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On Applications of the Yano–Ako Operator*

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Abstract

In this paper we consider a method by which a skew-symmetric tensor field of type (1,2) in M_n can be extended to the tensor bundle $T_q^0(M_n)$ $(q > 0)$ on the *pure cross-section.* The results obtained are to some extend similar to results previously established for cotangent bundles $T_1^0(M_n)$. However, there are various important differences and it appears that the problem of lifting tensor fields of type (1,2) to the tensor bundle $T_q^0(M_n)$ $(q > 1)$ on the *pure cross-section* presents difficulties which are not encountered in the case of the cotangent bundle.

Key words: Lift; tensor bundle; pure tensor; operator Yano–Ako.

2000 Mathematics Subject Classification: 53C15, 53C25, 53C55

1 Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let $T_q^0(M_n)$ $(q > 0)$ be the bundle over M_n of tensors of type $(0, q)$:

$$T_q^0(M_n) = \bigcup_{P \in M_n} T_q^0(P),$$

where $T_q^0(P)$ denotes the tensor spaces of tensors of type $(0, q)$ at $P \in M_n$.

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i. $\pi : T^0_q(M_n) \to M_n$ is the projection $T^0_q(M_n)$ onto M_n.

ii. The indices $i, j, \dots$ run from 1 to n, the indices $\bar{\imath}, \bar{\jmath}, \dots$ from $n+1$ to $n + n^q = \dim T^0_q(M_n)$ and the indices $I = (i, \bar{\imath}), J = (j, \bar{\jmath}), \dots$ from 1 to $n + n^q$. The so-called Einsteins summation convention is used.

iii. $\Im(M)$ is the ring of real-valued C^∞ functions on M_n. $T^p_q(M_n)$ is the module over $\Im(M)$ of C^∞ tensor fields of type (p, q).

iv. Vector fields in M_n are denoted by $V, W, \dots$ The Lie derivation with respect to V is denoted by L_V.

Denoting by x^j the local coordinates of $P = \pi(\tilde{P})$ ($\tilde{P} \in T^0_q(M_n)$) in a neighborhood $U \subset M_n$ and if we make $(x^j, t_{j_1 \dots j_q}) = (x^j, x^{\bar{\jmath}})$ correspond to the point $\tilde{P} \in \pi^{-1}(U)$, we can introduce a system of local coordinates $(x^j, x^{\bar{\jmath}})$ in a neighborhood $\pi^{-1}(U) \subset T^0_q(M_n)$, where $t_{j_1 \dots j_q} \overset{\text{def}}{=} x^{\bar{\jmath}}$ are components of $t \in T^0_q(P)$ with respect to the natural frame ∂_i.

If $\alpha \in T^0_q(M_n)$, it is regarded, in a natural way (by contraction), as a function in $T^0_q(M_n)$, which we denote by $i\alpha$. If α has the local expression $\alpha = \alpha^{j_1 \cdots j_q} \partial_{j_1} \otimes \dots \otimes \partial_{j_q}$ in a coordinate neighborhood $U(x^i) \subset M_n$, then $i\alpha$ has the local expression $i\alpha = \alpha(t) = \alpha^{j_1 \cdots j_q} t_{j_1 \dots j_q}$ with respect to the coordinates $(x^j, x^{\bar{\jmath}})$ in $\pi^{-1}(U)$.

Suppose that $A \in T^0_q(M_n)$. We define the vertical lift ${}^V\!A \in T^1_0(T^0_q(M_n))$ of A to $T^0_q(M_n)$ (see [1]) by ${}^V\!A(i\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A))$, where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \Im(M_n)$. The vertical lift ${}^V\!A$ of A to $T^0_q(M_n)$ has components

$$ {}^V\!A = \begin{pmatrix} {}^V\!A^j \\ V A^{\bar{\jmath}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1 \dots j_q} \end{pmatrix} \tag{1.1} $$

with respect to the coordinates $(x^j, x^{\bar{\jmath}})$ in $T^0_q(M_n)$.

We define the complete lift ${}^C V = \bar{L}_V$ of V to $T^0_q(M_n)$ (see [1]) by ${}^C V(i\alpha) = i(L_V \alpha)$, $\alpha \in T^q_0(M_n)$. The complete lift ${}^C V$ of V to $T^0_q(M_n)$ has components

$$ {}^C V^k = V^k, \qquad {}^C V^{\bar{k}} = -\sum_{\lambda=1}^{q} t_{k_1 \dots s \dots k_q} \partial_{k_\lambda} V^s \tag{1.2} $$

with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T^0_q(M_n)$.

Suppose that there is given a tensor field $\xi \in T^0_q(M_n)$. Then the correspondence $x \to \xi_x$, ξ_x being the value of ξ at $x \in M_n$, determines a mapping $\sigma_\xi : M_n \to T^0_q(M_n)$ such that $\pi \circ \sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T^0_q(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1 \dots k_q}(x^k)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by $x^k = x^k$, $x^{\bar{k}} = \xi_{k_1 \dots k_q}(x^k)$ with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T^0_q(M_n)$. Differentiating by x^j, we see that the n tangent vector fields B_j to $\sigma_\xi(M_n)$have components

$$ (B^K_j) = \left(\frac{\partial x^K}{\partial x^j} \right) = \begin{pmatrix} \delta^k_j \\ \partial_j \xi_{k_1 \dots k_q} \end{pmatrix} \tag{1.3} $$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T_q^0(M_n)$.

On the other hand, the fibre is locally expressed by $x^k = \text{const}$, $t_{k_1\dots k_q} = t_{k_1\dots k_q}$, $t_{k_1\dots k_q}$ being consider as parameters. Thus, on differentiating with respect to $x^{\bar{j}} = t_{j_1\dots j_q}$, we see that the n^q tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$(C_{\bar{j}}^K) = \left(\frac{\partial x^K}{\partial x^{\bar{j}}}\right) = \begin{pmatrix} 0 \\ \delta_{k_1}^{j_1} \dots \delta_{k_q}^{j_q} \end{pmatrix} \tag{1.4}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T_q^0(M_n)$.

We consider in $\pi^{-1}(U) \subset T_q^0(M_n)$, $n+n^q$ local vector fields B_j and $C_{\bar{j}}$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j, C_{\bar{j}}\}$ along $\sigma_\xi(M_n)$, which is called the adapted (B,C)-frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. Taking account of (1.2), we can easily prove that , the complete lift ${}^CV$ has along $\sigma_\xi(M_n)$ components of the form

$${}^CV = \begin{pmatrix} {}^C\tilde{V}^j \\ {}^C\tilde{V}^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ -(L_V\xi)_{j_1\dots j_q} \end{pmatrix} \tag{1.5}$$

with respect to the adapted (B,C)-frame [2], where $(L_V\xi)_{j_1\dots j_q}$ are local components of $L_V\xi$ in M_n.

2 The vertical-vector lift of a tensor field of type (1,1)

Let $\varphi \in T_1^1(M_n)$. Making use of the Jacobian matrix of the coordinate transformation in $T_q^0(M_n)$:

$$x^{i'} = x^{i'}(x^i), x^{\bar{i}'} = t_{(i')} = A_{(i')}^{(i)} t_{(i)}$$
$$= A_{(i')}^{(i)} x^{\bar{i}} \big(t_{(i)} = t_{i_1\dots i_q}, A_{(i')}^{(i)} = A_{i_1'}^{i_1} \dots A_{i_q'}^{i_q}, A_{i'}^{i} = \frac{\partial x_i}{\partial x_{i'}}\big)$$

we can define a vector field $\gamma\varphi \in T_0^1(T_q^0(M_n))$ [3]:

$$\gamma\varphi = ((\gamma\varphi)^J) = \begin{pmatrix} 0 \\ t_{ji_2\dots i_q}\varphi_{i_1}^j \end{pmatrix},$$

where $\varphi_{i_1}^j$ are local components of φ in M_n. Clearly, we have $(\gamma\varphi)({}^Vf) = 0$ for any $f \in \Im(M_n)$, so that $\gamma\varphi$ is a vertical vector field. We call $\gamma\varphi$ the vertical-vector lift of the tensor field $\varphi \in T_1^1(M_n)$ to $T_q^0(M_n)$. We can easily verify that the vertical-vector lift $\gamma\varphi$ has along $\sigma_\xi(M_n)$ components

$$\gamma\varphi = ((\gamma\tilde{\varphi})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\dots i_q}\varphi_{i_1}^j \end{pmatrix}$$

with respect to the adapted (B,C)-frame, where $\xi_{i_1\dots i_q}$ are local components of ξ in M_n.

Let S be an element of $T_2^1(M_n)$ with local components S_{ij}^k in M_n. In a similar way, if $\gamma((L_{V_1}S)_{V_2})$, $\gamma((L_{V_2}S)_{V_1})$ and $\gamma(S_{[V_1,V_2]})$ are vertical-vector lifts of $(L_{V_1}S)_{V_2} = (v_2^m (L_{V_1}S)^j_{im}) \in T_1^1(M_n)$, $(L_{V_2}S)_{V_1} = (v_1^m (L_{V_2}S)^j_{im}) \in T_1^1(M_n)$ and $S_{[V_1,V_2]} = (S^j_{im}[V_1,V_2]^m) \in T_1^1(M_n)$, respectively, then $\gamma((L_{V_1}S)_{V_2})$, $\gamma((L_{V_2}S)_{V_1})$ and $\gamma(S_{[V_1,V_2]})$ have along $\sigma_\xi(M_n)$ respectively components of the form

$$\gamma((L_{V_1}S)_{V_2}) = (\gamma((\tilde{L}_{V_1}S)_{V_2})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q} v_2^m (L_{V_1}S)^j_{i_1 m} \end{pmatrix},$$

$$\gamma((L_{V_2}S)_{V_1}) = (\gamma((\tilde{L}_{V_2}S)_{V_1})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q} v_1^m (L_{V_2}S)^j_{i_1 m} \end{pmatrix},$$

$$\gamma(S_{[V_1,V_2]}) = (\gamma(\tilde{S}_{[V_1,V_2]})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q} S^j_{i_1 m}[V_1,V_2]^m \end{pmatrix}$$

with respect to the adapted (B,C)-frame, where $[V_1,V_2] = L_{V_1}V_2$.

3 The complete lift of a skew-symmetric tensor field of type (1,2)

Suppose now that $S \in T_2^1(M_n)$ is a skew-symmetric tensor field of type (1,2) with local components S^k_{ij}, that is $S(V,W) = -S(W,V)$, $\forall V, W \in T_0^1(M_n)$. A tensor field $\xi \in T_q^0(M_n)$ is called pure with respect to $S \in T_2^1(M_n)$, if [4]:

$$\begin{cases} S^r_{k_1 j_1}\xi_{r\ldots j_q} = \ldots = S^r_{k_1 j_q}\xi_{j_1\ldots r}, \\ S^r_{j_1 k_2}\xi_{r\ldots j_q} = \ldots = S^r_{j_q k_2}\xi_{j_1\ldots r}. \end{cases}$$

In particular, covector fields will be considered to be pure. Let $\overset{*}{T}{}_q^0(M_n)$denotes a module of all the tensor fields $\xi \in T_q^0(M_n)$ which are pure with respect to S. We consider a pure cross-section $\sigma_\xi^S(M_n)$ determined by $\xi \in \overset{*}{T}{}_q^0(M_n)$. We observe that the local vector fields

$${}^C X_{(i)} = {}^C (\frac{\partial}{\partial x^i}) = {}^C (\delta_i^h \frac{\partial}{\partial x^h}) = \begin{pmatrix} \delta_i^h \\ 0 \end{pmatrix}$$

and

$${}^V X^{(\bar{\imath})} = {}^V (dx^{i_1} \otimes \ldots \otimes dx^{i_q}) = {}^V (\delta_{h_1}^{i_1} \ldots \delta_{h_q}^{i_q} dx^{h_1} \otimes \ldots \otimes dx^{h_q}) = \begin{pmatrix} 0 \\ \delta_{h_1}^{i_1} \ldots . \delta_{h_q}^{i_q} \end{pmatrix}$$

$$i = 1, \ldots, n, \ \bar{\imath} = n+1, \ldots, n+n^q$$

span the module of vector fields in $\pi^{-1}(U) \subset T_q^0(M_n)$. Hence any tensor field is determined in $\pi^{-1}(U)$ by its action of ${}^C X_{(i)}$ and ${}^V X^{(\bar{\imath})}$. Then we define a tensor field ${}^CS \in T^1_2(T^0_q(M_n))$ along the pure cross-section $\sigma^S_\xi(M_n)$ by

$$\begin{cases} {}^CS({}^CV_1, {}^CV_2) = {}^C(S(V_1,V_2)) - \gamma((L_{V_2}S)_{V_1}) \\ \quad + \gamma((L_{V_1}S)_{V_2}) + \gamma(S_{[V_1,V_2]}), \quad \forall V_1, V_2 \in T^1_0(M_n) & \text{(i)} \\ {}^CS({}^VA, {}^CV_2) = {}^V(S_{V_2}(A)), \quad \forall A \in T^1_q(M_n), & \text{(ii)} \\ {}^CS({}^CV_1, {}^VB) = {}^V(S_{V_1}(B)), \quad \forall B \in T^1_q(M_n), & \text{(iii)} \\ {}^CS({}^VA, {}^VB) = 0, & \text{(iv)} \end{cases} \tag{3.1}$$

where $S_{V_2}(A), S_{V_1}(B) \in T^0_q(M_n)$ and call ${}^CS$ the complete lift of $S \in T^1_2(M_n)$ to $T^0_q(M_n)$ along $\sigma^S_\xi(M_n)$.

Let ${}^C\tilde{S}^J_{L_1L_2}$ be components of ${}^CS$ with respect to the adapted (B,C)-frame of the pure cross-section $\sigma^S_\xi(M_n)$. From (1.1), (1.3), (1.4) and ${}^VA = {}^V\tilde{A}^jB_j + {}^V\tilde{A}^{\bar{j}}C_{\bar{j}}$, we easily obtain ${}^V\tilde{A}^j = 0$, ${}^V\tilde{A}^{\bar{j}} = {}^VA^{\bar{j}} = A_{j_1\ldots j_q}$. Thus the vertical lift ${}^VA$ also has components of the form (1.1) with respect to the adapted (B,C)-frame of $\sigma^S_\xi(M_n)$. Then, from (3.1) we have

$$\begin{cases} {}^C\tilde{S}^J_{L_1L_2}\,{}^C\tilde{V}^{L_1}_1\,{}^C\tilde{V}^{L_2}_2 = {}^C(\tilde{S}(V_1,V_2))^J - \gamma((\tilde{L}_{V_2}S)_{V_1})^J \\ \quad + \gamma((\tilde{L}_{V_1}S)_{V_2})^J + \gamma(\tilde{S}_{[V_1,V_2]})^J, & \text{(i)} \\ {}^C\tilde{S}^J_{L_1L_2}\,{}^V\tilde{A}^{L_1}\,{}^C\tilde{V}^{L_2}_2 = {}^V(S_{V_2}(\tilde{A}))^J & \text{(ii)} \\ {}^C\tilde{S}^J_{L_1L_2}\,{}^C\tilde{V}^{L_1}_1\,{}^V\tilde{B}^{L_2} = {}^V(S_{V_1}(\tilde{B}))^J, & \text{(iii)} \\ {}^C\tilde{S}^J_{L_1L_2}\,{}^V\tilde{A}^{L_1}\,{}^V\tilde{B}^{L_2} = 0, & \text{(iv)} \end{cases} \tag{3.2}$$

where

$${}^V(S_{V_2}\tilde{(A)})^J = \begin{pmatrix} 0 \\ S^m_{j_1l}V^l_2A_{mj_2\ldots j_q} \end{pmatrix}, \qquad {}^V(S_{V_1}\tilde{(B)})^J = \begin{pmatrix} 0 \\ S^m_{lj_1}V^l_1B_{mj_2\ldots j_q} \end{pmatrix}.$$

When $J = j$, from (i) of (3.2) we have

$${}^C\tilde{S}^j_{l_1l_2} = S^j_{l_1l_2}, \quad {}^C\tilde{S}^j_{\bar{l}_1l_2} = {}^C\tilde{S}^j_{l_1\bar{l}_2} = {}^C\tilde{S}^j_{\bar{l}_1\bar{l}_2} = 0,$$

where $x^{\bar{l}_a} = t_{r_1\ldots r_q}$, $a = 1,2$.

When $J = \bar{j}$, (i) of (3.2) reduces to

$$\begin{aligned} {}^C\tilde{S}^{\bar{j}}_{l_1l_2}\,{}^C\tilde{V}^{l_1}_1\,{}^C\tilde{V}^{l_2}_2 + {}^C\tilde{S}^{\bar{j}}_{\bar{l}_1l_2}\,{}^C\tilde{V}^{\bar{l}_1}_1\,{}^C\tilde{V}^{l_2}_2 + {}^C\tilde{S}^{\bar{j}}_{l_1\bar{l}_2}\,{}^C\tilde{V}^{l_1}_1\,{}^C\tilde{V}^{\bar{l}_2}_2 \\ + {}^C\tilde{S}^{\bar{j}}_{\bar{l}_1\bar{l}_2}\,{}^C\tilde{V}^{\bar{l}_1}_1\,{}^C\tilde{V}^{\bar{l}_2}_2 + \xi_{ij_2\ldots j_q}v^m_1(L_{V_2}S)^i_{j_1m} \\ - \xi_{ij_2\ldots j_q}v^m_2(L_{V_1}S)^i_{j_1m} - \xi_{ij_2\ldots j_q}S^i_{j_1m}[V_1,V_2]^m = {}^C(\tilde{S}(V_1,V_2))^{\bar{j}} \end{aligned} \tag{3.3}$$

Now, using the Generalized Yano–Ako operator we will investigate components ${}^C\tilde{S}^{\bar{j}}_{l_1l_2}$. The Generalized Yano–Ako operator on the pure module $\overset{*}{T}{}^0_q(M_n)$ is given by [4], [5].

$$(\Phi_S\xi)_{l_1l_2j_1\ldots j_q} = S^m_{l_1l_2}\partial_m\xi_{j_1\ldots j_q} - \partial_{l_1}(S^m_{j_1l_2}\xi_{mj_2\ldots j_q}) - \partial_{l_2}(S^m_{l_1j_1}\xi_{mj_2\ldots j_q}) + \sum_{a=1}^{q}(\partial_{j_a}S^m_{l_1l_2})\xi_{j_1\ldots m\ldots j_q}.$$

After some calculations we have

$$\begin{aligned} &V_2^{l_2}V_1^{l_1}(\Phi_{S(V_1,V_2)}\xi)_{l_1l_2j_1\ldots j_q} + V_1^{l_1}S^m_{l_1j_1}L_{V_2}\xi_{mj_2\ldots j_q} + V_2^{l_2}S^m_{j_1l_2}L_{V_1}\xi_{mj_2\ldots j_q} \\ &+ V_2^{l_2}(L_{V_1}S^m_{j_1l_2})\xi_{mj_2\ldots j_q} - V_1^{l_1}(L_{V_2}S^m_{j_1l_1})\xi_{mj_2\ldots j_q} + (L_{V_1}V_2)^{l_1}S^m_{j_1l_1})\xi_{mj_2\ldots j_q} \\ &\quad = L_{S(V_1,V_2)}\xi_{j_1\ldots j_q} \end{aligned} \tag{3.4}$$

for any $V_1, V_2 \in T_0^1(M_n)$. Using (1.5), from (3.4) we have

$$\begin{aligned} &((\Phi_{S(V_1,V_2)}\xi)_{l_1l_2j_1\ldots j_q})^C\tilde{V}_1^{l_1}{}^C\tilde{V}_2^{l_2} - S^{r_1}_{l_1j_1}\delta^{r_2}_{j_2}\ldots\delta^{r_q}_{j_q}{}^C\tilde{V}_1^{l_1}{}^C\tilde{V}_2^{\bar{l}_2} \\ &- S^{r_1}_{j_1l_2}\delta^{r_2}_{j_2}\ldots\delta^{r_q}_{j_q}{}^C\tilde{V}_1^{\bar{l}_1}{}^C\tilde{V}_2^{l_2} + V_2^{l_2}(L_{V_1}S^m_{j_1l_2})\xi_{mj_2\ldots j_q} - V_1^{l_1}(L_{V_2}S^m_{j_1l_1})\xi_{mj_2\ldots j_q} \\ &+ (L_{V_1}V_2)^{l_1}S^m_{j_1l_1}\xi_{mj_2\ldots j_q} = -^C(\tilde{S}(V_1,V_2))^{\bar{J}}. \end{aligned} \tag{3.5}$$

Comparing (3.3) and (3.5), we get

$$^C\tilde{S}^{\bar{J}}_{l_1l_2} = -(\Phi_S\xi)_{l_1l_2j_1\ldots j_q}.$$

By similar devices, from (ii)–(iv) of (3.2) we have also

$$^C\tilde{S}^{\bar{J}}_{\bar{l}_1\bar{l}_2} = 0, \quad ^C\tilde{S}^{\bar{J}}_{\bar{l}_1l_2} = S^{r_1}_{j_1l_2}\delta^{r_2}_{j_2}\ldots\delta^{r_q}_{j_q}, \quad ^C\tilde{S}^{\bar{J}}_{l_1\bar{l}_2} = S^{r_1}_{l_1j_1}\delta^{r_2}_{j_2}\ldots\delta^{r_q}_{j_q}.$$

Thus the complete lift ^CS of $S \in T_2^1(M_n)$ $(S(V,W) = -S(W,V))$ has along the pure cross-section $\sigma_\xi^S(M_n)$ components

$$\begin{cases} ^C\tilde{S}^j_{l_1l_2} = S^j_{l_1l_2}, \ ^C\tilde{S}^j_{\bar{l}_1l_2} = {}^C\tilde{S}^j_{l_1\bar{l}_2} = {}^C\tilde{S}^j_{\bar{l}_1\bar{l}_2} = {}^C\tilde{S}^{\bar{J}}_{\bar{l}_1\bar{l}_2} = 0 \\ ^C\tilde{S}^{\bar{J}}_{\bar{l}_1l_2} = S^{r_1}_{j_1l_2}\delta^{r_2}_{j_2}\ldots\delta^{r_q}_{j_q}, \ ^C\tilde{S}^{\bar{J}}_{l_1\bar{l}_2} = S^{r_1}_{l_1j_1}\delta^{r_2}_{j_2}\ldots\delta^{r_q}_{j_q}, \\ ^C\tilde{S}^{\bar{J}}_{l_1l_2} = -(\Phi_S\xi)_{l_1l_2j_1\ldots j_q} \end{cases} \tag{3.6}$$

with respect to the adapted (B,C)-frame of $\sigma_\xi^S(M_n)$, where $\Phi_S\xi$ is the Generalized Yano–Ako operator.

Remark 1 ^CS in the form (3.6) is unique solution of (3.1). Therefore, if $\overset{*}{S}$ is element of $T_2^1(T_q^0(M_n))$, such that

$$\begin{cases} ^C\overset{*}{S}(^CV_1, {}^CV_2) = {}^C(S(V_1,V_2)) - \gamma((L_{V_2}S)_{V_1}) \\ \quad + \gamma((L_{V_1}S)_{V_2}) + \gamma(S_{[V_1,V_2]}), \\ ^C\overset{*}{S}(^VA, {}^CV_2) = {}^V(S_{V_2}(A)), \\ ^C\overset{*}{S}(^CV_1, {}^VB) = {}^V(S_{V_1}(B)), \\ ^C\overset{*}{S}(^VA, {}^VB) = 0, \end{cases}$$

then $\overset{*}{S} = {}^CS$.

Remark 2 The equation (3.1) is a useful extension of the equation $^CV(i\alpha) = i(L_V\alpha)$, $\alpha \in T_0^q(M_n)$ (see §1) to tensor fields of type (1,2) along the pure cross-section $\sigma_\xi^S(M_n)$.

In the case $\partial_m \xi_{j_1 \ldots j_q} = 0$, (B, C)-frame is considered as a natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma_\xi^S(M_n)$. Then, from (3.6) we obtain components of ${}^C S$ along the pure cross-section with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma_\xi^S(M_n)$ in $\pi^{-1}(U)$ (see [5]). The diagonal and horizontal lifts for tensor fields of special kinds to the tensor bundle have been studied in [6]–[8].

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