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Density of a Family of Linear Varietes*

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Abstract

The measurability of the family, made up of the family of plane pairs and the family of lines in 3-dimensional space A_3, is stated and its density is given.

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1 Introduction

A measure on a family of geometric objects can be introduced by assigning to each object a point of an auxiliary space and considering a suitable measure on that space. In general the dimension of the auxiliary space is equal to the number of parameters on which the geometric objects depend. A basic problem is to specify measures which are invariant with respect to a given group of transformations which map the family onto itself.

This problem was first considered by Crofton [3] who specified the invariant measure on the family of all straight lines in Euclidean 2-space E^2. This was extended to E^3 by Deltheil [4] and Chern [1] first considered families of geometric objects in projective space.

Santaló [9] calculated measures of certain families of varieties with respect to three different groups and found that these were equal. Stoka [10] studied the family of parabolas. He proved that a family is measurable if it is measurable with respect to its maximal group of invariance

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However Cirlincione [2] found a measurable family of varieties even though the family was not measurable with respect to the maximal group of invariance. This proves that the Stoka's condition is not necessary.

In Section 2 we provide background and definitions and in Section 3 we prove that the family of varieties, where each variety is a pair consisting of two hyperplanes and a straight line in 3-dimensional affine space A_3 is measurable.

2 Background

Let $\mathcal{H}_n$ be an n-dimensional space with coordinates $x_1, x_2, \ldots, x_n$ in which a Lie group of transformations acts.

Let G_r be one of its subgroups defined by the equations

$$y_i = f_i(x_{1,}x_2, \ldots, x_n; a_1, a_2, \ldots, a_r) \qquad (i = 1, 2, \ldots, n) \qquad (\#)$$

where $a_1, a_2, \ldots, a_r$ are basic parameters.

Definition 1 The function $F(x_{1,}x_2, \ldots, x_n)$ is an integral invariant function of the group (#), if

$$\int_{\mathcal{A}_x} F(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n = \int_{\mathcal{A}_y} F(y_1 y_2, \ldots, y_n)\, dy_1 dy_2 \ldots dy_n)$$

for each measurable set of points $\mathcal{A}_x$ of the space $\mathcal{H}_n$.

Theorem 1 *The integral invariant functions of the group (#) are the solutions of the following Deltheil's system of partial differential equations:*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\xi_h^i(x) F(x)\right] = 0 \qquad (h = 1, 2, \ldots, r),$$

where $\xi_h^i(x)$ are the coefficients of the infinitesimal transformations of the group (#) (see [4], p. 28).

Definition 2 A measurable Lie group of transformations is a group which admits only one integral invariant function (up to a multiplicative constant).

Let G be a group which leaves globally invariant a family $\Im$ of varietes in $\mathcal{H}_n$. To G there is associated a group H (isomorphic to G) of transformations acting on the (auxiliary) space of parameters of the family.

Definition 3 A family $\Im$ is measurable with respect to G if H is measurable in the sense of Definition 2. If Φ is its integral invariant function, then the measure of $\Im$ with respect to the group G is given by

$$\mu_G = \int_{\mathcal{A}_\alpha} \Phi(\alpha_1, \alpha_2, \ldots, \alpha_q)\, d\alpha_1 d\alpha_2 \ldots d\alpha_q,$$

where $\mathcal{A}_\alpha$ is the set of points of the auxiliary space which corresponds to the family $\Im$.

Definition 4 A family $\Im$ of varieties is measurable if the measures with respect to every group of invariance of the family are equal, if they exist.

Theorem 2 (Stoka's first condition) *If the group $\overline{H}$ associated to the maximal group of invariance of $\Im$ (where the only transformation, which leaves invariant each element of the family, is the identity) is measurable, the family is measurable.*

Theorem 3 (Stoka's second condition) *If $\overline{H}$ is not measurable and there are two measurable subgroups with different integral invariant functions, then $\Im$ is not measurable.*

3 Measurability of the family $\Im_{10}$

Theorem 4 *The family of varieties,where each variety is consisted of two planes and a straight line in 3-dimensional affine space A_3, is measurable.*

Let us consider the family of plane pairs and the family of lines in the affine space A_3 (suppose that planes and lines are in general position)

$$\Im_{10}: \begin{cases} b_1x_1 + b_2x_2 + b_3x_3 = 1, \\ c_1x_1 + c_2x_2 + c_3x_3 = 1, \\ x_1 = l_1x_3 + q_1 \\ x_2 = l_2x_3 + q_2 \end{cases}$$

which depend on 10 parameters $b_1, b_2, b_3, c_1, c_2, c_3, l_1, l_2, q_1, q_2$.

Let G_{12} be the affinity group given by the equations

$$G_{12}: \begin{cases} x_1 = p_{11}x'_1 + p_{12}x'_2 + p_{13}x'_3 + \alpha_1 \\ x_2 = p_{21}x'_1 + p_{22}x'_2 + p_{23}x'_3 + \alpha_2 \\ x_3 = p_{31}x'_1 + p_{32}x'_2 + p_{33}x'_3 + \alpha_3 \end{cases}$$

and let $\sum_{i=1}^{3} b_i\alpha_i \neq 1$, $\sum_{i=1}^{3} c_i\alpha_i \neq 1$.

We put

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

$$L = \begin{pmatrix} l_1 \\ l_2 \\ 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix}, \quad X' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix},$$

$$P = (p_{ij}) \ \ (i, j = 1, 2, 3) \text{ with } \det P \neq 0, \quad A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

so that we obtain

$$\begin{aligned} \Im_{10} &: {}^tB \cdot X = 1,\ {}^tC \cdot X = 1,\ X = L \cdot x_3 + Q \\ G_{12} &: X = P \cdot X' + A \end{aligned} \tag{1}$$

Now we see how the 10 parameters of the family $\Im_{10}$ change by applying any transformation T of G_{12}.

With a similar meaning of B', C', L', Q' the new variety is given by the equations

$$ {}^tB' \cdot X' = 1,\ {}^tC' \cdot X' = 1 \tag{2}$$

$$ X' = L' \cdot x_3' + Q'. \tag{3}$$

From (1) we have

$$\begin{aligned} {}^tB \cdot X &= {}^tB \cdot (P \cdot X' + A) = {}^tB \cdot P \cdot X' + {}^tB \cdot A = 1 \\ {}^tC \cdot X &= {}^tC \cdot (P \cdot X' + A) = {}^tC \cdot P \cdot X' + {}^tC \cdot A = 1 \end{aligned}$$

hence

$$ {}^tB \cdot P \cdot X' = 1 - {}^tB \cdot A, \qquad {}^tC \cdot P \cdot X' = 1 - {}^tC \cdot A.$$

Finally, dividing by $1-{}^tB \cdot A$ and $1-{}^tC \cdot A$ respectively, we obtain

$$\frac{1}{1 - {}^tB \cdot A}({}^tB \cdot P)X' = 1, \qquad \frac{1}{1 - {}^tC \cdot A}({}^tC \cdot P)X' = 1. \tag{4}$$

In the same way we obtain

$$\begin{aligned} X &= L \cdot x_3 + Q \Rightarrow P \cdot X' + A = L \cdot (p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3) + Q \\ &\Rightarrow P \cdot X' = L \cdot (p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3) + Q - A \end{aligned}$$

i.e.

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ 1 \end{pmatrix} \cdot (p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3) + \begin{pmatrix} q_1 - \alpha_1 \\ q_2 - \alpha_2 \\ 0 - \alpha_3 \end{pmatrix}$$

or equivalently

$$\begin{aligned} p_{11}x_1' + p_{12}x_2' + p_{13}x_3' &= l_1(p_{31}x_1' + p_{32}x_2' + p_{33}x_3') + l_1\alpha_3 + (q_1 - \alpha_1) \\ p_{21}x_1' + p_{22}x_2' + p_{23}x_3' &= l_2(p_{31}x_1' + p_{32}x_2' + p_{33}x_3') + l_2\alpha_3 + (q_2 - \alpha_2) \\ p_{33}x_3' &= p_{33}x_3' + \alpha_3 + (0 - \alpha_3) \end{aligned}$$

hence

$$\begin{aligned} (p_{11} - l_1p_{31})x_1' + (p_{12} - l_1p_{32})x_2' &= (l_1p_{33} - p_{13})x_3' + l_1\alpha_3 + (q_1 - \alpha_1) \\ (p_{21} - l_2p_{31})x_1' + (p_{22} - l_2p_{32})x_2' &= (l_2p_{33} - p_{23})x_3' + l_2\alpha_3 + (q_2 - \alpha_2) \\ p_{33}x_3' &= p_{33}x_3' \end{aligned}$$

Omitting the last identity and using the matrix form, we have

$$\begin{pmatrix} p_{11}-l_1p_{31} & p_{12}-l_1p_{32} \\ p_{21}-l_2p_{31} & p_{22}-l_2p_{32} \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} l_1p_{33}-p_{13} \\ l_2p_{33}-p_{23} \end{pmatrix} x_3' + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\alpha_3 + \begin{pmatrix} q_1-\alpha_1 \\ q_2-\alpha_2 \end{pmatrix} \tag{5}$$

Putting

$$R = \begin{pmatrix} p_{11}-l_1p_{31} & p_{12}-l_1p_{32} \\ p_{21}-l_2p_{31} & p_{22}-l_2p_{32} \end{pmatrix},$$

we have

$$R^{-1} = \frac{1}{\triangle}\begin{pmatrix} p_{22}-l_2p_{32} & -p_{12}+l_1p_{32} \\ -p_{21}+l_2p_{31} & p_{22}-l_1p_{31} \end{pmatrix}$$

where

$$\triangle = \|R\| = (p_{11}-l_1p_{31})(p_{22}-l_2p_{32}) - (p_{12}-l_2p_{32})(p_{21}-l_2p_{31}).$$

Then we can write (5) as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = R^{-1}\begin{pmatrix} l_1p_{33}-p_{13} \\ l_2p_{33}-p_{23} \end{pmatrix} x_3' + R^{-1}\left[\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\alpha_3 + \begin{pmatrix} q_1-\alpha_1 \\ q_2-\alpha_2 \end{pmatrix}\right]$$

or

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \frac{1}{\triangle}\begin{pmatrix} p_{22}-l_2p_{32} & -p_{12}+l_1p_{32} \\ -p_{21}+l_2p_{31} & p_{11}-l_1p_{31} \end{pmatrix}\cdot\begin{pmatrix} l_1p_{33}-p_{13} \\ l_2p_{33}-p_{23} \end{pmatrix} x_3' \\ &+ \frac{1}{\triangle}\begin{pmatrix} p_{22}-l_2p_{32} & -p_{12}+l_1p_{32} \\ -p_{21}+l_2p_{31} & p_{11}-l_1p_{31} \end{pmatrix}\cdot\begin{pmatrix} l_1\alpha_3+q_1-\alpha_1 \\ l_2\alpha_3+q_2-\alpha_2 \end{pmatrix} \end{aligned} \tag{6}$$

By comparing (2) and (3) with (4) and (6) respectively, we have the connections between the new parameters b_1', b_2', b_3', c_1', c_2', c_3', l_1', l_2', q_1', q_2' and the initial ones:

$$\begin{aligned} b_1' &= \sum_{i=1}^{3} b_ip_{i1}\cdot\frac{1}{1-\sum_{i=1}^{3} b_i\alpha_i} \qquad b_2' = \sum_{i=1}^{3} b_ip_{i2}\cdot\frac{1}{1-\sum_{i=1}^{3} b_i\alpha_i} \\ b_3' &= \sum_{i=1}^{3} b_ip_{i3}\cdot\frac{1}{1-\sum_{i=1}^{3} b_i\alpha_i} \qquad c_1' = \sum_{i=1}^{3} c_ip_{i1}\cdot\frac{1}{1-\sum_{i=1}^{3} c_i\alpha_i} \\ c_2' &= \sum_{i=1}^{3} c_ip_{i2}\cdot\frac{1}{1-\sum_{i=1}^{3} c_i\alpha_i} \qquad c_3' = \sum_{i=1}^{3} c_ip_{i3}\cdot\frac{1}{1-\sum_{i=1}^{3} c_i\alpha_i} \\ l_1' &= \frac{1}{\triangle}\left[(p_{22}-l_2p_{32})(l_1p_{33}-p_{13}) + (-p_{12}+l_1p_{32})(l_2p_{33}-p_{23})\right] \\ l_2' &= \frac{1}{\triangle}\left[(-p_{21}+l_2p_{31})(l_1p_{33}-p_{13}) + (p_{11}-l_1p_{31})(l_2p_{33}-p_{23})\right] \\ q_1' &= \frac{1}{\triangle}\left[(p_{22}-l_2p_{32})(l_1\alpha_3+q_1-\alpha_1) + (-p_{12}+l_1p_{32})(l_2\alpha_3+q_2-\alpha_2)\right] \\ q_2' &= \frac{1}{\triangle}\left[(-p_{21}+l_2p_{31})(l_1\alpha_3+q_1-\alpha_1) + (p_{11}-l_1p_{31})(l_2\alpha_3+q_2-\alpha_2)\right] \end{aligned} \tag{7}$$

In the 10-dimensional parameter space $\mathcal{A}_{10}$, (7) are the equations of group H_{12} which is associated to G_{12} (operating in 3-dimensional space A_3).

Also group H_{12} depends on *twelve* parameters p_{11}, p_{21}, p_{31}, p_{12}, p_{22}, p_{32}, p_{13}, p_{23}, p_{33}, α_1, α_2, α_3 and the *unit* $o \in H_{12}$ (as for G_{12}) corresponds to the values of the parameters

$$p_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{and} \quad \alpha_i = 0 \ \ (i, j = 1, 2, 3).$$

Now we construct the matrix whose elements are the coefficients of the infinitesimal transfomations of H_{12} and we note that the columns of this matrix are the derivates of b'_1, b'_2, b'_3, c'_1, c'_2, c'_3, l'_1, l'_2, q'_1, q'_2 with respect to parameters p_{ij}, $i, j = 1, 2, 3$ and α_i, $i = 1, 2, 3$:

$$\begin{array}{lll}
\left(\frac{\partial b'_1}{\partial p_{11}}\right)_o = b_1, & \left(\frac{\partial b'_2}{\partial p_{11}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{11}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{21}}\right)_o = b_2, & \left(\frac{\partial b'_2}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{21}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{31}}\right)_o = b_3, & \left(\frac{\partial b'_2}{\partial p_{31}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{31}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{12}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{12}}\right)_o = b_1, & \left(\frac{\partial b'_3}{\partial p_{12}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{22}}\right)_o = b_2, & \left(\frac{\partial b'_3}{\partial p_{22}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{32}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{32}}\right)_o = b_3, & \left(\frac{\partial b'_3}{\partial p_{32}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{13}}\right)_o = b_1, \\
\left(\frac{\partial b'_1}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{23}}\right)_o = b_2, \\
\left(\frac{\partial b'_1}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{33}}\right)_o = b_3, \\
\left(\frac{\partial b'_1}{\partial \alpha_1}\right)_o = b_1^2, & \left(\frac{\partial b'_2}{\partial \alpha_1}\right)_o = b_2 b_1, & \left(\frac{\partial b'_3}{\partial \alpha_1}\right)_o = b_3 b_1, \\
\left(\frac{\partial b'_1}{\partial \alpha_2}\right)_o = b_1 b_2, & \left(\frac{\partial b'_2}{\partial \alpha_2}\right)_o = b_2^2, & \left(\frac{\partial b'_3}{\partial \alpha_2}\right)_o = b_3 b_2, \\
\left(\frac{\partial b'_1}{\partial \alpha_3}\right)_o = b_1 b_3, & \left(\frac{\partial b'_2}{\partial \alpha_3}\right)_o = b_2 b_3, & \left(\frac{\partial b'_3}{\partial \alpha_3}\right)_o = b_3^2, \\
\left(\frac{\partial c'_1}{\partial p_{11}}\right)_o = c_1, & \left(\frac{\partial c'_2}{\partial p_{11}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{11}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{21}}\right)_o = c_2, & \left(\frac{\partial c'_2}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{21}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{31}}\right)_o = c_3, & \left(\frac{\partial c'_2}{\partial p_{31}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{31}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{12}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{12}}\right)_o = c_1, & \left(\frac{\partial c'_3}{\partial p_{12}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{22}}\right)_o = c_2, & \left(\frac{\partial c'_3}{\partial p_{22}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{32}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{32}}\right)_o = c_3, & \left(\frac{\partial c'_3}{\partial p_{32}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{13}}\right)_o = c_1,
\end{array}$$

$$
\begin{array}{llll}
\left(\frac{\vartheta c_1'}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial c_2'}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial c_3'}{\partial p_{23}}\right)_o = c_2, & \\
\left(\frac{\partial c_1'}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial c_2'}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial c_3'}{\partial p_{33}}\right)_o = c_3, & \\
\left(\frac{\partial c_1'}{\partial \alpha_1}\right)_o = c_1^2, & \left(\frac{\partial c_2'}{\partial \alpha_1}\right)_o = c_2c_1, & \left(\frac{\partial c_3'}{\partial \alpha_1}\right)_o = c_3c_1, & \\
\left(\frac{c_1'}{\partial \alpha_2}\right)_o = c_1c_2, & \left(\frac{\partial c_2'}{\partial \alpha_2}\right)_o = c_2^2, & \left(\frac{\partial c_3'}{\partial \alpha_2}\right)_o = c_3c_2, & \\
\left(\frac{\partial c_1'}{\partial \alpha_3}\right)_o = c_1c_3, & \left(\frac{\partial c_2'}{\partial \alpha_3}\right)_o = c_2c_3, & \left(\frac{\partial c_3'}{\partial \alpha_3}\right)_o = c_3^2, & \\
\left(\frac{\partial l_1'}{\partial p_{11}}\right)_o = -l_1, & \left(\frac{\partial l_2'}{\partial p_{11}}\right)_o = 0, & \left(\frac{\partial q_1'}{\partial p_{11}}\right)_o = -q_1, & \left(\frac{\partial q_2'}{\partial p_{11}}\right)_o = 0, \\
\left(\frac{\partial l_1'}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial l_2'}{\partial p_{21}}\right)_o = -l_1, & \left(\frac{\partial q_1'}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial q_2'}{\partial p_{21}}\right)_o = -q_1, \\
\left(\frac{\partial l_1'}{\partial p_{31}}\right)_o = l_1^2, & \left(\frac{\partial l_2'}{\partial p_{31}}\right)_o = l_1l_2, & \left(\frac{\partial q_1'}{\partial p_{31}}\right)_o = l_1q_1, & \left(\frac{\partial q_2'}{\partial p_{31}}\right)_o = l_2q_1, \\
\left(\frac{\partial l_1'}{\partial p_{12}}\right)_o = -l_2, & \left(\frac{\partial l_2'}{\partial p_{12}}\right)_o = 0, & \left(\frac{\partial q_1'}{\partial p_{12}}\right)_o = -q_2, & \left(\frac{\partial q_2'}{\partial p_{12}}\right)_o = 0, \\
\left(\frac{\partial l_1'}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial l_2'}{\partial p_{22}}\right)_o = -l_2, & \left(\frac{\partial q_1'}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial q_2'}{\partial p_{22}}\right)_o = -q_2, \\
\left(\frac{\partial l_1'}{\partial p_{32}}\right)_o = l_1l_2, & \left(\frac{\partial l_2'}{\partial p_{32}}\right)_o = l_2^2, & \left(\frac{\partial q_1'}{\partial p_{32}}\right)_o = l_1q_2, & \left(\frac{\partial q_2'}{\partial p_{32}}\right)_o = l_2q_2, \\
\left(\frac{\partial l_1'}{\partial p_{13}}\right)_o = -1, & \left(\frac{\partial l_2'}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial q_1'}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial q_2'}{\partial p_{13}}\right)_o = 0, \\
\left(\frac{\partial l_1'}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial l_2'}{\partial p_{23}}\right)_o = -1, & \left(\frac{\partial q_1'}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial q_2'}{\partial p_{23}}\right)_o = 0, \\
\left(\frac{\partial l_1'}{\partial p_{33}}\right)_o = l_1, & \left(\frac{\partial l_2'}{\partial p_{33}}\right)_o = l_2, & \left(\frac{\partial q_1'}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial q_2'}{\partial p_{33}}\right)_o = 0, \\
\left(\frac{\partial l_1'}{\partial \alpha_1}\right)_o = 0, & \left(\frac{\partial l_2'}{\partial \alpha_1}\right)_o = 0, & \left(\frac{\partial q_1'}{\partial \alpha_1}\right)_o = -1, & \left(\frac{\partial q_2'}{\partial \alpha_1}\right)_o = 0, \\
\left(\frac{\partial l_1'}{\partial \alpha_2}\right)_o = 0, & \left(\frac{\partial l_2'}{\partial \alpha_2}\right)_o = 0, & \left(\frac{\partial q_1'}{\partial \alpha_2}\right)_o = 0, & \left(\frac{\partial q_2'}{\partial \alpha_2}\right)_o = -1, \\
\left(\frac{\partial l_1'}{\partial \alpha_3}\right)_o = 0, & \left(\frac{\partial l_2'}{\partial \alpha_3}\right)_o = 0, & \left(\frac{\partial q_1'}{\partial \alpha_3}\right)_o = l_1, & \left(\frac{\partial q_2'}{\partial \alpha_3}\right)_o = l_2.
\end{array}
$$

So, the matrix of the coefficients of the infinitesimal transformations of H_{12} is given by

$$
\zeta_{ij} = \begin{pmatrix}
b_1 & 0 & 0 & c_1 & 0 & 0 & -l_1 & 0 & -q_1 & 0 \\
b_2 & 0 & 0 & c_2 & 0 & 0 & 0 & -l_1 & 0 & -q_1 \\
b_3 & 0 & 0 & c_3 & 0 & 0 & l_1^2 & l_1l_2 & l_1q_1 & l_2q_1 \\
0 & b_1 & 0 & 0 & c_1 & 0 & -l_2 & 0 & -q_2 & 0 \\
0 & b_2 & 0 & 0 & c_2 & 0 & 0 & -l_2 & 0 & -q_2 \\
0 & b_3 & 0 & 0 & c_3 & 0 & l_1l_2 & l_2^2 & l_1q_2 & l_2q_2 \\
0 & 0 & b_1 & 0 & 0 & c_1 & -1 & 0 & 0 & 0 \\
0 & 0 & b_2 & 0 & 0 & c_2 & 0 & -1 & 0 & 0 \\
0 & 0 & b_3 & 0 & 0 & c_3 & l_1 & l_2 & 0 & 0 \\
b_1^2 & b_2b_1 & b_3b_1 & c_1^2 & c_2c_1 & c_3c_1 & 0 & 0 & -1 & 0 \\
b_1b_2 & b_2^2 & b_3b_2 & c_1c_2 & c_2^2 & c_3c_2 & 0 & 0 & 0 & -1 \\
b_1b_3 & b_2b_3 & b_3^2 & c_1c_3 & c_2c_3 & c_3^2 & 0 & 0 & l_1 & l_2
\end{pmatrix}. \quad (8)
$$

Our aim is to find functions $\Phi(b_1, b_2, b_3, c_1, c_2, c_3, l_1, l_2, q_1, q_2)$ which satisfy the following (Deltheil) system:

$$\begin{aligned}
b_1\tfrac{\partial\Phi}{\partial b_1} + c_1\tfrac{\partial\Phi}{\partial c_1} + (-l_1)\tfrac{\partial\Phi}{\partial l_1} + (-q_1)\tfrac{\partial\Phi}{\partial q_1} &= 0\\
b_2\tfrac{\vartheta\Phi}{\vartheta b_1} + c_2\tfrac{\vartheta\Phi}{\vartheta c_1} + (-l_1)\tfrac{\partial\Phi}{\partial l_2} + (-q_1)\tfrac{\partial\Phi}{\partial q_2} &= 0\\
b_3\tfrac{\partial\Phi}{\partial b_1} + c_3\tfrac{\partial\Phi}{\partial c_1} + l_1^2\tfrac{\partial\Phi}{\partial l_1} + l_1l_2\tfrac{\partial\Phi}{\partial l_2} + l_1q_1\tfrac{\partial\Phi}{\partial q_1} + l_2q_1\tfrac{\partial\Phi}{\partial q_2} &= -4l_1\Phi\\
b_1\tfrac{\partial\Phi}{\partial b_2} + c_1\tfrac{\partial\Phi}{\vartheta c_2} + (-l_2)\tfrac{\partial\Phi}{\partial l_1} + (-q_2)\tfrac{\partial\Phi}{\partial q_1} &= 0\\
b_2\tfrac{\partial\Phi}{\partial b_2} + c_2\tfrac{\partial\Phi}{\partial c_2} + (-l_2)\tfrac{\partial\Phi}{\partial l_2} + (-q_2)\tfrac{\partial\Phi}{\partial q_2} &= 0\\
b_3\tfrac{\partial\Phi}{\partial b_2} + c_3\tfrac{\partial\Phi}{\partial c_2} + l_1l_2\tfrac{\partial\Phi}{\partial l_1} + l_2^2\tfrac{\partial\Phi}{\partial l_2} + l_1q_2\tfrac{\partial\Phi}{\partial q_1} + l_2q_2\tfrac{\partial\Phi}{\partial q_2} &= -4l_2\Phi\\
b_1\tfrac{\partial\Phi}{\partial b_3} + c_1\tfrac{\partial\Phi}{\partial c_3} + (-\tfrac{\partial\Phi}{\partial l_1}) &= 0\\
b_2\tfrac{\partial\Phi}{\partial b_3} + c_2\tfrac{\partial\Phi}{\partial c_3} + (-\tfrac{\partial\Phi}{\partial l_2}) &= 0\\
b_3\tfrac{\partial\Phi}{\partial b_3} + c_3\tfrac{\partial\Phi}{\partial c_3} + l_1\tfrac{\partial\Phi}{\partial l_1} + l_2\tfrac{\partial\Phi}{\partial l_2} &= -4\Phi\\
b_1^2\tfrac{\partial\Phi}{\partial b_1} + b_1b_2\tfrac{\partial\Phi}{\partial b_2} + b_1b_3\tfrac{\partial\Phi}{\partial b_3} + c_1^2\tfrac{\partial\Phi}{\partial c_1} + c_1c_2\tfrac{\partial\Phi}{\partial c_2} + c_1c_3\tfrac{\partial\Phi}{\partial c_3} + (-\tfrac{\partial\Phi}{\partial q_1}) &= -4(b_1+c_1)\Phi\\
b_1b_2\tfrac{\partial\Phi}{\partial b_1} + b_2^2\tfrac{\partial\Phi}{\partial b_2} + b_2b_3\tfrac{\partial\Phi}{\partial b_3} + c_1c_2\tfrac{\partial\Phi}{\partial c_1} + c_2^2\tfrac{\partial\Phi}{\partial c_2} + c_2c_3\tfrac{\partial\Phi}{\partial c_3} + (-\tfrac{\partial\Phi}{\partial q_2}) &= -4(b_2+c_2)\Phi\\
b_1b_3\tfrac{\partial\Phi}{\partial b_1} + b_2b_3\tfrac{\partial\Phi}{\partial b_2} + b_3^2\tfrac{\partial\Phi}{\partial b_3} + c_1c_3\tfrac{\partial\Phi}{\partial c_1} + c_2c_3\tfrac{\partial\Phi}{\partial c_2} + c_3^2\tfrac{\partial\Phi}{\partial c_3} + l_1\tfrac{\partial\Phi}{\partial q_1} + l_2\tfrac{\partial\Phi}{\partial q_2} &= -4(b_3+c_3)\Phi
\end{aligned} \tag{9}$$

System (9) has $\Phi = 0$ as the trivial solution, obviously. Then by dividing any equation of (12) by Φ, it becomes a (linear non-homogeneous) system of 12 algebraic equations with *ten* unknown quantities:

$$\frac{\partial \ln\Phi}{\partial b_1}, \frac{\partial \ln\Phi}{\partial b_2}, \frac{\partial \ln\Phi}{\partial b_3}, \frac{\partial \ln\Phi}{\partial c_1}, \frac{\partial \ln\Phi}{\partial c_2}, \frac{\partial \ln\Phi}{\partial c_3}, \frac{\partial \ln\Phi}{\partial l_1}, \frac{\partial \ln\Phi}{\partial l_2}, \frac{\partial \ln\Phi}{\partial q_1}, \frac{\partial \ln\Phi}{\partial q_2}.$$

The incomplete and complete matrix (respectively) of the previous system are given by:

$$\begin{pmatrix}
b_1 & 0 & 0 & c_1 & 0 & 0 & -l_1 & 0 & -q_1 & 0\\
b_2 & 0 & 0 & c_2 & 0 & 0 & 0 & -l_1 & 0 & -q_1\\
b_3 & 0 & 0 & c_3 & 0 & 0 & l_1^2 & l_1l_2 & l_1q_1 & l_2q_1\\
0 & b_1 & 0 & 0 & c_1 & 0 & -l_2 & 0 & -q_2 & 0\\
0 & b_2 & 0 & 0 & c_2 & 0 & 0 & -l_2 & 0 & -q_2\\
0 & b_3 & 0 & 0 & c_3 & 0 & l_1l_2 & l_2^2 & l_1q_2 & l_2q_2\\
0 & 0 & b_1 & 0 & 0 & c_1 & -1 & 0 & 0 & 0\\
0 & 0 & b_2 & 0 & 0 & c_2 & 0 & -1 & 0 & 0\\
0 & 0 & b_3 & 0 & 0 & c_3 & l_1 & l_2 & 0 & 0\\
b_1^2 & b_2b_1 & b_3b_1 & c_1^2 & c_2c_1 & c_3c_1 & 0 & 0 & -1 & 0\\
b_2b_1 & b_2^2 & b_3b_2 & c_2c_1 & c_2^2 & c_3c_2 & 0 & 0 & 0 & -1\\
b_3b_1 & b_3b_2 & b_3^2 & c_3c_1 & c_3c_2 & c_3^2 & 0 & 0 & l_1 & l_2
\end{pmatrix}$$

$$\begin{pmatrix}
b_1 & 0 & 0 & c_1 & 0 & 0 & -l_1 & 0 & -q_1 & 0 & 0 \\
b_2 & 0 & 0 & c_2 & 0 & 0 & 0 & -l_1 & 0 & -q_1 & 0 \\
b_3 & 0 & 0 & c_3 & 0 & 0 & l_1^2 & l_1l_2 & l_1q_1 & l_2q_1 & -4l_1 \\
0 & b_1 & 0 & 0 & c_1 & 0 & -l_2 & 0 & -q_2 & 0 & 0 \\
0 & b_2 & 0 & 0 & c_2 & 0 & 0 & -l_2 & 0 & -q_2 & 0 \\
0 & b_3 & 0 & 0 & c_3 & 0 & l_1l_2 & l_2^2 & l_1q_2 & l_2q_2 & -4l_2 \\
0 & 0 & b_1 & 0 & 0 & c_1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & b_2 & 0 & 0 & c_2 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & b_3 & 0 & 0 & c_3 & l_1 & l_2 & 0 & 0 & -4 \\
b_1^2 & b_2b_1 & b_3b_1 & c_1^2 & c_2c_1 & c_3c_1 & 0 & 0 & -1 & 0 & -4(b_1+c_1) \\
b_2b_1 & b_2^2 & b_3b_2 & c_2c_1 & c_2^2 & c_3c_2 & 0 & 0 & 0 & -1 & -4(b_2+c_2) \\
b_3b_1 & b_3b_2 & b_3^2 & c_3c_1 & c_3c_2 & c_3^2 & 0 & 0 & l_1 & l_2 & -4(b_3+c_3)
\end{pmatrix}.$$

We consider the 10×10 submatrix of the incomplete matrix which is obtained by deleting the ninth and the twelfth rows. Its determinant is not zero. Therefore, the incomplete matrix has rank 10. As that submatrix is also contained in the complete matrix, adding first the ninth row and then the twelfth row (always considering the last column, obviously), we obtain two 11×11 submatrices.Their determinants are both zero; therefore the complete matrix has rank 10.

We conclude that system (9) is solvable, so there exsists only one not trivial solution given by the function

$$\Phi = k(\sigma_2\rho_1 - \sigma_1\rho_2)^{-4} \quad \text{with } k \in R^*$$

where $\sigma_1 = b_1q_1 + b_2q_2 - 1$, $\rho_2 = c_1q_1 + c_2q_2 - 1$, $\sigma_1 = l_1b_1 + l_2b_2 + b_3$, $\sigma_2 = l_1c_1 + l_2c_2 + c_3$.

We leave out the calculus.

So group H_{12} associated to G_{12} is measurable by Theorem 2. Hence family $\Im_{10}$ is measurable and its density is given by

$$d\Phi = (\sigma_2\rho_1 - \sigma_1\rho_2)^{-4} db_1 \wedge db_2 \wedge db_3 \wedge dc_1 \wedge dc_2 \wedge dc_3 \wedge dl_1 \wedge dl_2 \wedge dq.$$

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