at a contradiction as in cases (iii) and (iv). In each case, we obtain a contradiction to the fact that (17) is disconjugate. Hence the theorem is proved. □

Theorem 4.12. *Let* $p_n < 0$. *If there exists a positive number* k *such that*

$$k^2 + (r_n+1)k - p_n = 0, \quad n \geqslant 0,$$

then (15) *is disconjugate on* $[0,\infty)$.

This follows from Theorem 4.11 and Corollary 6.9 in [3].

E x a m p l e 6. All conditions of Theorem 4.12 hold for $k = 2$ for the equation

$$y_{n+3} - 6y_{n+2} + 11y_{n+1} - 6y_n = 0, \quad n \geqslant 0.$$

Hence it is disconjugate on $[0,\infty)$. As $\{\{1\},\ \{2^n\},\ \{3^n\}\}$ is a basis of the solution space of the equation, any solution $\{y_n\}$ of the equation can be written as $y_n = C_1 + C_2 2^n + C_3 3^n$, where C_1, C_2, C_3 are constants such that $C_1^2 + C_2^2 + C_3^2 \neq 0$. It cannot have more than two generalized zeros in $[0,\infty)$.

In [1], Henderson and Peterson obtained sufficient conditions for disconjugacy of

$$\Delta^3 y_{n-1} + p_n \Delta y_n + q_n y_n = 0.$$

However, our results and techniques are different from theirs.

Results similar to Theorems 4.8 and 4.11 also hold for the equations of the form

$$\Delta(b_{n-1}\Delta^2 y_{n-1}) + c_n \Delta y_n = 0, \quad n \geqslant 1, \tag{19}$$

where $b_n > 0$. Indeed, we have the following theorem.

Theorem 4.13. (i) *If all solutions of*

$$\Delta(b_{n-1}\Delta u_{n-1}) + c_n u_n = 0, \quad n \geqslant 1, \tag{20}$$

are nonoscillatory, then all solutions of (19) *are nonoscillatory.* (ii) *If* (20) *is disconjugate on* $[0, \infty)$, *then* (19) *is disconjugate on* $[0, \infty)$.

The proof is similar to those of Theorems 4.8 and 4.11 if we observe that $\{\Delta y_n\}$ is a solution of (20) if $\{y_n\}$ is a solution of (19).

5. Conclusions

There are many results concerning oscillation of (1) which are yet to be generalized to (2). It seems that there is no result in literature which would provide sufficient conditions for nonoscillation of all solutions of (2) although we have such results for (1). Indeed, the following results hold for (1) (See [6]).

Theorem 5.1. *If* $p < 0$, $q = \frac{1}{3}r^2$, $r \neq 0$ *and* $p - \frac{1}{3}qr + \frac{2}{27}r^3 = 0$, *then all solutions of* (1) *are nonoscillatory.*

Theorem 5.2. *If* $p < 0$, $q < \frac{1}{3}r^2$, $r < 0$ *and*

$$0 < p - \frac{qr}{3} + \frac{2r^3}{27} < \frac{2}{3\sqrt{3}}\left(\frac{r^2}{3} - q\right)^{3/2},$$

then all solutions of (1) *are nonoscillatory.*

Theorem 5.3. *If $p < 0$, $r < 0$, $p/r \leqslant q < \frac{1}{3}r^2$ and*

$$\frac{2}{3\sqrt{3}}\left(\frac{r^2}{3} - q\right)^{3/2} > \frac{qr}{3} - p - \frac{2r^3}{27} > 0,$$

then all solutions of (1) *are nonoscillatory.*

Theorem 5.4. *If $p < 0$, $r < 0$, $p/r \leqslant q < \frac{1}{3}r^2$ and*

$$0 < \frac{2}{3\sqrt{3}}\left(\frac{r^2}{3} - q\right)^{3/2} = \frac{qr}{3} - p - \frac{2r^3}{27},$$

then all solutions of (1) *are nonoscillatory.*

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