THE LONG-TIME BEHAVIOUR OF THE SOLUTIONS TO SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS ON THE WHOLE SPACE

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Abstract. The Cauchy problem for a stochastic partial differential equation with a spatial correlated Gaussian noise is considered. The "drift" is continuous, one-sided linearily bounded and of at most polynomial growth while the "diffusion" is globally Lipschitz continuous. In the paper statements on existence and uniqueness of solutions, their pathwise spatial growth and on their ultimate boundedness as well as on asymptotical exponential stability in mean square in a certain Hilbert space of weighted functions are proved.

Keywords: Cauchy problem, nuclear and cylindrical noise, existence and uniqueness of the solution, spatial growth, ultimate boundedness, asymptotic mean square stability

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1. Introduction and preliminaries

We will investigate the formal initial value problem (C)

$$\frac{\partial}{\partial t}u(t,x) = (\Delta - mI)u(t,x) + f(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x),\ t>0,\ x\in\mathbb{R}^d,$$
$$u(0,x) = \vartheta(x),\quad x\in\mathbb{R}^d,$$

where $m \geqslant 1$ is a real number. By $\dot{W}$ we denote a Gaussian noise which is in general supposed to be white in time and correlated in space. It is given as follows. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space which is fixed in this paper. Let $(w_n)_{n\in\mathbb{N}}$ denote a sequence of independent standard one-dimensional Wiener processes which are supposed to be independent of the initial datum ϑ. Let $(h_n)_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathbb{L}^2(\mathbb{R}^d)$ and choose a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers.

Define

$$B_t := \sum_{n\in\mathbb{N}} \sqrt{a_n}\cdot w_n(t)\cdot h_n,\ t \geqslant 0$$

and require additionally

$$a := \sum_{n\in\mathbb{N}} a_n \cdot \|h_n\|_\infty^2 < \infty.$$

Note that there exists an orthonormal basis $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{L}^2(\mathbb{R}^d)$ such that

$$\sup_{n\in\mathbb{N}} \|h_n\|_\infty < \infty,$$

where $\|.\|_\infty$ denotes the $\mathbb{L}^\infty$-norm (cf. Manthey and Mittmann, 1996; Manthey and Zausinger, 1996; Ovsepian and Pelczynski, 1975). Therefore, the above mentioned condition can be always fulfilled by choosing $(a_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} a_n < \infty$ if the basis is already taken. Then $B = (B_t)_{t\geqslant 0}$ is a so-called Q-Wiener process on $\mathbb{L}^2(\mathbb{R}^d)$. Since under this choice the covariance operator Q of B is nuclear we are in the "nuclear case". In contrast to that one could deal also with the "cylindrical case" where $a_n = 1$, $n \in \mathbb{N}$. Then B represents a cylindrical Wiener process which no longer exists in $\mathbb{L}^2(\mathbb{R}^d)$. (In this situation the basis $(h_n)_{n\in\mathbb{N}}$ can be taken arbitrarily. For details see Da Prato and Zabczyk, 1992.) In both cases we have

$$B_t(h) = \sum_{n\in\mathbb{N}} \sqrt{a_n}\cdot w_n(t)\cdot (h_n, h).$$

Except for the uniqueness assertion below we will deal in this paper only with the (generalized) nuclear case described above.

We will use the standard (i.e. right continuous and complete) filtration generated by $(B(h_n))_{n\in\mathbb{N}}$ and ϑ and denote it by $\mathbb{F} = (\mathfrak{F}_t)_{t\geqslant 0}$. Defining

$$W(t,x) := B_t(\mathbf{1}_{(-\infty,x]}),$$

$(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we get a centered Gaussian random field possessing a space-time continuous version which is chosen from now on. Taking

$$\dot{W} = \frac{\partial^{d+1}}{\partial t \partial x_1 \ldots \partial x_d} W$$

in the generalized sense we arrive at the driving noise in (C).

Let $\varphi\colon\ \Omega \times [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be progressively measurable and such that

$$\mathbb{E}\int_0^t \left[\int_{\mathbb{R}^d} |\varphi(s,x)|\,\mathrm{d}x\right]^2 \mathrm{d}s < \infty$$

in the nuclear case and

$$\mathbb{E}\int_0^t\int_{\mathbb{R}^d}|\varphi(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s<\infty$$

in the cylindrical case. For these classes of mappings the stochastic integral

$$\int_0^t\int_{\mathbb{R}^d}\varphi(s,x)\,\mathrm{d}W(s,x):=\sum_{n\in\mathbb{N}}\int_0^t\left[\int_{\mathbb{R}^d}\varphi(s,x)h_n(x)\,\mathrm{d}x\right]\mathrm{d}B_s(h_n),\ t\in[0,\infty),$$

is well-defined.

We will always assume that the mappings $f,\sigma\colon\ \mathbb{R}\to\mathbb{R}$ are continuous. Later on we will impose more restrictive conditions.

Introduce the mapping $g\colon\ (t,x)\to(4\pi t)^{-d/2}\cdot\exp(-|x|^2/4t)$ for $t>0$, $x\in\mathbb{R}^d$ and denote $\mathrm{e}^{-mt}\cdot g(t,x)$ by $G(t,x)$. Consider

$$\text{(I)}\qquad \begin{aligned} u(t,x)&=\int_{\mathbb{R}^d}G(t,x-y)\vartheta(y)\,\mathrm{d}y+\int_0^t\int_{\mathbb{R}^d}G(t-s,x-y)f(u(s,y))\,\mathrm{d}y\,\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}^d}G(t-s,x-y)\sigma(u(s,y))\,\mathrm{d}W(s,y)\\ &=:\ \Theta(t,x)+\Phi(t,x,u)+\mathcal{S}(t,x,u),\ t>0,\ x\in\mathbb{R}^d. \end{aligned}$$

Let ℓ belong to the set $\mathcal{P}$ of mappings $\ell\colon\ \mathbb{R}^d\to[1,\infty)$ possessing the representation $\ell(x)=1+|x|^n$ for some $n\geqslant 1$ and let us introduce the Banach space

$$\mathbb{C}_\ell=\mathbb{C}_\ell(\mathbb{R}^d):=\left\{\varphi\in\mathbb{C}(\mathbb{R}^d)\colon\ \sup_{x\in\mathbb{R}^d}\frac{|\varphi(x)|}{\ell(x)}<\infty\right\}$$

with the norm $|\varphi|_\ell:=\sup\limits_{x\in\mathbb{R}^d}\ell^{-1}(x)|\varphi(x)|$. Finally, set

$$\mathbb{C}_\ell^T=\mathbb{C}_\ell^T(\mathbb{R}^d):=\big\{u\in\mathbb{C}([0,T]\times\mathbb{R}^d,\mathbb{R})\colon\ \sup_{t\in[0,T]}|u(t,.)|_\ell<\infty\big\}.$$

Two mappings $\psi,\zeta\colon\ \Omega\times\mathbb{R}^d\to\mathbb{R}$ are called *equivalent* if $\psi(x)=\zeta(x)$ holds $\mathbb{P}$ a.s. for every $x\in\mathbb{R}^d$. Let $\varepsilon\colon\ \mathbb{R}^d\to[1,\infty)$ be a continuous function. Note that for $\kappa\geqslant 1$ the set $\mathbb{H}_{\varepsilon,\kappa}$ of classes of equivalent measurable mappings $\psi\colon\ \Omega\times\mathbb{R}^d\to\mathbb{R}$ with

$$\|\psi\|_{\varepsilon,\kappa}:=\sup_{x\in\mathbb{R}^d}\frac{(\mathbb{E}|\psi(x)|^\kappa)^{1/\kappa}}{\varepsilon(x)}<\infty$$

is a Banach space with a norm $\|.\|_{\varepsilon,\kappa}$. Obviously, $\psi\in\mathbb{H}_{\varepsilon,\kappa}$ implies $\psi\in\mathbb{H}_{\varepsilon,\alpha}$ provided $\kappa>\alpha$. Finally, denote the set $[0,T]\times\mathbb{R}^d$ by $\mathbb{D}_T$ and $[0,\infty)\times\mathbb{R}^d$ by $\mathbb{D}$.

Definition 1.1. A pathwise continuous random field $u\colon \Omega \times \mathbb{D} \to \mathbb{R}$ is called a *solution to* (C) *on* $[0,\infty)$ if the following holds.

(i) $u(0,x) = \vartheta(x)$ $\mathbb{P}$ a.s. for any $x \in \mathbb{R}^d$.

(ii) For any $x \in \mathbb{R}^d$ the process $u(.,x)$ is $\mathbb{F}$-adapted.

(iii) There exists an $\ell \in \mathcal{P}$ such that the paths of $u(t,.)$ belong to $\mathbb{C}_\ell$ for any $t \geqslant 0$.

(iv) There exist $\nu \geqslant 1$, $\ell \in \mathcal{P}$ such that $u(t,.)$ belong to $\mathbb{H}_{\ell,\nu}$ for any $t \geqslant 0$.

(v) u solves (I) $\mathbb{P}$ a.s. for any $(t,x) \in \mathbb{D}$.

In the cylindrical case existence of a solution was already investigated in Da Prato and Zabczyk (1992), Iwata (1987), Manthey (1988, 1996), Manthey and Mittmann (1997), Manthey and Zausinger (1996), Marcus (1979) and Shiga (1994). Except Manthey (1996), Manthey and Zausinger (1996) and Shiga (1994) the methods used in these papers are heavily dependent on the fact that σ is either a constant mapping or bounded. We need no such restrictions here. Moreover, we will prove a uniqueness theorem which seems to be completely new. It covers the physically interesting situation when f is an odd polynomial with a negative leading coefficient. Furthermore, this statement is also true in the cylindrical case. The pathwise spatial growth of solutions to the Cauchy problem was first investigated in Manthey (1996) for the cylindrical case. We will formulate the corresponding assertions for the nuclear case here. As far as we know, stability investigations for the solution of problems we are interested in are done only for Lipschitz continuous coefficients. We will show that in the case of the existence and uniqueness theorem mentioned above the solution is exponentially ultimately bounded in mean square. Under a natural additional condition the "equilibrium solution" ($u = 0$) is even asymptotically exponentially stable in mean square. Both assertions are true in a certain Hilbert space of weighted functions. In particular, this result shows that the only invariant measure in this case is δ_0.

The paper is organized in a simple way. After this introduction of the problem and some preliminaries the next section contains the formulation of the main results. Section 3 is devoted to some auxiliary results while in the final section the main results are proved.

2. Main results

2.1. Existence of a solution and growth in space. The initial datum ϑ is supposed to satisfy the following conditions:

(ϑ1) There exist $\ell \in \mathcal{P}$ and $p \geqslant 2$ such that $\vartheta \in \mathbb{H}_{\ell,p}$.

(ϑ2) For some $\ell \in \mathcal{P}$ and each $\omega \in \Omega$, ϑ belongs to $\mathbb{C}_\ell$.

To prove existence of a solution we will consider the problem under the following conditions on f:

(f1) The mapping $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

(f2) There exists a nonnegative constant c_f such that

$$f(u) \geqslant -c_f(1-u),\ \ u \leqslant 0,$$

and

$$f(u) \leqslant c_f(1+u),\ \ u \geqslant 0.$$

(f3) There exist constants $\nu \in \{1, 2, \ldots\}$ and $c_\nu > 0$ such that

$$|f(u)| \leqslant c_\nu \cdot (1 + |u|^\nu)$$

for every $u \in \mathbb{R}$.

Property (f2) is sometimes called the "one-sided linear growth condition", while (f3) is naturally a "polynomial growth condition".

A mapping $\Lambda\colon \mathbb{R} \to \mathbb{R}$ is called *globally Lipschitz continuous* if there is a constant $L > 0$ such that

$$|\Lambda(u) - \Lambda(v)| \leqslant L \cdot |u - v| \tag{L}$$

for any $u, v \in \mathbb{R}$. Obviously, (L) yields

$$|\Lambda(u)| \leqslant C(1 + |u|)$$

for some $C > 0$ and any $u \in \mathbb{R}$. To simplify the notation we will always assume $C = L$ which is clearly not a restriction.

Throughout this paper σ is supposed to have the following property.

(σ1) The mapping $\sigma\colon \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous.

R e m a r k. One could allow the mappings f and σ to depend additionally on $(t, x) \in [0, T] \times \mathbb{R}^d$ for an arbitrary $T > 0$. In this case the constants in (f2), (f3) and (σ1) must be independent of (t, x). This generalization leads only to a more complicated notation. Therefore, we omit it.

The main tool for estimating the spatial growth of the desired solution is the following assertion.

Theorem 2.1.1. *Let $F, \Sigma\colon \Omega \times \mathbb{D}_T \to \mathbb{R}$ be progressively measurable functions with the property that there exist $p > 2$, $k \geqslant 0$ such that*

$$\mathbb{E}[|F(t,x)|^{2p} + |\Sigma(t,x)|^{2p}] \leqslant c_0(p, T) \cdot (1 + |x|^k)$$

for any $(t,x) \in \mathbb{D}_T$, *where* $c_0(p,T)$ *is a certain constant. Then* $V\colon \Omega \times \mathbb{D}_T \to \mathbb{R}$ *defined for* $t \in (0,T]$ *by*

$$V(t,x) := \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)F(s,y)\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\Sigma(s,y)\,\mathrm{d}W(s,y)$$

and for $t=0$ *by* $V(0,x)=0$, $x \in \mathbb{R}^d$, *possesses a pathwise locally Hölder continuous version that is again denoted by V. This version satisfies*

$$|V(t,x)| \leqslant \Xi(a,k,m,p,c_0(p,T),T) \cdot \left[1+|x|^{\frac{1}{4}+\frac{2+k}{2p}}\right]$$

for $(t,x) \in \mathbb{D}_T$, *where* Ξ *is a random variable with* $\mathbb{P}(\{0<\Xi<\infty\})=1$.

A quite large class of nonlinearities ("reaction functions") f is covered by the following existence result.

Theorem 2.1.2. *Let* f *and* σ *satisfy* (f1)–(f3) *and* (σ1), *respectively. Suppose that* ϑ *has properties* (ϑ1) *with* $p > \nu$ *and* (ϑ2). *Then for each* $T>0$ *there exists at least one solution* u *to* (C) *on* $[0,T]$ *such that* $\vartheta \in \mathbb{C}_\ell$, $\ell(x) = 1+|x|^k$, *implies* $u \in \mathbb{C}^T_{\ell^*}$, *where*

$$\ell^*(x) = 1+|x|^{\frac{1}{4}+\frac{1}{p}+k}, \; x \in \mathbb{R}^d.$$

Moreover, $\sup_{t\in[0,T]} \|u(t,.)\|_{\ell,p} < \infty$ *if* $\vartheta \in \mathbb{H}_{\ell,p}$.

This shows that under the just formulated conditions there exists at least one solution to (C) on [0,T] not growing faster than the solution under Lipschitz conditions on f (compare Theorem 3.2.1 below). Furthermore, as in the Lipschitz case this solution cannot leave the Banach space $\mathbb{H}_{\ell,p}$ if it starts in it. Under an additional condition u also stays in $\mathbb{C}_\ell$.

Theorem 2.1.3. *In addition to the conditions of Theorem* 2.1.2 *let* σ *be bounded and* $p \geqslant 3 \vee \nu$. *Then all paths of the solution obtained in Theorem* 2.1.2 *belong to* $\mathbb{C}^T_\ell$.

2.2. Uniqueness. Let $T>0$ be arbitrary but fixed and let us introduce the following condition.

(f4) There exists constants $\nu \geqslant 1$ and $K>0$ such that

$$|f(u)-f(v)| \leqslant K \cdot |u-v| \cdot (1+|u|^{\nu-1}+|v|^{\nu-1})$$

for any $u,v \in \mathbb{R}$.

R e m a r k s .

(i) Obviously, (f4) implies that f is locally Lipschitz continuous.

Notice that the usual procedure of proving uniqueness does not work here. First, we will not assume a linear growth condition on f and second, ϑ may be unbounded on $\mathbb{R}^d$. $\mathcal{S}(t, .)$ is already unbounded in $x \in \mathbb{R}^d$ even if σ is a constant mapping. Hence local Lipschitz continuity alone is useless here.

(ii) Condition (f4) implies the polynomial growth condition (f3).

(iii) Condition (f4) is satisfied by

$$f(u) = \sum_{k=0}^{\nu} a_k u^k$$

provided $a_\nu < 0$ and ν is odd.

We shall prove the following uniqueness assertion.

Theorem 2.2.1. *Let* $u = (u(t,x))_{(t,x)\in\mathbb{D}_T}$ *and* $v = (v(t,x))_{(t,x)\in\mathbb{D}_T}$ *be two solutions of* (C) *on* [0,T] *with paths belonging to* $\mathbb{C}_\ell^T$ *and corresponding to the same initial condition. If conditions* (f4) *and* (σ1) *are satisfied, then*

$$\mathbb{P}(\{\sup_{(t,x)\in\mathbb{D}} |u(t,x) - v(t,x)| = 0\}) = 1$$

holds.

Note that the existence theorem in Section 2.1 gives us a solution u in each fixed time interval $[0, T]$. At worst these solutions differ for different T. Theorem 2.2.2 guarantees the existence of only one solution for $t \in [0, \infty)$. Namely, we have

Theorem 2.2.2. *Let f and σ satisfy* (f2), (f4) *and* (σ1)*, respectively. Suppose that ϑ has the properties* (ϑ1) *with* $p > \nu$ *and* (ϑ2). *Then there exists a pathwise unique solution u to* (C) *on* $[0, \infty)$ *such that* $\vartheta \in \mathbb{C}_\ell$, $\ell(x) = 1 + |x|^k$, *implies* $u(t, .) \in \mathbb{C}_{\ell^*}$, $t \geqslant 0$, *where*

$$\ell^*(x) = 1 + |x|^{\frac{1}{4}+\frac{1}{p}+k}, \; x \in \mathbb{R}^d.$$

Moreover, $\sup_{t\in[0,T]} \|u(t, .)\|_{\ell,p} < \infty$ *provided* $\vartheta \in \mathbb{H}_{\ell,p}$. *If, additionally, σ is bounded and p in* (ϑ1) *satisfies* $p \geqslant 3 \vee \nu$, *then* $\ell^* = \ell$ *holds.*

2.3. Comparison of solutions. Let us consider (C) with two different reactions $f^{(1)}$ and $f^{(2)}$, two different initial data $\vartheta^{(1)}$ and $\vartheta^{(2)}$ and the same noise intensity σ and denote the corresponding solutions by $u^{(i)}$, $i = 1, 2$.

Definition 2.3.1. We say that the comparison principle holds for (C) if the following conclusion is true. There exist solutions $u^{(i)}$, $i = 1,2$, of (C) and the inequalities

$$\vartheta^{(1)}(x) \geqslant \vartheta^{(2)}(x)$$

$\mathbb{P}$ a.s. for every $x \in \mathbb{R}^d$ and

$$f^{(1)}(u) \geqslant f^{(2)}(u),$$

$u \in \mathbb{R}$, imply

$$u^{(1)}(t,x) \geqslant u^{(2)}(t,x)$$

$\mathbb{P}$ a.s. for any $(t,x) \in \mathbb{D}$.

Combining the pathwise space-time continuity (see Theorem 3.2.1 below) and the comparison theorem of [9] in the Lipschitz case one easily checks the validity of the following assertion.

Theorem 2.3.2. *Let $f^{(1)}$, $f^{(2)}$ and σ satisfy* (L) *and suppose that $\vartheta^{(i)}$, $i = 1,2$, have properties* (ϑ1) *and* (ϑ2). *Then the comparison principle holds for* (C).

This theorem, the construction of the solution obtained by Theorem 2.1.2 and the uniqueness of that solution lead to the following conclusion.

Theorem 2.3.3. *Let $f^{(1)}$, $f^{(2)}$ satisfy* (f2) *and* (f4) *and let σ have the property* (σ1). *Moreover, suppose that $\vartheta^{(i)}$, $i = 1,2$, have properties* (ϑ1) *with $p > \nu$ and* (ϑ2). *Then the comparison principle holds for* (C).

2.4. Asymptotical exponential mean square stability of the equilibrium solution. Let $\lambda\colon \mathbb{R}^d \to \mathbb{R}$ be given by

$$\lambda(x) := (1 + |x|^2)^{1/2}$$

and introduce a finite measure μ by

$$\mu(\mathrm{d}x) := \lambda^{-\varrho}(x)\,\mathrm{d}x$$

with $\varrho > d$. The set of all Borel measurable mappings $\psi\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$|\psi|_{\varrho,\kappa}^{\kappa} := \int_{\mathbb{R}^d} |\psi(x)|^{\kappa} \lambda^{-\varrho}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |\psi(x)|^{\kappa} \mu(\mathrm{d}x) < \infty$$

will be denoted in the sequel by $\mathbb{L}_{\varrho}^{\kappa}(\mathbb{R}^d)$. We shall show next that the solution constructed in Theorem 2.2.2 belongs to some $\mathbb{L}_{\varrho}^{p}(\mathbb{R}^d)$. Let ϑ belong to $\mathbb{C}_{\ell}$ and $\mathbb{H}_{\ell,p}$, where $\ell(x) = 1 + |x|^k$. Notice that Theorems 2.1.2 and 2.2.2 guarantee $u \in \mathbb{C}_{\bar{\ell}}^T$, where $\bar{\ell}(x) = 1 + |x|^{k+1}$, and

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{\mathbb{E}|u(t,x)|^p}{\ell^p(x)} \leqslant c(p,T) < \infty.$$

Using

$$1 + |x|^q \leqslant (1 + |x|^2)^{q/2} \leqslant 2^{(q/2)-1} \cdot (1 + |x|^q),$$

$q \geqslant 2$, $x \in \mathbb{R}^d$, we get

$$\begin{aligned}
|u(t,.)|^p_{(k+1)p+d+1,p} &= \int_{\mathbb{R}^d} |u(t,x)|^p \cdot \lambda^{-(k+1)p-d-1}(x)\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^d} |u(t,x)|^p \cdot (1 + |x|^{(k+1)p+d+1})^{-1}\,\mathrm{d}x \\
&\leqslant 2^{(k+1)p+d} \cdot \int_{\mathbb{R}^d} |u(t,x)|^p \cdot (1 + |x|)^{-(k+1)p-d-1}\,\mathrm{d}x \\
&\leqslant 2^{(k+1)p+d} \cdot \int_{\mathbb{R}^d} \frac{|u(t,x)|^p}{(1 + |x|^{k+1})^p} \cdot (1 + |x|)^{-(d+1)}\,\mathrm{d}x \\
&\leqslant c(k,p,d) \cdot \sup_{t\in[0,T]} |u(t,.)|^p_{\bar{\ell}} \cdot \int_{\mathbb{R}^d} (1 + |x|)^{-(d+1)}\,\mathrm{d}x < \infty.
\end{aligned}$$

Hence $u(t,.) \in \mathbb{L}^p_\varrho(\mathbb{R}^d)$ for any $t \geqslant 0$, where $\varrho = (k+1)p + d + 1$. Obviously, the same conclusion goes through if we additionally take the expectation. Moreover, u constructed in Theorem 2.2.2 has the property $u(t,.) \in \mathbb{H}_{\ell,2}$, $t \geqslant 0$. Therefore, we may put $p = 2$ and have $u(t,.) \in \mathbb{L}^2_\varrho(\mathbb{R}^d)$ with $\varrho = 2k + d + 3$. In the sequel we fix this ϱ. To simplify the notation we will use $|\psi|_\varrho$ for $|\psi|_{\varrho,2}$.

For the stability assertion we need a further condition on σ, namely,

(σ2) $\sigma(0) = 0$.

Definition 2.4.1. A solution u of (C) with an initial condition $\vartheta \in \mathbb{H}_{\ell,2}$ is called exponentially ultimately bounded in $\mathbb{L}^2_\varrho(\mathbb{R}^d)$ in mean square if there exist positive constants α, β and M such that

$$\mathbb{E}|u(t,.)|^2_\varrho \leqslant \alpha \cdot \mathbb{E}|\vartheta|^2_\varrho \mathrm{e}^{-\beta t} + M$$

holds.

Definition 2.4.2. A solution u of (C) with an initial condition $\vartheta \in \mathbb{H}_{\ell,2}$ is called asymptotically exponentially stable in $\mathbb{L}^2_\varrho(\mathbb{R}^d)$ in mean square if there exist positive constants α and β such that

$$\mathbb{E}|u(t,.)|^2_\varrho \leqslant \alpha \cdot \mathbb{E}|\vartheta|^2_\varrho \mathrm{e}^{-\beta t}$$

holds.

To formulate the main result of this section we introduce the abbreviation

$$\overline{\psi}_j(\varrho) := 2^{\varrho-1} \cdot \Big[1 + \frac{\Gamma(\frac{\varrho+d}{2})}{\Gamma(\frac{d}{2})} \cdot \Big(\frac{\varrho}{j\mathrm{e}}\Big)^{\frac{\varrho}{2}}\Big],$$

$j = \frac{1}{2}, 1$. Moreover, we need a condition that is stronger than (f2), namely:

(f2)* there exists a nonnegative constant c_f such that

$$f(u) \geqslant c_f u, \ \ u \leqslant 0,$$

and

$$f(u) \leqslant c_f u, \ \ u \geqslant 0.$$

Obviously, this condition implies $f(0) = 0$.

Theorem 2.4.3. *Let u be a solution to* (C) *on* $[0, \infty)$ *such that* $u(t, .) \in \mathbb{C}_\ell^T$ *and*

$$\sup_{t\in[0,T]} \|u(t, .)\|_{\ell,p} \leqslant c(p, T) < \infty$$

for $T > 0$ and some $p \geqslant 1$. Suppose that f and σ satisfy (f4) *and* (σ1), (σ2), *respectively. Moreover, let*

$$m > 4c_f^2\overline{\psi}_{\frac{1}{2}}(\varrho) + c_\sigma^2 a\overline{\psi}_1(\varrho).$$

Then the following holds:

(i) *If f satisfies* (f2) *then u is exponentially ultimately bounded in* $\mathbb{L}_\varrho^2(\mathbb{R}^d)$.

(ii) *If f satisfies* (f2)* *then u is asymptotically exponentially mean square stable in* $\mathbb{L}_\varrho^2(\mathbb{R}^d)$.

Remark. The proof below will show that the condition (f4) in Theorem 2.4.3 can be replaced by a condition which is weaker but lengthy and not easy to verify. Namely, it is enough to require that the comparison principle for (C) holds for f and several modifications of f as the positive and negative part respectively linear transformations of f.

Example. Let us consider the so-called cubic nonlinearity $f(u) = -u^3 + \gamma u$. We already know that this mapping satisfies (f4). If we take $c_f = \gamma$ then f fulfils also (f2)*. To simplify the result let $\sigma(u) = u$. Thus $c_\sigma = 1$. If $d = 3$ and $k = 1$ the values $|u(t, x)|$ of the corresponding solution are bounded by a constant times $1 + |x|^{7/4}$. Moreover, if m is strictly larger than $2^6\psi_{1/2}(8)\gamma^2 + 2^4 a\psi_1(8)$ we observe the asymptotic mean square stability of u in $\mathbb{L}_8^2(\mathbb{R}^3)$.

Remark. To simplify notation in the proofs below constants of different values will be denoted by the same symbol if their value is not essential.

3.1. Estimates. We shall frequently use a series of estimates that we will now present.

Lemma 3.1.1 (cf. Manthey and Stiewe (1992)). *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $q_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$q_n(t) \leqslant a + b\cdot \int_0^t (t-s)^{-\delta}\cdot q_{n-1}(s)\,\mathrm{d}s,$$

$n\in\mathbb{N}$, $\delta\in[0,1)$, $b>0$, $a\geqslant 0$, $t\in[0,T]$. *Then*

$$q_n(t) \leqslant a\cdot \sum_{k=o}^{n-1} a_k t^{k(1-\delta)} + a_n t^{n(1-\delta)}\cdot \sup_{t\in[0,T]} q_0(t)$$

holds, where $\sum_{k=0}^{\infty} a_n T^{(1-\delta)n} < \infty$.

Our main tool for estimating the pathwise growth of random fields is

Lemma 3.1.2 (cf. Manthey and Mittmann (1996)). *Let $Z = (Z(r))_{r\in\mathbb{R}^k}$ be a random field with the property that there exist constants $\gamma>1$, $\kappa\geqslant 0$, $\alpha>k$ and $c>0$ such that*

$$\mathbb{E}|Z(r)-Z(s)|^\gamma \leqslant c\cdot n^\kappa\cdot |r-s|^\alpha$$

for any $r,s\in[-n,n]^k$, $n\in\mathbb{N}$. Then the following holds:

(i) *There exists a pathwise locally Hölder continuous version Y of Z.*

(ii) *For any $\delta>1$ this version has the property*

$$|Y(r)| \leqslant \Xi_\delta\cdot (1+|r|^{\frac{\alpha+\delta+\kappa}{\gamma}})$$

for any $r\in\mathbb{R}^k$, where Ξ_δ is a random variable with $\mathbb{P}(\{0<\Xi_\delta<\infty\})=1$.

Space-time continuity of the desired solution and an estimate of the spatial growth will be consequences of the previous lemma and the following simple estimates for G and $G^k * \ell$.

Lemma 3.1.3. *For any $0\leqslant s\leqslant t\leqslant T$, $x,y\in\mathbb{R}^d$ and $\mu\in(0,\frac{1}{2})$ we have*

$$\int_0^t \left[\int_{\mathbb{R}^d} |G(t-r,x-z) - \mathbf{1}_{[0,s)}(r)G(s-r,y-z)|\,\mathrm{d}z\right]^2 \mathrm{d}r \leqslant c(\mu,T)(|t-s|^{2\mu} + |x-y|^{4\mu}).$$

Idea of the proof. This estimate is a consequence of the mean value theorem and the following inequality that can be directly checked:

For any $T > 0$, $t \in [0,T]$, $\ell, n \in \mathbb{N}$ with $2\ell + n \leqslant 2$ one has

$$|D_t^\ell D_x^n G(t,x)| \leqslant c_1(T) t^{-\frac{d+n}{2}-\ell} \cdot \exp(-c_2(T)\cdot |x|^2/t).$$

Lemma 3.1.4. *Let $\ell(x) = 1 + |x|^k$, $x \in \mathbb{R}^d$. There exists a constant $c(k)$ such that*

$$0 \leqslant \int_{\mathbb{R}^d} G(r, x-y)\ell^p(y)\,\mathrm{d}y \leqslant c(p,k)\ell^p(x)$$

holds for any $r > 0$, $x \in \mathbb{R}^d$, $k \in \{1,2,\ldots\}$.

Proof. Obviously, we have

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^d} G(r,x-y)\ell^p(y)\,\mathrm{d}y \\ &\leqslant c(p)\cdot \left[\int_{\mathbb{R}^d} G(r,y)\,\mathrm{d}y + \int_{\mathbb{R}^d} G(r,x-y)|y|^{pk}\,\mathrm{d}y\right] \\ &\leqslant c(p,k)\left[1 + |x|^{pk} + \mathrm{e}^{-mr}\cdot \int_{\mathbb{R}^d} g(r,z)|z|^{pk}\,\mathrm{d}z\right]. \end{aligned}$$

Note that

$$\int_{\mathbb{R}^d} g(r,z)|z|^\varrho\,\mathrm{d}z = \frac{2^{\varrho/2}\cdot\Gamma(\frac{\varrho+d}{2})}{\Gamma(\frac{d}{2})}\cdot r^{\varrho/2},$$

d, $\varrho \in \{1,2,\ldots\}$, $r > 0$. Consequently, we arrive at

$$\leqslant c(d,p,k)\big[1 + |x|^{pk} + \mathrm{e}^{-mr}\cdot r^{pk/2}\big].$$

Note that the mapping $r \to \mathrm{e}^{-mr} r^{dpk/2}$, $r \in \mathbb{R}_+$, is bounded. Hence we can continue by

$$\leqslant c(d,p,k)(1 + |x|^{pk}) \leqslant c(d,p,k)\ell^p(x),$$

proving the assertion. □

Lemma 3.1.5. *The estimate*

$$\int_{\mathbb{R}^d} g(r,z)\lambda^\varrho(z)\,\mathrm{d}z \leqslant 2^{(\varrho/2)-1}\cdot\left[1 + \frac{2^{\varrho/2}\cdot\Gamma(\frac{\varrho+d}{2})}{\Gamma(\frac{d}{2})}\cdot r^{\varrho/2}\right]$$

holds for $r > 0$, $\varrho \in \{1,2,\ldots\}$.

Proof. Because of $\varrho > d \geqslant 1$ we have

$$\int_{\mathbb{R}^d} g(r,z)\lambda^\varrho(z)\,\mathrm{d}z \leqslant 2^{(\varrho/2)-1} \cdot \int_{\mathbb{R}^d} g(r,z)(1+|z|^\varrho)\,\mathrm{d}z$$
$$= 2^{(\varrho/2)-1} \cdot \left[1 + \frac{2^{\varrho/2} \cdot \Gamma(\frac{\varrho+d}{2})}{\Gamma(\frac{d}{2})} \cdot r^{\varrho/2}\right].$$

□

Corollary 3.1.6. *We have*

$$\int_{\mathbb{R}^d} \mathrm{e}^{-jr} g(r,z)\lambda^\varrho(z)\,\mathrm{d}z \leqslant 2^{(\varrho/2)-1} \cdot \left[1 + \frac{\Gamma(\frac{\varrho+d}{2})}{\Gamma(\frac{d}{2})} \cdot \left(\frac{\varrho}{j\mathrm{e}}\right)^{\varrho/2}\right] =: \psi_j(\varrho)$$

for $j, r > 0$, $\varrho \in \{1, 2, \ldots\}$.

Proof. One easily checks that $r \to 2^{\varrho/2} \cdot \mathrm{e}^{-jr} r^{\varrho/2}$ has a maximum of size $\mathrm{e}^{-\varrho/2}(\varrho/j)^{\varrho/2}$. Putting this and Lemma 3.1.5 together one immediately obtains the claim. □

Lemma 3.1.7. *The inequality*

$$\lambda^\varrho(y) \leqslant 2^{\varrho/2} \cdot \lambda^\varrho(x-y)\lambda^\varrho(x)$$

holds for $x, y \in \mathbb{R}^d$, $\varrho \geqslant 1$.

Proof. We have

$$\lambda^\varrho(y) \leqslant (1 + 2 \cdot |x|^2 + 2 \cdot |y-x|^2)^{\varrho/2}$$
$$\leqslant 2^{\varrho/2} \cdot (1 + |x|^2 + |y-x|^2 + |x|^2|y-x|^2)^{\varrho/2}$$
$$= 2^{\varrho/2} \cdot \lambda^\varrho(x-y)\lambda^\varrho(x).$$

□

3.2. The case of Lipschitz continuous and linearly bounded coefficients. We need existence and uniqueness of a solution to (C) in the case when both f and σ satisfy (L) for the forthcoming construction. Though the proof is more or less standard we will demonstrate it very briefly because we need some of its particular conclusions.

Theorem 3.2.1. *Suppose that* ϑ *fulfils* (ϑ1) *and* (ϑ2) *and* f *and* σ *satisfy* (L). *Then the following holds:*

(i) (C) *possesses a unique solution* u *with* $u(t,.) \in \mathbb{H}_{\ell,p}$, $t \geqslant 0$, *where* ℓ *and* p *are given by* $(\vartheta 1)$.

(ii) $\vartheta \in \mathbb{C}_\ell$ *with* $\ell(x) = 1 + |x|^k$, $x \in \mathbb{R}^d$, *implies* $u \in \mathbb{C}^T_{\ell^*} \supset \mathbb{C}^T_\ell$ *for any* $T > 0$, *where*

$$\ell^*(x) = 1 + |x|^{\frac{1}{4}+\frac{1}{p}+k}.$$

P r o o f. 1° To point out the power of the various conditions required in the theorem let us assume first that ϑ satisfies $(\vartheta 1)$ and both f and σ are globally Lipschitz continuous. This already implies the existence of a random field u having all properties of Definition 1.1 except (iii): We will assume here for simplicity that both coefficients have the same Lipschitz constant L. Let $n \in \mathbb{N}$ and put

$$u_{n+1}(t,x) \colon = \Theta(t,x) + \varphi(t,x,u_n) + \mathcal{S}(t,x,u_n),\ (t,x) \in \mathbb{D}$$

and $u_0 = \Theta$. Let $p \geqslant 1$, $\ell(x) = 1 + |x|^k$ and let us note that

$$\mathbb{E}|u_{n+1}(t,x)|^p \leqslant 3^{p-1} \cdot \big[\mathbb{E}|\Theta(t,x)|^p + \mathbb{E}|\varphi(t,x,u_n)|^p + \mathbb{E}|\mathcal{S}(t,x,u_n)|^p\big].$$

Hölder's inequality and Lemma 3.1.4 lead to

$$\mathbb{E}|\Theta(t,x)|^p \leqslant c(m,p,k) \cdot \|\vartheta\|^p_{\ell,p} \cdot \ell^p(x),$$

proving $\sup\limits_{t\geqslant 0} \|\Theta(t,.)\|_{\ell,p} < \infty$ if $\vartheta \in \mathbb{H}_{\ell,p}$. Suppose now $\sup\limits_{t\geqslant 0} \|u_n(t,.)\|_{\ell,p} < \infty$. A simple estimate similar to the previous one leads to

$$\begin{aligned}\mathbb{E}|\Phi(t,x,u_n)|^p &\leqslant (2L)^p \cdot \left[1 + \int_0^t\!\!\int_{\mathbb{R}^d} G(t-s,x-y)\ell^p(y) \cdot \|u_n(s,.)\|^p_{\ell,p}\,\mathrm{d}y\,\mathrm{d}s\right]\\ &\leqslant c(L,p,m) \cdot (1 \vee \sup_{t\geqslant 0} \|u_n(t,.)\|^p_{\ell,p}) \cdot \ell^p(x),\end{aligned}$$

proving $\sup\limits_{t\geqslant 0} \|\Phi(t,.,u_n)\|_{\ell,p} < \infty$. Finally, the Burkholder-Gundy inequality and (L) lead similarly to

$$\sup_{t\geqslant 0} \|\mathcal{S}(t,.,u_n)\|_{\ell,p} \leqslant c(a,\ell,p,L,m) < \infty.$$

Hence we have also $\sup\limits_{t\geqslant 0} \|u_{n+1}(t,.)\|_{\ell,p} < \infty$. Let $j \in \mathbb{N}$ be arbitrary but fixed. The previous considerations imply

$$\psi_0(t) := \|u_j(t,.) - u_0(t,.)\|^p_{\ell,p} \leqslant c(a,\ell,p,L,m)$$

for any $t \geqslant 0$. Note that

$$\psi_n(t) := \|u_{n+j}(t,.) - u_n(t,.)\|_{\ell,p}^p \leqslant c(a,\ell,p,L,m) \cdot \int_0^t \psi_{n-1}(s)\,\mathrm{d}s,$$

$n \in \mathbb{N}$, $t \geqslant 0$, and $\sup_{t\geqslant 0} \psi_n(t) < \infty$, $n \in \mathbb{N}$. This inequality implies that $(u_n(t,.))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}_{\ell,p}$, $t \geqslant 0$, see Manthey, 1996, for details. It is routine to show that the corresponding limit u solves (I). Conditions (i), (ii), (iv) and (v) of Definition 1.1 follow by construction. Thus it remains to show (iii).

2° The above mentioned conclusions yield

$$\sup_{t\geqslant 0} \|u(t,.)\|_{\ell,p} < \infty.$$

Hence

$$\mathbb{E}\big[|\Phi(t,x,u)|^{2p} + |\mathcal{S}(t,x,u)|^{2p}\big] \leqslant c(a,\ell,L,m,p)\ell^{2p}(x).$$

If now $\ell(x) = 1 + |x|^k$, then Theorem 2.1.1 immediately implies the claim.

3° The uniqueness obviously follows from Theorem 2.2.1. This completes the proof of Theorem 3.2.1. □

4. Proofs

4.1. Proof of theorem 2.1.1. Let (t,x), $(s,y) \in \mathbb{D}_T$, $t \geqslant s$, be arbitrary but fixed. Introduce the abbreviations

$$H(r,z) := |G(t-r,x-z) - \mathbf{1}_{[0,s)}(r)G(s-r,y-z)|$$

and

$$H_+(r,z) := G(t-r,x-z) + \mathbf{1}_{[0,s)}(r)G(s-r,y-z).$$

(i) First we shall estimate the stochastic integral and then in an analogous way the remaining part containing F. The Burkholder-Gundy inequality yields

$$\mathbb{E}\bigg[\int_0^t\int_{\mathbb{R}^d} (G(t-r,x-z) - \mathbf{1}_{[0,s)}(r)G(s-r,y-z))\Sigma(r,z)\,\mathrm{d}W(r,z)\bigg]^{2p}$$
$$\leqslant c(p)a^p \cdot \mathbb{E}\bigg|\int_0^t \bigg[\int_{\mathbb{R}^d} |G(t-r,x-z) - \mathbf{1}_{[0,s)}(r)G(s-r,y-z)||\Sigma(r,z)|\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^p.$$

Due to the Cauchy-Schwarz inequality we further have

$$\leqslant c(a,p)\bigg|\int_0^t \bigg[\int_{\mathbb{R}^d} H(r,z)\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^{p/2} \cdot \mathbb{E}\bigg|\int_0^t \bigg[\int_{\mathbb{R}^d} H_+(r,z)\Sigma^2(r,z)\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^{p/2}.$$

To avoid additional variables let us put $\mu = \frac{1}{4}$ in Lemma 3.1.3 and obtain

$$\leqslant c(T,p)(|t-s|^{\frac{1}{2}} + |x-y|)^{p/2}$$

for the first integral term above. Furthermore, we have

$$\begin{aligned}
\mathbb{E}\bigg|\int_0^t \bigg[\int_{\mathbb{R}^d} H_+(r,z)\Sigma^2(r,z)\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^{p/2} \\
&\leqslant c(p)\cdot\bigg[\mathbb{E}\bigg|\int_0^s \bigg[\int_{\mathbb{R}^d} H_+(r,z)\Sigma^2(r,z)\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^{p/2} \\
&\quad + \mathbb{E}\bigg|\int_s^t \bigg[\int_{\mathbb{R}^d} H_+(r,z)\Sigma^2(r,z)\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^{p/2}\bigg] \\
&=: I_1(t,s,x,y) + I_2(t,s,x,y) =: I(t,s,x,y)
\end{aligned}$$

and

$$\begin{aligned}
I_1(t,s,x,y) \leqslant \mathbb{E}\bigg|\int_0^s \mathrm{e}^{-m(s-r)} \times \\
\times\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{m}{2}(s-r)}(\mathrm{e}^{-m(t-s)}g(t-r,x-z) + g(s-r,y-z))\Sigma^2(r,z)\,\mathrm{d}z\bigg]^2 \mathrm{d}r\bigg|^{p/2}.
\end{aligned}$$

Using twice Hölder's inequality we get

$$\begin{aligned}
&\leqslant \mathbb{E}\int_0^s \mathrm{e}^{-\frac{mp}{2}(s-r)}\bigg[\int_{\mathbb{R}^d}(\mathrm{e}^{-m(t-s)}g(t-r,x-z) + g(s-r,y-z))\Sigma^2(r,z)\,\mathrm{d}z\bigg]^p \mathrm{d}r \\
&\leqslant \int_0^s \mathrm{e}^{-\frac{mp}{2}(s-r)}\int_{\mathbb{R}^d}(\mathrm{e}^{-m(t-s)}g(t-r,x-z) + g(s-r,y-z))\cdot\mathbb{E}\Sigma^{2p}(r,z)\,\mathrm{d}z\,\mathrm{d}r.
\end{aligned}$$

Now we can apply the condition of the theorem to obtain

$$\leqslant c_0(p,T)\cdot\int_0^s \mathrm{e}^{-\frac{mp}{2}(s-r)}\int_{\mathbb{R}^d}(\mathrm{e}^{-m(t-s)}g(t-r,x-z) + g(s-r,y-z))\cdot\ell_*^k(z)\,\mathrm{d}z\,\mathrm{d}r,$$

where $\ell_*(z) := 1 + |z|$. Noting that

$$\exp(-m(t-s))\exp(-\tfrac{mp}{4}(s-r)) \leqslant \exp(-m(t-s))\exp(-\tfrac{m}{4}(s-r)) \leqslant \exp(-\tfrac{m}{4}(t-r))$$

we can continue the estimate with

$$\begin{aligned}
\leqslant c_0(p,T)\cdot\int_0^s \mathrm{e}^{\frac{mp}{4}(s-r)}\int_{\mathbb{R}^d}(\mathrm{e}^{-\frac{m}{4}(t-r)}g(t-r,x-z) \\
+ \mathrm{e}^{-\frac{m}{4}(s-r)}g(s-r,y-z))\cdot\ell_*^k(z)\,\mathrm{d}z\,\mathrm{d}r.
\end{aligned}$$

By Lemma 3.1.4 this is less than

$$\leqslant c_0(p,T)c(k)(\ell_*^k(x)+\ell_*^k(y)) \leqslant c(p,k,c_0(p,T))(1+|x|^k+|y|^k).$$

For the remaining term we obtain analogously

$$\begin{aligned} I_2(t,s,x,y) &\leqslant c(k)\cdot\int_s^t \mathrm{e}^{-\frac{m}{2}(t-r)}\cdot\mathbb{E}\Big[\int_{\mathbb{R}^d} g(t-r,x-z)\Sigma^2(r,z)\,\mathrm{d}z\Big]^p\,\mathrm{d}r \\ &\leqslant c(p,k,c_0(p,T))(1+|x|^k). \end{aligned}$$

Hence

$$I_2(t,s,x,y)\leqslant c(p,k,c_0(p,T))(1+|x|^k+|y|^k).$$

On the other hand, we observe that

$$\begin{aligned} (|t-s|^{\frac12}+|x-y|)^{p/2} &\leqslant (2n)^{p/4}\cdot(|t-s|^{\frac12}+|x-y|^{\frac12})^{p/2} \\ &\leqslant c\cdot n^{p/4}\cdot|(t,x)-(s,y)|^{p/4} \end{aligned}$$

provided $x,y\in[-n,n]^d$. Thus, putting $V(t,.\,)=V(T,.\,)$ for $t>T$ and $V(t,.\,)=0$ for $t\leqslant 0$ we have defined V on the whole $\mathbb{R}^{d+1}$. Moreover, one observes that

$$V(t,x)-V(s,y)=V(0\vee t\wedge T,x)-V(0\vee s\wedge T,y).$$

Redefining in the same way the term we have just estimated, we obtain by summing up

$$\leqslant c(a,k,m,p,c_0(p,T))\cdot|(t,x)-(s,y)|^{p/4}\cdot n^{\frac{p}{4}+k},$$

$(t,x),(s,y)\in[-n,n]^{d+1}$.

(ii) Next we deal with the F-term in a similar way. We observe that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}^d}(G(t-r,x-z)-\mathbf{1}_{[0,s)}(r)G(s-r,y-z))F(r,z)\,\mathrm{d}z\,\mathrm{d}r\Big]^{2p} \\ &\quad\leqslant \mathbb{E}\Big|\int_0^t\Big[\int_{\mathbb{R}^d}|(\mathrm{e}^{-\frac{m}{2}(t-r)}g(t-r,x-z)-\mathbf{1}_{[0,s)}(r)\mathrm{e}^{-\frac{m}{2}(s-r)} \\ &\qquad\qquad\times g(s-r,y-z)||F(r,z)|\,\mathrm{d}z\Big]^2\mathrm{d}r\Big|^{p/2}, \end{aligned}$$

i.e. the same initial situation as in the Σ-term with the only unessential difference that the exponential is $\mathrm{e}^{-\frac{m}{2}\cdot\tau}$ instead of $\mathrm{e}^{-m\tau}$. Hence we obtain the same estimate and conclude that

$$\leqslant c(k,m,p,c_0(p,T))\cdot|(t,x)-(s,y)|^{p/4}\cdot n^{\frac{p}{4}+k}.$$

Applying now Lemma 3.1.2 with $\delta = 2$ we have finally

$$|V(t,x)| \leqslant \Xi(a,k,m,p,c_0(p,T))(1+|(t,x)|^{\frac{1}{4}+\frac{2+k}{2p}})$$

and hence also

$$|V(t,x)| \leqslant \Xi(a,k,m,p,c_0(p,T),T)(1+|x|^{\frac{1}{4}+\frac{2+k}{2p}})$$

$\mathbb{P}$ a.s. for any $(t,x) \in \mathbb{D}_T$, proving the claim.

Remark. In the course of the proof we have frequently used Hölder's inequality where exponents $p/(p-2)$ appear. The corresponding factors are less than one and were therefore immediately omitted. However, here the requirement $p > 2$ is needed.

4.2. Proof of the existence and comparison results. Proof of Theorems 2.1.2 and 2.1.3. The conclusion basically follows from the respective proofs of Theorems 2.4 and 2.5 of Manthey (1996). Therefore, we will very briefly describe only the main ideas. First, f is pointwise monotonously approximated by $f_N := f \vee (-N)$. The mapping f_N can be obtained as the pointwise monotonous limit of Lipschitz continuous mappings for which existence, uniqueness and spatial growth estimates of the corresponding solutions are proved in Theorem 3.2.1 above. Condition (f2) combined with the Comparison Theorem 2.3.1 guarantees the monotonous convergence of the approximating solutions to the desired solution in Theorem 2.1.2. Because of the difference in the noise the various steps of the proof in Manthey (1996) differ in details from those needed here. But this difference can be easily handled by the procedures and estimates described in the proofs of Theorems 2.1.1 and 3.2.1 above. We will not repeat them here and omit further details.

Proof of Theorem 2.3.3. Here we use again the above described procedure and the pathwise uniqueness. Namely, first we obtain for the solution of the Lipschitz approximation $f^{(i)}_{N,M}$ of $f^{(i)}_N$ the relation

$$u^{(1)}_{N,M}(t,x) \geqslant u^{(2)}_{N,M}(t,x)$$

$\mathbb{P}$ a.s. for any $(t,x) \in \mathbb{D}$. There are subsequences of these solutions converging $\mathbb{P}$ a.s. pointwise to $u^{(i)}_N$ and preserving the above relation for the limits. Repeating this conclusion for these limits and using the uniqueness we immediately arrive at the desired result. For details of the corresponding constructions see again [5].

4.3. Proof of the uniqueness theorem. Let $N \geqslant 1$, define

$$\tau_N := \inf\{t \in [0,T]\colon\ |u(t,.)|^{\nu-1}_\ell \vee |v(t,.)|^{\nu-1}_\ell > N\} \wedge T,$$

where $\inf\emptyset=\infty$, and introduce the notation

$$z(t,x) :\, = \mathbb{E}\mathbf{1}_{\{t\leqslant\tau_N\}}(u(t,x)-v(t,x))^2.$$

Using (f4) we have

$$\begin{aligned}
z(t,x) \leqslant c(K)\cdot\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}^d} &G(t-s,x-y)\mathbf{1}_{\{s\leqslant\tau_N\}}|u(s,y)-v(s,y)| \\
&\times(1+|u(s,y)|^{\nu-1}+|v(s,y)|^{\nu-1})\,\mathrm{d}y\,\mathrm{d}s\Big]^2 \\
&+\mathbb{E}\mathbf{1}_{\{t\leqslant\tau_N\}}\Big[\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)(\sigma(u(s,y)-\sigma(v(s,y)))\,\mathrm{d}W(s,y)\Big]^2.
\end{aligned}$$

Note that

$$\mathbf{1}_{\{s\leqslant\tau_N\}}(1+|u(s,y)|^{\nu-1}+|v(s,y)|^{\nu-1})\leqslant 3\cdot\ell^{\nu-1}(y)\cdot N.$$

Now we can continue the estimate by

$$\begin{aligned}
&\leqslant c(N,K,\nu)\cdot\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\ell^{2\nu-2}(y)\,\mathrm{d}y\,\mathrm{d}s\times \\
&\quad\times\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\mathbb{E}\mathbf{1}_{\{s<\tau_N\}}|u(s,y)-v(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad+\mathbb{E}\mathbf{1}_{\{t\leqslant\tau_N\}}\Big[\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)(\sigma(u(s,y)-\sigma(v(s,y)))\,\mathrm{d}W(s,y)\Big]^2.
\end{aligned}$$

To handle the last term we consider it first in the cylindrical case. In this situation we have

$$\begin{aligned}
\mathbb{E}\mathbf{1}_{\{t\leqslant\tau_N\}}&\Big[\int_0^t\int_{\mathbb{R}} G(t-s,x-y)(\sigma(u(s,y)-\sigma(v(s,y)))\,\mathrm{d}W(s,y)\Big]^2 \\
&\leqslant c\cdot\int_0^t\int_{\mathbb{R}} G^2(t-s,x-y)\mathbb{E}\mathbf{1}_{\{s<\tau_N\}}|u(s,y)-v(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant c\cdot\int_0^t(t-s)^{-1/2}\int_{\mathbb{R}} G(t-s,x-y)z(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

In the nuclear case we arrive at

$$\begin{aligned}
\mathbb{E}\mathbf{1}_{\{t\leqslant\tau_N\}}&\Big[\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)(\sigma(u(s,y)-\sigma(v(s,y)))\,\mathrm{d}W(s,y)\Big]^2 \\
&\leqslant c\cdot\int_0^t\mathbb{E}\Big[\int_{\mathbb{R}^d} G(t-s,x-y)\mathbf{1}_{\{s<\tau_N\}}|u(s,y)-v(s,y)|\,\mathrm{d}y\Big]^2\,\mathrm{d}s \\
&\leqslant c\cdot\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)z(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Because of $1+(t-s)^{-1/2} \leqslant c(T)\cdot(t-s)^{-1/2}$ we have in both cases

$$\leqslant c\cdot \int_0^t (t-s)^{-1/2}\int_{\mathbb{R}^d} G(t-s,x-y)z(s,y)\,\mathrm{d}y\,\mathrm{d}s$$

with $d=1$ in the cylindrical case. Hence

$$z(t,x)\leqslant c\cdot \ell^{2\nu-2}(x)\cdot \int_0^t (t-s)^{-1/2}\int_{\mathbb{R}^d} G(t-s,x-y)z(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$

where $c=c(T,N,K,\nu,a,c_\sigma)$. Now we iterate this inequality and obtain

$$\leqslant c^n\cdot \ell^{2\nu-2}(x)\cdot \int_0^{s_0}\dots\int_0^{s_{n-1}}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}\prod_{i=0}^{n-1}(s_i-s_{i+1})^{-1/2}$$
$$\times G(s_i-s_{i+1},x_i-x_{i+1})\ell^{2\nu-2}(x_i)z(s_{i+1},x_{i+1})\,\mathrm{d}x_n\dots\mathrm{d}x_1\,\mathrm{d}s_n\dots\mathrm{d}s_1,$$

where $s_0=t$ and $x_0=x$. Using $z(t,x)\leqslant \ell^2(x)N^2, t\in[0,T]$, and Lemma 3.1.4 and trivial estimates we observe that

$$\leqslant c^n\cdot \ell^{2(n+1)\nu}(x)\cdot N^2\cdot \int_0^{s_0}\dots\int_0^{s_{n-1}}\prod_{i=0}^{n-1}(s_i-s_{i+1})^{-1/2}\,\mathrm{d}s_n\dots\mathrm{d}s_1.$$

For the integral with respect to $\mathrm{d}s_n$ we get $\frac{1}{2}\cdot s_{n-1}^{1/2}$. Then going further step by step we always arrive at integrals of the type

$$s_{n-k}^{k/2}\cdot\int_0^1(1-r)^{-1/2}r^{(k-1)/2}\,\mathrm{d}r = s_{n-k}^{k/2}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}.$$

This leads to the estimate

$$z(t,x)\leqslant c\cdot \ell^{4\nu}(x)\cdot N^2\cdot\frac{(\ell^{2\nu}(x)c(T))^{n-1}}{(n-1)!}\to 0,\ n\to\infty.$$

This implies

$$\mathbb{P}(\{|u(t,x)-v(t,x)|=0\})+\mathbb{P}(\{t>\tau_N\})\geqslant 1.$$

Because of $\lim_{N\to\infty}\mathbb{P}(\{t>\tau_N\})=0$ and the pathwise continuity of both u and v we obtain

$$\mathbb{P}\Big(\Big\{\sup_{x\in[-M,M]}|u(t,x)-v(t,x)|=0\Big\}\Big)=1$$

for any $M \in \mathbb{N}$ and hence

$$\mathbb{P}\Big(\Big\{\sup_{x\in\mathbb{R}^d} |u(t,x) - v(t,x)| = 0\Big\}\Big) = 1,$$

proving the theorem.

4.4. Proof of the stability theorem. First we shall explain the simple strategy of the proof that is a combination of a comparison technique with (f2) or (f2)*.

For a real-valued mapping ψ we will use the notation $\psi^+ := \psi \vee 0$ and $\psi^- := \psi \wedge 0$. The solution of (C) with $(\vartheta^-, f^-, \sigma)$ will be denoted by $\underline{u}$ and $\bar{u}$ stands for the solution of (C) with $(\vartheta^+, f^+, \sigma)$.

Both assertions in Theorem 2.4.3 will be proved by the same method. Therefore, the only difference will be indicated at the corresponding places. To show the assertions we distinguish between two cases.

(i) Suppose $f(0) \leqslant 0$. Introduce the mapping $f_*(u) := f(u) - f(0) \geqslant f(u)$. Because of $f_*(0) = 0$ Theorem 2.3.1 leads to $\bar{u}_* \geqslant 0$. Moreover, the same theorem ensures $\bar{u}_* \geqslant \bar{u} \geqslant u \geqslant \underline{u}$ and $\underline{u} \leqslant 0$. Hence

$$|u(t,.)|_\varrho^2 \leqslant |\bar{u}_*(t,.)|_\varrho^2 \vee |\underline{u}(t,.)|_\varrho^2.$$

In this way, it is enough to estimate both terms on the right hand side in mean square. Because both $\bar{u}_*$ and $\underline{u}$ are sign-stable we can use for this purpose respectively (f2) or (f2)* in the corresponding equations.

(ii) In the case $f(0) \geqslant 0$ we have $f_*(u) \leqslant f(u)$ and get analogously $\underline{u}_* \leqslant \underline{u} \leqslant u \leqslant \bar{u}$ and $\underline{u}_* \leqslant 0$ as well as $\bar{u} \geqslant 0$. Therefore, we are in principle in the same situation as before and consequently, it suffices to handle only one of them.

Note that if f has properties (f2) and (f4) then $f^\pm$ and f_* have them, too. Consequently, the corresponding problems (C) have pathwise unique solutions. Moreover, due to Theorem 2.3.3 the comparison principle holds correspondingly.

We shall show now the estimate stated above. We have

$$\begin{aligned}\bar{u}_*(t,x) = &\int_{\mathbb{R}^d} G(t,x-y)\vartheta^+(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y) f_*^+(\bar{u}_*(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\sigma(\bar{u}_*(s,y))\,\mathrm{d}W(s,y),\end{aligned}$$

$t > 0$, $x \in \mathbb{R}^d$. Next we shall estimate separately the three terms on the right hand side in mean square in $\mathbb{L}^2_\varrho(\mathbb{R}^d)$. Obviously, for the first term we observe

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} G(t,x-y)\vartheta^+(y)\,\mathrm{d}y\right]^2 \mu(\,\mathrm{d}x) \\ &\quad\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t,x-y)\,\mathrm{d}y \cdot \int_{\mathbb{R}^d} G(t,x-y)\mathbb{E}(\vartheta^+(y))^2\,\mathrm{d}y\lambda^{-\varrho}(x)\,\mathrm{d}x.\end{aligned}$$

Using Lemma 3.1.7 we get

$$\leqslant 2^{\varrho/2} \cdot \mathrm{e}^{-mt} \cdot \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t, x-y)\mathbb{E}(\vartheta^+(y))^2 \cdot \lambda^{\varrho}(x-y)\lambda^{-\varrho}(y)\,\mathrm{d}y\,\mathrm{d}x,$$

which is by Lemma 3.1.5

$$\leqslant 2^{\varrho/2} \cdot \psi_1(\varrho)\mathrm{e}^{-mt} \cdot \int_{\mathbb{R}^d} \mathbb{E}(\vartheta^+(y))^2 \mu(dy) \leqslant 2^{\varrho/2} \cdot \psi_1(\varrho)\mathrm{e}^{-mt} \cdot \mathbb{E}|\vartheta|^2_{\varrho}.$$

We have to estimate the second term first with f_*^+ and then with f^- using in both cases the sign stability of the corresponding solution (i.e. $\bar{u}_* \geqslant 0, \underline{u} \leqslant 0$). In the first case we obtain by using the Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d}\Big[\int_0^t \exp\Big(-\frac{m}{4}(t-s)\Big)\int_{\mathbb{R}^d}\exp\Big(-\frac{3m}{4}(t-s)\Big) \\
&\qquad\times g(t-s, x-y) f_*^+(\bar{u}_*(s,y))\,\mathrm{d}y\,\mathrm{d}s\Big]^2 \mu(\,\mathrm{d}x) \\
&\quad\leqslant \mathbb{E}\int_{\mathbb{R}^d}\int_0^t \exp\Big(-\frac{m}{2}(t-s)\Big)\,\mathrm{d}s \\
&\qquad\times \int_0^t \Big[\int_{\mathbb{R}^d}\exp\Big(-\frac{3m}{4}(t-s)\Big) g(t-s,x-y) f_*^+(\bar{u}_*(s,y))\,\mathrm{d}y\Big]^2 \mathrm{d}s\mu(\,\mathrm{d}x) \\
&\quad\leqslant 2\cdot\int_{\mathbb{R}^d}\int_0^t \exp\Big(-\frac{3m}{2}(t-s)\Big)\int_{\mathbb{R}^d} g(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad\times \int_{\mathbb{R}^d} g(t-s,x-y)\mathbb{E}(f_*^+(\bar{u}_*(s,y)))^2\,\mathrm{d}y\,\mathrm{d}s\mu(\,\mathrm{d}x) \\
&\quad= 2\cdot \mathrm{e}^{-mt}\int_0^t \mathrm{e}^{ms}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\exp\Big(-\frac{m}{2}(t-s)\Big) \\
&\qquad\times g(t-s,x-y)\mathbb{E}(f_*^+(\bar{u}_*(s,y)))^2\,\mathrm{d}y\mu(\,\mathrm{d}x)\,\mathrm{d}s.
\end{aligned}$$

Lemma 3.1.7 leads to

$$\begin{aligned}
\leqslant 2^{(\varrho/2)+1} \cdot \mathrm{e}^{-mt} \cdot \int_0^t \mathrm{e}^{ms}\int_{\mathbb{R}^d} &\mathbb{E}(f_*^+(\bar{u}_*(s,y)))^2 \\
&\times \int_{\mathbb{R}^d} g(t-s,x-y)\exp(-\frac{m}{2}(t-s))\lambda^{\varrho}(x-y)\lambda^{-\varrho}(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

while Corollary 3.1.6 gives

$$\leqslant 2^{(\varrho/2)+1} \cdot \mathrm{e}^{-mt} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot \int_0^t\int_{\mathbb{R}^d} \mathrm{e}^{ms}\mathbb{E}(f_*^+(\bar{u}_*(s,y)))^2\mu(\,\mathrm{d}y)\,\mathrm{d}s.$$

At this place we have to distinguish between the two cases formulated in Theorem 2.4.3. Because of $\bar{u}_* \geqslant 0$ we can apply (f2) to obtain

$$\leqslant 2^{(\varrho/2)+2} \cdot \mathrm{e}^{-mt} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 \cdot \int_0^t \int_{\mathbb{R}^d} \mathrm{e}^{ms}(1 + \mathbb{E}\bar{u}_*^2(s,y)))\mu(\,\mathrm{d}y)\,\mathrm{d}s.$$

In case (f2)* there is no additional summand of size one under the intergral, i.e., in this situation we have simply

$$\leqslant 2^{(\varrho/2)+2} \cdot \mathrm{e}^{-mt} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 \cdot \int_0^t \int_{\mathbb{R}^d} \mathrm{e}^{ms} \cdot \mathbb{E}\bar{u}_*^2(s,y)\mu(\,\mathrm{d}y)\,\mathrm{d}s.$$

To handle the additional summand we introduce the abbreviation $c_\mu := \mu(\mathbb{R}^d)$ in case (f2) and $c_\mu := 0$ in the remaining case. Now we can continue by

$$\leqslant 2^{(\varrho/2)+2} \cdot \mathrm{e}^{-mt} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 \cdot \int_0^t \mathrm{e}^{ms}\left[c_\mu + \int_{\mathbb{R}^d} \mathbb{E}\bar{u}_*^2(s,y)\mu(\,\mathrm{d}y)\right]\mathrm{d}s.$$

Note that in the case of $\underline{u}$ we have $f^- \leqslant 0$ and $\underline{u} \leqslant 0$. Consequently,

$$(f^-)^2(\underline{u}(t,x)) \leqslant c_f^2(1 - \underline{u}(t,x))^2 \leqslant 2c_f^2(1 + \underline{u}^2(t,x)).$$

That means simply that we are just in the situation we have already discussed. Both cases give the same estimate. In this way, we finally have

$$\leqslant 2^{(\varrho/2)+2} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 \cdot \mathrm{e}^{-mt} \cdot \left[c_\mu \cdot \mathrm{e}^{mt} + \int_0^t \mathrm{e}^{ms} \cdot \mathbb{E}|\bar{u}_*^2(s,.)|_\varrho^2\,\mathrm{d}s\right].$$

It remains to estimate the third term of the initial relation above. We get

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d}\left[\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\sigma(\bar{u}_*(s,y))\,\mathrm{d}W(s,y)\right]^2 \mu(\,\mathrm{d}x)\\
&\leqslant a \cdot c_\sigma^2 \cdot \int_{\mathbb{R}^d}\int_0^t \mathbb{E}\left[\int_{\mathbb{R}^d} G(t-s,x-y)\bar{u}_*^2(s,y)\,\mathrm{d}y\right]^2 \mu(\,\mathrm{d}x)\\
&\leqslant a \cdot c_\sigma^2 \cdot \int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\,\mathrm{d}y \cdot \int_{\mathbb{R}^d} G(t-s,x-y) \cdot \mathbb{E}\bar{u}_*^2(s,y)\,\mathrm{d}y\,\mathrm{d}s\mu(\,\mathrm{d}x)\\
&\leqslant 2^{\varrho/2} \cdot a \cdot c_\sigma^2 \cdot \int_{\mathbb{R}^d}\int_0^t \mathrm{e}^{-m(t-s)} \cdot \int_{\mathbb{R}^d} G(t-s,x-y)\lambda^\varrho(x-y)\lambda^{-\varrho}(y)\mathbb{E}\bar{u}_*^2(s,y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\\
&\leqslant 2^{\varrho/2} \cdot a \cdot c_\sigma^2 \cdot \psi_1(\varrho)\mathrm{e}^{-mt} \cdot \int_0^t \mathrm{e}^{ms} \cdot \mathbb{E}|\bar{u}_*(s,.)|_\varrho^2\,\mathrm{d}s.
\end{aligned}$$

Because we have only used ($\sigma 1$) and ($\sigma 2$), there is no difference from the case when we have to estimate the terms with $\underline{u}$. Hence in both cases we get in all three terms the same bounds.

To sum up let

$$\varphi(t) := \mathrm{e}^{mt} \cdot \mathbb{E}|\bar{u}_*(t,.)|_\varrho^2, \quad t \geqslant 0.$$

We have

$$\varphi(t) \leqslant 2^{\varrho/2} \cdot \psi_1(\varrho) \cdot \mathbb{E}|\vartheta|_\varrho^2 + 2^{(\varrho/2)+2} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 \cdot \left[\mathrm{e}^{mt} c_\mu + \int_0^t \varphi(s)\,\mathrm{d}s\right]$$
$$+ 2^{\varrho/2} \cdot a \cdot c_\sigma^2 \cdot \psi_1(\varrho) \cdot \int_0^t \varphi(s)\,\mathrm{d}s.$$

Denoting

$$c_1 := 2^{\varrho/2} \cdot \psi_1(\varrho) \cdot \mathbb{E}|\vartheta|_\varrho^2,$$
$$c_2 := 2^{(\varrho/2)+2} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 \cdot c_\mu,$$

and

$$c_3 := 2^{(\varrho/2)+2} \cdot \psi_{\frac{1}{2}}(\varrho) \cdot c_f^2 + 2^{\varrho/2} \cdot a \cdot c_\sigma^2 \cdot \psi_1(\varrho)$$

we get

$$\varphi(t) \leqslant c_1 + c_2 \mathrm{e}^{mt} + c_3 \cdot \int_0^t \varphi(s)\,\mathrm{d}s.$$

Since φ is bounded on each finite interval we can apply Gronwall's lemma to obtain

$$\varphi(t) \leqslant c_1 + c_2 \mathrm{e}^{mt} + c_3 \cdot \int_0^t (c_1 + c_2 \mathrm{e}^{ms}) \exp(c_3(t-s))\,\mathrm{d}s$$
$$\leqslant c_1 \cdot \exp(c_3 t) + c_2\Big(1 + \frac{c_3}{m - c_3}\Big)\mathrm{e}^{mt}, \ t \geqslant 0.$$

Consequently,

$$\mathbb{E}|\bar{u}_*(t,.)|_\varrho^2 \leqslant c_2\Big(1 + \frac{c_3}{m - c_3}\Big) + c_1 \exp(-(m - c_3)t), \ t \geqslant 0.$$

Thus in the case $m > c_3$ there exist positive constants $\beta = m - c_3$ and

$$M = c_2\Big(1 + \frac{c_3}{m - c_3}\Big)$$

such that

$$\mathbb{E}|\bar{u}_*(t,.)|_\varrho^2 \leqslant 2^{\varrho/2} \cdot \psi_1(\varrho) \cdot \mathbb{E}|\vartheta|_\varrho^2 \cdot \mathrm{e}^{-\beta t} + M.$$

The same holds for $\underline{u}$ and hence for u. Therefore, the ultimate exponential boundedness is proved.

Under condition (f2)* we get $c_\mu = 0$. This leads to

$$\mathbb{E}|u(t,.)|_\varrho^2 \leqslant 2^{\varrho/2} \cdot \psi_1(\varrho) \cdot \mathbb{E}|\vartheta|_\varrho^2 \cdot \mathrm{e}^{-\beta t},$$

proving part (ii) of the assertion.

Remark. Note that we need the assumption $m \geqslant 1$ only for the proof of the stability results. In fact, we have used Corollary 3.1.6 to obtain the factors ψ_m and $\psi_{m/2}$ in the previous proof, which are smaller than ψ_1 and $\psi_{1/2}$, respectively, if $m \geqslant 1$. In this way we avoid the additional dependence of c_1, c_2 and especially c_3 on m.

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