Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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*Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, Vol. 45 (2006), No. 1, 7--29

Persistent URL: `http://dml.cz/dmlcz/133445`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **45** (2006) 7–29

# Almost-Periodic Solutions in Various Metrics of Higher-Order Differential Equations with a Nonlinear Restoring Term

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(Received November 18, 2005)

**Abstract**

Almost-periodic solutions in various metrics (Stepanov, Weyl, Besicovitch) of higher-order differential equations with a nonlinear Lipschitz-continuous restoring term are investigated. The main emphasis is focused on a Lipschitz constant which is the same as for uniformly almost-periodic solutions treated in [A1] and much better than those from our investigations for differential systems in [A2], [A3], [AB], [ABL], [AK]. The upper estimates of $\varepsilon$ for $\varepsilon$-almost-periods of solutions and their derivatives are also deduced under various restrictions imposed on the constant coefficients of the linear differential operator on the left-hand side of the given equation. Besides the existence, uniqueness and localization of almost-periodic solutions and their derivatives are established.

**Key words:** Almost-periodic solutions; various metrics; higher-order differential equation; nonlinear restoring term; existence and uniqueness criteria.

**2000 Mathematics Subject Classification:** 34C27, 34C15, 42A75

*Supported by the Council of Czech Government (MSM 6198959214).

# 1 Introduction

We shall consider the differential equation

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} = f(y) + p(t), \tag{1}$$

where $a_j \in \mathbb{R}$, $j = 1, \ldots, n$, are real constants such that the real parts of the roots of the characteristic polynomial associated with the linear operator on the left-hand side of (1), namely

$$\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}, \tag{2}$$

are at least nonzero, i.e. $\operatorname{Re}\lambda_j \neq 0$, $j = 1, \ldots, n$. It is well-known that the related Routh–Hurwitz conditions are necessary and sufficient for $\operatorname{Re}\lambda_j < 0$, $j = 1, \ldots, n$, i.e. in order polynomial (2) to be stable. In this case, all coefficients $a_j \in \mathbb{R}$ in (1) must be positive, i.e. $a_j > 0$, $j = 1, \ldots, n$. One can also find necessary and sufficient conditions in order all roots of (2) to be negative, but for characteristic polynomials of a higher degree these conditions are rather cumbersome (see e.g. [AG, Chapter III.5]). Assume, furthermore, that the restoring term $f \in \mathrm{Lip}(\mathbb{R}, \mathbb{R})$ is a bounded Lipschitz-continuous function with constant $L < |a_n|$, and that the forcing term $p \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbb{R})$ is an essentially bounded, locally Lebesgue integrable function which will be successively supposed to be almost-periodic (a.p.) in the sense of Stepanov, Weyl or Besicovitch.

The main aim of the present paper is to extend appropriately sufficient conditions for the existence of uniformly almost-periodic solutions and their derivatives, obtained for (1) in [A1] (cf. also [AG, Chapter III.10]), provided the forcing term $p$ is almost-periodic in a more general sense (Stepanov, Weyl, Besicovitch). Although the existence criteria for such a.p. solutions and their derivatives can be deduced from our earlier results for differential systems, namely for Stepanov a.p. solutions in [AB], for Weyl a.p. solutions in [A2], [A3], and for Besicovitch a.p. solutions in [ABL] (cf. also [AG, Chapter III.10]), the upper estimates for Lipschitz constant $L$ related to $f$ would be very rough (cf. e.g. [AK]). Another purpose therefore consists in obtaining much sharper inequality for $L$, namely $L < |a_n|$. Since this is possible only if the roots of (2) are at least nonzero real (otherwise, the desired estimates for $L$ would explicitly depend on them), we shall still assume that the coefficients $a_j$, $j = 1, \ldots, n$, yield nonzero real roots.

Higher-order differential equations of the type (1), where $n > 2$, have not been treated w.r.t. the existence of a.p. solutions so often (see e.g. [Kh], [KBK], [L]). The investigations of the other authors of more general than uniformly a.p. solutions were also quite rare (see e.g. [BFSD1]–[BFSD3], [BFH], [DHS], [DM], [H], [Ku], [LZ], [P], [ZL]). As far as we know, apart from our mentioned papers [A2], [A3], [AB], [ABL] and [LZ], [P], [R], [ZL], almost-periodic solutions in the generalized sense of (1), where $n > 2$, have not yet been studied with the indicated respect.

The paper is organized as follows. After some preliminaries, the main existence results are formulated. Roughly speaking, as much as we impose on the coefficients $a_j$, $j = 1, \ldots, n$, on the left-hand side of (1), as good estimates of $\varepsilon$ for $\varepsilon$-almost-periods of a.p. solutions and their derivatives we obtain. Moreover, more transparent estimates of (entirely bounded) a.p. solutions allow us to replace global boundedness assumption on $f$ by restrictions localized only on certain domains. This will be done, besides another, in concluding remarks, jointly with extending our results to differential inclusions, on the basis of selection theorems in [HP] and [D1]–[D3], [DS].

## 2 Some preliminaries

At first, we recall various types of almost-periodicity.

**Definition 1** Let us introduce the following (pseudo-) metrics:
*(Stepanov)*

$$D_{S_l}(f,g) := \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t) - g(t)|\, dt,$$

*(Weyl)*

$$D_W(f,g) := \lim_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t) - g(t)|\, dt = \lim_{l\to\infty} D_{S_l}(f,g),$$

*(Besicovitch)*

$$D_B(f,g) := \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t) - g(t)|\, dt,$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are measurable functions. Denoting by $D_G$ any of the above (pseudo-) metrics, by the metric space $(G, D_G)$, we understand the related quotient space in the sense that we identify such elements $f_1, f_2$, for which $D_G(f_1, f_2) = 0$.

**Definition 2** A function $f \in \mathrm{L}^1_{loc}(\mathbb{R},\mathbb{R})$ is said to be *G-almost-periodic* (G-a.p.) if

$$\forall\varepsilon > 0\ \exists k > 0\ \forall a \in \mathbb{R}\ \exists\tau \in [a, a+k] : D_G(f(t+\tau), f(t)) < \varepsilon.$$

The above $\tau$ is called an *$\varepsilon$-almost-period* in the respective sense.

Instead of $D_{S_1}$-a.p. or $D_W$-a.p. or $D_B$-a.p. function, we shall write $S_1$-a.p. or $W$-a.p. or $B$-a.p., respectively.

The following definition uses curiously the Stepanov metric for the almost-periodicity in the sense of H. Weyl.

**Definition 3** A function $f \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbb{R})$ is said to be *equi-Weyl-almost-periodic* (equi-$W$-a.p.) if

$$\forall \varepsilon > 0\ \exists k, l_0(\varepsilon) > 0\ \forall a \in \mathbb{R}\ \exists \tau \in [a, a+k] :$$
$$D_{S_l}(f(t+\tau), f(t)) < \varepsilon, \quad \forall l \geq l_0(\varepsilon).$$

**Remark 1** It is well-known (see e.g. [ABG], [L], [LZ]) that, without any loss of generality, we can take $l_0 \geq 1$ in Definition 3.

**Definition 4** A function $f : \mathbb{R} \to \mathbb{R}$ is called *uniformly G-continuous* if

$$\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0 : |h| < \delta \Longrightarrow D_G(f(t+h), f(t)) < \varepsilon.$$

If, in particular, the above implication holds for a continuous function $f$ with $D_G$ replaced by the sup-norm, then we simply speak about *uniform continuity* of $f$.

In the following sections, the existence of almost-periodic solutions and their derivatives in various metrics will be proved by three different techniques for differential equation (1).

Hence, consider the differential equation (1), i.e.

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} = f(y) + p(t),$$

where $a_j \in \mathbb{R}$, $j = 1, \ldots n$, $f \in \mathrm{Lip}(\mathbb{R}, \mathbb{R})$ and $p \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbb{R})$.

Assume, furthermore, that

(i) all roots $\lambda_j$, $j = 1, \ldots n$, of the characteristic polynomial (2), i.e. of

$$\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j},$$

are nonzero and real;

(ii) $f$ is bounded and Lipschitz on $\mathbb{R}$, i.e. there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R};$$

(iii) $p$ is an essentially bounded $\bar{G}$-a.p. function, where $\bar{G}$ means either $S$ or $W$ or $B$ or equi-$W$ case;

(iv) there exists a positive constant $D_0$ s.t.

$$\operatorname*{supess}_{t \in \mathbb{R}} |p(t)| + \sup_{y \in \mathbb{R}} |f(y)| \leq D_0.$$

In the entire text, by a *solution* $y(\cdot)$ of (1), we shall mean the one in the sense of *Carathéodory*, i.e. such that $y^{(n-1)}(\cdot)$ is locally absolutely continuous.

The following lemma guarantees the existence of a unique bounded solution of (1), including its suitable representation for our application, and the same for its derivatives.

**Lemma 1** *Assume that all roots of the characteristic polynomial (2) are nonzero and real (i.e. (i)). Under the assumption (iv), and (ii) with $L < |a_n|$, equation (1) has exactly one (Carathéodory) entirely bounded solution $y(\cdot)$ given by the formula*

$$y(t) = \int_{\Lambda_1}^{t}\int_{\Lambda_2}^{t_1}\dots\int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t+(\lambda_2-\lambda_1)t_1+\ldots+(\lambda_n-\lambda_{n-1})t_{n-1}-\lambda_n t_n}[f(y(t_n))+p(t_n)]\,dt_n\dots dt_1,$$

*where $\Lambda_j = +\infty\cdot\lambda_j$, $j = 1,\dots,n$.*

*Denoting the right-hand side of the preceding formula by $[1,\dots,n]$, the $k$-th derivatives ($k = 1,\dots,n-1$) of solution $y(\cdot)$ satisfy*

$$\begin{aligned} y^{(k)}(t) = \frac{d^k([1,\dots,n])}{dt^k} &= [k+1,\dots,n] + \sum_{c_1=1}^{k}\lambda_{c_1}[c_1,k+1,\dots,n] \\ &+ \sum_{\substack{c_1,c_2=1\\ c_1<c_2}}^{k}\lambda_{c_1}\lambda_{c_2}[c_1,c_2,k+1,\dots,n]+\dots \\ + \sum_{\substack{c_1,\dots,c_p=1\\ c_1<\dots<c_p}}^{k}&\left(\prod_{i=1}^{p}\lambda_{c_i}\right)[c_1,\dots,c_p,k+1,\dots,n]+\dots+\left(\prod_{i=1}^{k}\lambda_i\right)[1,\dots,n], \end{aligned}$$

*where*

$$[c,\dots,n] = \int_{\Lambda_c}^{t}\int_{\Lambda_{c+1}}^{t_c}\dots\int_{\Lambda_n}^{t_{n-1}} e^{\lambda_c t+(\lambda_{c+1}-\lambda_c)t_c+\ldots+(\lambda_n-\lambda_{n-1})t_{n-1}-\lambda_n t_n} \times [f(y(t_n))+p(t_n)]\,dt_n\dots dt_{c+1}\,dt_c.$$

**Proof** The complete proof can be found in [AG]. The existence of a bounded solution is verified at page 554 (cf. also pp. 329–330). The representation formula is given at p. 321 (Lemma 5.45) and the formula for the $k$-th derivative is derived at pp. 324–325 (Lemma 5.61). The uniqueness is proved at p. 556. □

**Remark 2** The solution $y(\cdot)$ in Lemma 1 satisfies

$$\sup_{t\in\mathbb{R}}|y(t)| \le \frac{D_0}{|a_n|}$$

(see [AG, p. 323]) and its $k$-th derivative ($k = 1,\dots,n-1$) can be estimated by

a) $\sup_{t\in\mathbb{R}}|y^{(k)}(t)| \le \frac{2^k D_0}{|a_n|}\prod_{j=1}^{k}|\alpha_j|$, when the characteristic polynomial has only real nonzero roots (see [AG, Lemma 5.63 at pp. 325–326]);

b) $\sup_{t\in\mathbb{R}} |y^{(k)}(t)| \leq \dfrac{2^k D_0}{|a_{n-k}|}$, provided each of the shifted polynomials

$$\lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-p-j}, \quad p = 0, \dots, n-1,$$

admits real nonzero roots (see [AG, Lemma 5.70 at p. 327]);

c) $\sup_{t\in\mathbb{R}} |y^{(k)}(t)| \leq \dfrac{2^k a_k D_0}{\binom{n}{k} a_n}$, whenever all roots of the characteristic polynomial are negative (see [AG, Lemma 5.67 at p. 326]).

The meaning of constant $D_0$ can be seen in (iv).

Moreover, the estimates for the $k$-th derivatives are independent of the permutation of the roots (see [AG, p. 326]).

**Remark 3** Observe that, under the assumptions (i), (iv), a bounded solution of (1) with its derivatives, up to the $(n-1)$-th order, are uniformly continuous, and subsequently also uniformly $G$-continuous.

**Remark 4** The existence and representation parts of Lemma 1 are true if only the real parts of roots of (2) are assumed to be nonzero (cf. [AG, Chapter III.5]). On the other hand, the related estimates for solutions $y(\cdot)$ and their derivatives $y^{(k)}(\cdot)$, $k = 1, \dots, n-1$, do not depend explicitly on the coefficients $a_k$, but only on the real parts of the roots of (2) (cf. again [AG, Chapter III.5]).

## 3 Existence of a.p. solutions: case of nonzero real roots

The following main theorem is stated under the most general assumptions, when comparing with other main results of this paper.

**Theorem 1** *Let the above conditions (i)–(iv) be satisfied. If $L < |a_n|$, then equation (1) admits a unique bounded $\bar{G}$-a.p. solution with bounded $\bar{G}$-a.p. derivatives, up to the $(n-1)$-th order.*

*Moreover, the $\varepsilon$-almost-period of $p(\cdot)$ implies the $\frac{1}{|a_n|-L}\,\varepsilon$-almost-period of the solution $y(\cdot)$ and the $\frac{2^k|\lambda_1 \dots \lambda_k|}{|a_n|-L}\,\varepsilon$-almost-period of the $k$-th derivative $y^{(k)}(\cdot)$ of the solution in the $\bar{G}$-(pseudo-)metric, for $k = 1, \dots, n-1$, where $\lambda_1, \dots, \lambda_n$ are the roots of the characteristic polynomial $\lambda^n + \sum_{j=1}^n a_j \lambda^{n-j}$.*

**Proof** It follows from Lemma 1 that equation (1) admits a unique bounded solution of the form as above. Using the appropriate representation of this solution, one can obtain by means of (ii):

$$|y(t+\tau) - y(t)| \leq$$

$$\leq \Bigg| \int_{\Lambda_1}^{t} \int_{\Lambda_2}^{t_1} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2-\lambda_1)t_1 + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n} \left[ |f(y(t_n+\tau)) - f(y(t_n))| \right.$$

$$+\, |p(t_n+\tau)-p(t_n)|\,]\; dt_n\, dt_{n-1}\dots dt_1\Bigg|$$

$$\leq \Bigg|\int_{\Lambda_1}^{t}\int_{\Lambda_2}^{t_1}\dots\int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t+(\lambda_2-\lambda_1)t_1+\ldots+(\lambda_n-\lambda_{n-1})t_{n-1}-\lambda_n t_n}\,(L|y(t_n+\tau)-y(t_n)|$$

$$+\, |p(t_n+\tau)-p(t_n)|\,)\, dt_n\, dt_{n-1}\dots dt_1\Bigg|$$

$$=\Big|(-\frac{1}{\lambda_n})\dots(-\frac{1}{\lambda_1})\Big|\int_0^1\int_0^1\dots\int_0^1 L\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\, ds_n\dots ds_2\, ds_1,$$

where the last equality can be obtained by virtue of successive substitutions $s_j=e^{\lambda_j(t_{j-1}-t_j)}$, for $j=n,n-1\dots,2$, and $s_1=e^{\lambda_1(t-t_1)}$.

Now, we shall prove the $\bar{G}$-almost-periodicity of solution $y(\cdot)$, when applying assumption (iii). To employ all of the considered (pseudo-) metrics, we will need the following estimate (for $a<b$, $a,b\in\mathbb{R}$):

$$\int_a^b |y(t+\tau)-y(t)|\, dt \leq$$

$$\leq \frac{1}{|\lambda_n\cdot\ldots\cdot\lambda_1|}\int_a^b\int_0^1\dots\int_0^1 L\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\, ds_n\dots ds_2\, ds_1\, dt$$

$$=\frac{L}{|a_n|}\int_0^1\dots\int_0^1\int_a^b\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\, dt\, ds_n\dots ds_1$$

$$+\frac{1}{|a_n|}\int_0^1\dots\int_0^1\int_a^b\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\, dt\, ds_n\dots ds_1.$$

Using the Stepanov metric, we get ($a:=u$, $b:=u+1$):

$$\sup_{u\in\mathbb{R}}\int_u^{u+1} |\, y(t+\tau)-y(t)\,|\;\, dt \leq$$

$$\leq \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \ldots \int_0^1 \int_u^{u+1} \Big| y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t)\Big| \, dt\, ds_n \ldots ds_1$$

$$+ \frac{1}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \ldots \int_0^1 \int_u^{u+1} \Big| p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t)\Big| \, dt\, ds_n \ldots ds_1$$

$$< \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \int_u^{u+1} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \ldots \int_0^1 ds_n \, \ldots \, ds_1.$$

Hence,

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} |y(t+\tau) - y(t)| \, dt < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}.$$

Thus, under the assumption $|a_n| > L$, the $\widehat{\varepsilon}$-almost period of solution $y(\cdot)$ corresponds to an $\varepsilon$-almost period of function $p(\cdot)$ (in the sense of Stepanov).

For the equi-Weyl case, we get ($a := u$, $b := u + l$):

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt \leq$$

$$\leq \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \ldots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t)\Big| \, dt\, ds_n \ldots ds_1$$

$$+ \frac{1}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \ldots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t)\Big| \, dt\, ds_n \ldots ds_1$$

$$< \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \ldots \int_0^1 ds_n \ldots ds_1,$$

which implies

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}, \qquad \forall l \geq l_0.$$

By the above estimate, we can also obtain the following inequalities for the $W$-almost-periodicity:

$$\lim_{l\to\infty} \Big[ \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt \Big] \leq$$

$$\leq \frac{L}{|a_n|} \lim_{l\to\infty} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt\, ds_n \dots ds_1$$

$$+ \frac{1}{|a_n|} \lim_{l\to\infty} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt\, ds_n \dots ds_1$$

$$< \frac{L}{|a_n|} \lim_{l\to\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \dots \int_0^1 ds_n \dots ds_1.$$

Thus,

$$\lim_{l\to\infty} \left[ \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} \mid y(t+\tau) - y(t) \mid \; dt \right] < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}$$

holds for the $W$-almost-periodicity of $y(\cdot)$.

The proof for $B$-almost-periodicity is again based on the application of the inequality derived above. Hence, ($a := -T$, $b := T$):

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t+\tau) - y(t)| \, dt \leq$$

$$\leq \frac{L}{|a_n|} \limsup_{T\to\infty} \frac{1}{2T} \int_0^1 \dots \int_0^1 \int_{-T}^{T} \Big| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt\, ds_n \dots ds_1$$

$$+ \frac{1}{|a_n|} \limsup_{T\to\infty} \frac{1}{2T} \int_0^1 \dots \int_0^1 \int_{-T}^{T} \Big| p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt\, ds_n \dots ds_1$$

$$< \frac{L}{|a_n|} \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \dots \int_0^1 ds_n \dots ds_1.$$

Repeating the procedure as in the preceding cases, one arrives at

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t+\tau) - y(t)|\, dt < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}.$$

We could see that the almost-periodicity of solution $y(\cdot)$ was verified in all given (pseudo-)metrics, whenever $L < |a_n|$. Moreover, to $\varepsilon$-almost period of $p(\cdot)$, there corresponds the $\frac{\varepsilon}{|a_n|-L}$-almost period of solution $y(\cdot)$ (in the related pseudo-metric).

To prove the $\bar{G}$-almost-periodicity of the derivatives $y^{(k)}(\cdot)$, we use the formula from Lemma 1. Hence, applying (ii) and making successive substitutions as in the preceding part of the proof, we get

$$|y^{(k)}(t+\tau) - y^{(k)}(t)| \leq$$

$$\leq \Bigg| \int_{\Lambda_{k+1}}^{t} \int_{\Lambda_{k+2}}^{t_{k+1}} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_{k+1}t + (\lambda_{k+2}-\lambda_{k+1})t_{k+1} + \ldots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, |f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)|\,\right] dt_n \dots dt_{k+1} \Bigg|$$

$$+ \sum_{j=1}^{k} \Bigg| \lambda_j \int_{\Lambda_j}^{t} \int_{\Lambda_{k+1}}^{t_j} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2 - \lambda_1)t_1 + \ldots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, |f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)|\,\right] dt_n \dots dt_j \Bigg|$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{k} \Bigg| \lambda_i \lambda_j \int_{\Lambda_i}^{t} \int_{\Lambda_j}^{t_i} \int_{\Lambda_{k+1}}^{t_j} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_i t + (\lambda_j - \lambda_i)t_i + \ldots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, |f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)|\,\right] dt_n \dots dt_j\, dt_i \Bigg| + \dots$$

$$+ \Bigg| \left( \prod_{j=1}^{k} \lambda_j \right) \int_{\Lambda_1}^{t} \int_{\Lambda_2}^{t_1} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2 - \lambda_1)t_1 + \ldots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, |f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)|\,\right] dt_n \dots dt_1 \Bigg|$$

$$\leq \Bigg| \int_{\Lambda_{k+1}}^{t} \int_{\Lambda_{k+2}}^{t_{k+1}} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_{k+1}t + (\lambda_{k+2}-\lambda_{k+1})t_{k+1} + \ldots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[ L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)| \right] dt_n \dots dt_{k+1} \Bigg|$$

$$+ \sum_{j=1}^{k} \Bigg| \lambda_j \int\limits_{\Lambda_j}^{t} \int\limits_{\Lambda_{k+1}}^{t_j} \dots \int\limits_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2-\lambda_1)t_1 + \dots + (\lambda_n-\lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[ L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)| \right] dt_n \dots dt_j \Bigg|$$

$$+ \sum_{\substack{i,j=1\\ i<j}}^{k} \Bigg| \lambda_i\lambda_j \int\limits_{\Lambda_i}^{t} \int\limits_{\Lambda_j}^{t_i} \int\limits_{\Lambda_{k+1}}^{t_j} \dots \int\limits_{\Lambda_n}^{t_{n-1}} e^{\lambda_i t + (\lambda_j-\lambda_i)t_i + \dots + (\lambda_n-\lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[ L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)| \right] dt_n \dots dt_j\, dt_i \Bigg| + \dots$$

$$+ \Bigg| \left( \prod_{j=1}^{k} \lambda_j \right) \int\limits_{\Lambda_1}^{t} \int\limits_{\Lambda_2}^{t_1} \dots \int\limits_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2-\lambda_1)t_1 + \dots + (\lambda_n-\lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[ L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)| \right] dt_n \dots dt_1 \Bigg|$$

$$= \frac{1}{\left| \prod\limits_{i=k+1}^{n} (-\lambda_i) \right|} \int\limits_0^1 \int\limits_0^1 \dots \int\limits_0^1 L \Bigg| y(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Bigg|$$

$$+ \Bigg| p(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Bigg| ds_n \dots ds_{k+2}\, ds_{k+1}$$

$$+ \frac{1}{\left| \prod\limits_{i=k+1}^{n} (-\lambda_i) \right|} \sum_{i=1}^{k} \int\limits_0^1 \int\limits_0^1 \dots \int\limits_0^1 L \Bigg| y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Bigg|$$

$$+ \Bigg| p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Bigg| ds_n \dots ds_{k+1}\, ds_i$$

$$+ \dots + \frac{1}{\left| \prod\limits_{i=k+1}^{n} (-\lambda_i) \right|} \int\limits_0^1 \int\limits_0^1 \dots \int\limits_0^1 L \Bigg| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Bigg|$$

$$+\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,ds_n\dots ds_2\,ds_1.$$

Thus, for arbitrary $a<b$, the following inequality holds:

$$\int_a^b |y^{(k)}(t+\tau)-y^{(k)}(t)|\,dt \le$$

$$\le \frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\int_a^b\int_0^1\dots\int_0^1 L\Big|y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,ds_n\dots ds_{k+1}\,dt$$

$$+\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\sum_{i=1}^{k}\int_a^b\int_0^1\int_0^1\dots\int_0^1 L\Big|y(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)$$

$$-y(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,ds_n\dots ds_{k+1}\,ds_i\,dt$$

$$+\dots+\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\int_a^b\int_0^1\dots\int_0^1 L\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,ds_n\dots ds_1\,dt$$

$$=\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\int_0^1\dots\int_0^1\int_a^b L\Big|y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,dt\,ds_n\dots ds_{k+1}$$

$$+\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\sum_{i=1}^{k}\int_0^1\int_0^1\dots\int_0^1\int_a^b L\Big|y(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)$$

$$- y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big| p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$\times\, dt\, ds_n \,\ldots\, ds_{k+1}\, ds_i$$

$$+ \ldots + \frac{1}{\prod_{i=k+1}^{n} |\lambda_i|} \int_0^1 \ldots \int_0^1 \int_a^b L\Big| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big| p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big| dt\, ds_n \ldots ds_1.$$

Now, the $\bar{G}$-almost-periodicity of derivatives $y^{(k)}(\cdot)$ will be verified for single cases separately. Applying (iii) and employing to the correspondence between the $\varepsilon$-almost-period of $p(\cdot)$ and the $\frac{\varepsilon}{|a_n|-L}$-almost-period of solution $y(\cdot)$ (in the given pseudo-metric), one obtains e.g. in the Stepanov case (taking $a := u$, $b := u + 1$):

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt \le$$

$$\le \sup_{u\in\mathbb{R}} \frac{1}{\prod_{i=k+1}^{n} |\lambda_i|} \Bigg( \int_0^1 \ldots \int_0^1 \int_u^{u+1} L\Big| y(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big| p(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|\, dt\, ds_n \ldots ds_{k+1}$$

$$+ \sum_{i=1}^{k} \int_0^1 \int_0^1 \ldots \int_0^1 \int_u^{u+1} L\Big| y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big| p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$\times\, dt\, ds_n \ldots ds_{k+1}\, ds_i$$

$$+ \ldots + \int_0^1 \ldots \int_0^1 \int_u^{u+1} L\Big| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big| p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|\, dt\, ds_n \ldots ds_1 \Bigg)$$

$$< \frac{1}{\prod\limits_{i=k+1}^{n} |\lambda_i|} \cdot \left\{ \frac{\varepsilon\, L}{|a_n| - L} + \varepsilon + \binom{k}{1} \left[ \frac{\varepsilon\, L}{|a_n| - L} + \varepsilon \right] + \binom{k}{2} \left[ \frac{\varepsilon\, L}{|a_n| - L} + \varepsilon \right] \right.$$

$$\left. + \ldots + \binom{k}{k} \left[ \frac{\varepsilon\, L}{|a_n| - L} + \varepsilon \right] \right\} = \frac{|a_n|\, \varepsilon}{(|a_n| - L) \cdot \prod\limits_{i=k+1}^{n} |\lambda_i|} \sum_{j=0}^{k} \binom{k}{j}$$

$$= \frac{2^k |a_n|\, \varepsilon}{(|a_n| - L) \prod\limits_{i=k+1}^{n} |\lambda_i|}.$$

Following the similar way as above, we can prove quite analogously the equi-$W$-almost-periodicity of derivatives (taking $a := u$, $b := u + l$).

To verify the Weyl-almost-periodicity or the Besicovitch-almost-periodicity of derivatives $y^{(k)}(\cdot)$, we use the above integral estimate. Integrands contain the Weyl-a.p. or Besicovitch-a.p. function $p(\cdot)$ and entirely bounded, $W$-a.p. or $B$-a.p. solution $y(\cdot)$, respectively. Thus, we can verify by the similar manner as above the Weyl-almost-periodicity (putting $a := u$, $b := u + l$) as well as the Besicovitch-almost-periodicity ($a := -T$, $b = T$) of derivatives.

After all, to $\varepsilon$-almost-period of function $p(\cdot)$, there corresponds the $\frac{2^k |\lambda_1 \ldots \lambda_k|}{|a_n| - L} \varepsilon$-almost-period of $k$-th derivative ($k = 1, \ldots, n - 1$) of solution $y(\cdot)$, in the given (pseudo-)metric, provided $L < |a_n|$. $\square$

## 4 Existence of a.p. solutions: shifted polynomials approach

It is not very convenient that the almost-periods of the derivatives of a $\bar{G}$-a.p. solution depended on the roots of the characteristic polynomial (2). The shifted polynomials approach will allow us to avoid this handicap.

**Theorem 2** *Let the above conditions (i)–(iv) be satisfied. Assume, furthermore, that $a_j \neq 0$, for $j = 1, \ldots, n - 1$, and that all shifted polynomials $\lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-p-j}$, $p = 0, \ldots, n - 1$, have real nonzero roots. If $|a_n| > L$, then equation (1) admits a unique bounded $\bar{G}$-a.p. solution with $\bar{G}$-a.p. derivatives, up to the $(n - 1)$-th order.*

*Moreover, the $\varepsilon$-almost-period of $p(\cdot)$ implies the $\frac{2^k |a_n|}{|a_{n-k}|(|a_n| - L)}\, \varepsilon$-almost-period of the $k$-th derivative of the solution in the $\bar{G}$-(pseudo-)metric, for $k = 0, \ldots, n - 1$.*

**Proof** The existence of a unique bounded solution $y(\cdot)$ of (1) follows from Lemma 1. Its $\bar{G}$-almost-periodicity can be proved exactly in the same way as in the proof of Theorem 1. So, it remains to prove the $\bar{G}$-almost-periodicity of derivatives $y^{(k)}(\cdot)$. Putting $y(t)$ into $f$ and substituting $\phi = y'$, one can write (1) in the form

$$\phi^{(n-1)} + \sum_{j=1}^{n-1} a_j \phi^{(n-j-1)} = f(y(t)) - a_n y(t) + p(t),$$

with exactly one bounded solution (again, according to Lemma 1). Applying the same procedure as at the beginning of the proof of Theorem 1, we can write the following inequality:

$$|\phi(t+\tau) - \phi(t)| \leq$$

$$\leq \frac{1}{|\widehat{\lambda}_{n-1}| \dots |\widehat{\lambda}_1|} \int_0^1 \dots \int_0^1 (L + |a_n|) \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big|$$

$$+ \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1,$$

where $\widehat{\lambda}_j \in \mathbb{R}$, $j = 1, \dots, n-1$, are nonzero roots of the corresponding characteristic polynomial $\lambda^{n-1} + \sum_{j=1}^{n-1} a_j \lambda^{n-1-j}$. Thus, for arbitrary $a < b$, the following estimate holds

$$\int_a^b |y'(t+\tau) - y'(t)| \, dt = \int_a^b |\phi(t+\tau) - \phi(t)| \, dt \leq$$

$$\leq \frac{L + |a_n|}{|\widehat{\lambda}_{n-1}| \dots |\widehat{\lambda}_1|} \int_a^b \int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt$$

$$+ \frac{1}{|\widehat{\lambda}_{n-1}| \dots |\widehat{\lambda}_1|} \int_a^b \int_0^1 \dots \int_0^1 \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt$$

$$= \frac{L + |a_n|}{|a_{n-1}|} \int_a^b \int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt$$

$$+ \frac{1}{|a_{n-1}|} \int_a^b \int_0^1 \dots \int_0^1 \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt.$$

To prove the $\bar{G}$-almost-periodicity of $y'(\cdot)$, denote by $\tau$ the $\varepsilon$-almost period of $p$ and, subsequently, the $\frac{\varepsilon}{|a_n|-L}$-almost period of $y$ (in the $\bar{G}$ (pseudo-)metric).

Concretely, for the $S$-almost-periodicity of $y'(\cdot)$, we apply the preceding inequality with $a = u$, $b = u+1$ and the fact that $\tau$ is the Stepanov $\varepsilon$-almost period of $p$ as well as the Stepanov $\frac{\varepsilon}{|a_n|-L}$-almost period of $y$. Therefore,

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} |y'(t+\tau) - y'(t)|\, dt < \frac{L+|a_n|}{|a_{n-1}|} \sup_{u\in\mathbb{R}} \int_u^{u+1}\int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1}\dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|}$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \cdot \frac{\varepsilon}{|a_n|-L} + \frac{\varepsilon}{|a_{n-1}|} = \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

Repeating the procedure with the equi-Weyl pseudo-metric, we obtain for $a = u$, $b = u+l$, where $l \geq l_0$, and for the equi-Weyl $\varepsilon$-almost period of $p$ denoted by $\tau$ (which is the $\frac{\varepsilon}{|a_n|-L}$-almost period of solution $y$) that

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y'(t+\tau) - y'(t)|\, dt <$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l}\int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1}\dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|}$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \cdot \frac{\varepsilon}{|a_n|-L} + \frac{\varepsilon}{|a_{n-1}|} = \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

This inequality holds for $\forall l \geq l_0$, where $l_0$ is connected with $p$.

The $W$-almost-periodicity of $y'(\cdot)$ will be proved in the same way. Denoting by $\tau$ the Weyl $\varepsilon$-almost period of $p$, one can derive:

$$\lim_{l\to+\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y'(t+\tau) - y'(t)|\, dt <$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \lim_{l\to+\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l}\int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1}\dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|}$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \cdot \frac{\varepsilon}{|a_n|-L} + \frac{\varepsilon}{|a_{n-1}|} = \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

Finally, let us concentrate on the Besicovitch case. Thanks to the above integral estimate, we can verify the $B$-almost periodicity of the derivative $y'(\cdot)$:

$$\limsup_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{+T} |y'(t+\tau) - y'(t)|\,dt <$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \limsup_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{+T} \int\limits_0^1 \dots \int\limits_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1} \dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|} < \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

Hence, to $\varepsilon$-almost-period of $p$, there corresponds the $\frac{2|a_n|}{|a_{n-1}|(|a_n|-L)}$ $\varepsilon$-almost-period of $y'$, in the $\bar{G}$-(pseudo-)metric.

Putting $\psi = \phi'$, we arrive at the equation

$$\psi^{(n-2)} + \sum_{j=1}^{n-2} a_j \psi^{(n-j-2)} = f(y(t)) - a_n y(t) - a_{n-1} y'(t) + p(t).$$

In view of Lemma 1, this equation has exactly one entirely bounded solution. Proceeding by the similar way as above and denoting the roots of the corresponding characteristic polynomial by $\widehat{\widehat{\lambda}}_j, j = 1, \dots, n-2$, one gets the estimate

$$|\psi(t+\tau) - \psi(t)| \le \frac{1}{|\widehat{\widehat{\lambda}}_{n-2}| \dots |\widehat{\widehat{\lambda}}_1|} \int\limits_0^1 \dots \int\limits_0^1 (L+|a_n|) \Big| y(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t)\Big| + |a_{n-1}| \Big| y'(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t + \tau) - y'(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t)\Big|$$

$$+ \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t + \tau) - p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t)\Big|\, ds_{n-2} \dots ds_1,$$

which leads (for $a < b$) to

$$\int\limits_a^b |y''(t+\tau) - y''(t)|\,dt = \int\limits_a^b |\psi(t+\tau) - \psi(t)|\,dt$$

$$\leq \frac{L+|a_n|}{|a_{n-2}|}\int_a^b\int_0^1\dots\int_0^1\left|y(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t+\tau)-y(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t)\right|$$
$$\times\, ds_{n-2}\dots ds_1\,dt$$

$$+\frac{|a_{n-1}|}{|a_{n-2}|}\int_a^b\int_0^1\dots\int_0^1\left|y'(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t+\tau)-y'(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t)\right|ds_{n-2}\dots ds_1\,dt$$

$$+\frac{1}{|a_{n-2}|}\int_a^b\int_0^1\dots\int_0^1\left|p(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t+\tau)-p(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t)\right|ds_{n-2}\dots ds_1\,dt.$$

Proceeding by the same way as in the case of $y'(\cdot)$, we can analyze all kinds of (pseudo-)metrics separately. The $\bar{G}$-almost-periodicity of $y''(\cdot)$ can be verified by means of the $\varepsilon$-almost-period of $p$ (denoted by $\tau$, as usual), which coincides with the $\frac{\varepsilon}{|a_n|-L}$-almost period of solution $y$ and the $\frac{2|a_n|}{|a_{n-1}|(|a_n|-L)}$ $\varepsilon$-almost-period of $y'$, in the $\bar{G}$-(pseudo-)metric. Repeating the procedure as above, we get that the mentioned almost-period $\tau$ coincides with the $\frac{4|a_n|}{|a_{n-2}|(|a_n|-L)}$ $\varepsilon$-almost-period of $y''$, in the $\bar{G}$-(pseudo-)metric.

By the same manner, we can verify the $\bar{G}$-almost-periodicity of higher-order derivatives $y^{(k)}$. The essential estimate takes now the form

$$\int_a^b |y^{(k)}(t+\tau)-y^{(k)}(t)|\,dt\leq$$

$$\leq\frac{L+|a_n|}{\prod_{j=1}^{n-k}|\widetilde{\lambda}_j|}\int_a^b\int_0^1\dots\int_0^1\left|y(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t+\tau)-y(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t)\right|ds_{n-k}\dots ds_1\,dt$$
$$+\sum_{l=1}^{k-1}\left(\frac{|a_{n-l}|}{\prod_{j=1}^{n-k}|\widetilde{\lambda}_j|}\int_a^b\int_0^1\dots\int_0^1\left|y^{(l)}(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t+\tau)\right.\right.$$
$$\left.\left.-\,y^{(l)}(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t)\right|ds_{n-k}\dots ds_1\,dt\right)$$
$$+\frac{1}{\prod_{j=1}^{n-k}|\widetilde{\lambda}_j|}\int_a^b\int_0^1\dots\int_0^1\left|p(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t+\tau)-p(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t)\right|$$
$$\times\, ds_{n-k}\dots ds_1\,dt,$$

for $a<b$, where $\widetilde{\lambda}_j\in\mathbb{R}$ denote the nonzero roots of the related shifted polynomial ($j=1,\dots,n-k$).

Studying all cases separately, we obtain the $\bar{G}$-almost-periodicity of $y^{(k)}$. Moreover, the relationship between $\bar{G}$-almost-periods of $p$ and of derivatives $y^{(k)}$ can be described as follows: to $\varepsilon$-almost-period of $p$, there corresponds the $\frac{2^k|a_n|}{|a_{n-k}|(|a_n|-L)}\,\varepsilon$-almost-period of $y^{(k)}$, in the $\bar{G}$-(pseudo-)metric, for $k = 0,\dots,n-1$, provided $L < |a_n|$, and $|a_j| \neq 0$, for $j = 0,\dots,n$. This completes the proof. $\square$

## 5 Existence of a.p. solutions: case of negative roots

Another way how to come to almost-periods of the derivatives not depending on the roots of the characteristic polynomial $\lambda^n + \sum_{j=1}^n a_j\lambda^{n-j}$ is to assume that all roots are negative. This implies that all coefficients $a_j$, $j = 1,\dots,n$, must be positive.

**Theorem 3** *Let the above conditions (i)–(iv) be satisfied. Assume additionally that all roots of the characteristic polynomial (2) are negative. If $L < a_n$, then there exists a unique bounded $\bar{G}$-a.p. solution $y(\cdot)$ of equation (1) with $\bar{G}$-a.p. derivatives, up to the $(n-1)$-th order.*

*Moreover, the $\varepsilon$-almost-period of $p(\cdot)$ implies the $\frac{2^k a_k}{\binom{n}{k}(a_n-L)}\,\varepsilon$-almost-period of the $k$-th derivative $y^{(k)}(\cdot)$ of the solution $y(\cdot)$, in the $\bar{G}$-(pseudo-)metric, for $k = 0,\dots,n-1$, where $a_0 := 1$.*

**Proof** According to Lemma 1, equation (1) admits exactly one bounded solution. Its representation formula can be now written in the form:

$$y(t) = \int_{-\infty}^{t}\int_{-\infty}^{t_1}\dots\int_{-\infty}^{t_{n-1}} e^{\lambda_1 t+(\lambda_2-\lambda_1)t_1+\dots+(\lambda_n-\lambda_{n-1})t_{n-1}-\lambda_n t_n} \times [f(y(t_n)) + p(t_n)]\,dt_n\dots dt_1.$$

Analogously as in the proof of Theorem 1, we can prove the $\bar{G}$-almost-periodicity of solution $y(\cdot)$.

Proceeding by the same way as in the proof of Theorem 1, we can check the $\bar{G}$-almost-periodicity of derivatives $y^{(k)}(\cdot)$, $k = 1,\dots,n-1$. More precisely, we can specify that the $\varepsilon$-almost-period of $p(\cdot)$ implies the $\frac{2^k(-1)^k\lambda_1\dots\lambda_k}{a_n-L}\,\varepsilon$-almost-period of $k$-th derivative of solution $y(\cdot)$, $k = 1,\dots,n-1$, in the $\bar{G}$-sense. Due to the independence of the preceding term under the permutation of roots (see [AG, p. 326]), one has $\binom{n}{k}$ choices of $\lambda_{i_1},\dots,\lambda_{i_k}$ for $n$ roots of the characteristic polynomial (2). Let us sum up the following $\binom{n}{k}$ inequalities:

$$\sup_{u\in\mathbb{R}}\int_u^{u+1} |y^{(k)}(t+\tau) - y^{(k)}(t)|\,dt < \frac{2^k(-1)^k\,\lambda_{i_1}\dots\lambda_{i_k}}{a_n - L}\,\varepsilon,$$

for the $S$-metric,

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_{u}^{u+l} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt < \frac{2^k(-1)^k\, \lambda_{i_1} \dots \lambda_{i_k}}{a_n - L}\, \varepsilon,$$

in the equi-Weyl case (for all $l \geq l_0$, where $l_0$ is connected with $p$),

$$\lim_{l\to+\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_{u}^{u+l} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt < \frac{2^k(-1)^k\, \lambda_{i_1} \dots \lambda_{i_k}}{a_n - L}\, \varepsilon,$$

for the Weyl pseudo-metric, and

$$\limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt < \frac{2^k(-1)^k\, \lambda_{i_1} \dots \lambda_{i_k}}{a_n - L}\, \varepsilon,$$

in the Besicovitch case.

Divide these sums by $\binom{n}{k}$. Now, application of the Vieta formula

$$\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} (-1)^k \prod_{j=1}^{k} \lambda_{i_j} = a_k$$

leads to the desired simplification: every $\varepsilon$-almost-period of $p(\cdot)$ implies the $\frac{2^k a_k}{\binom{n}{k}(a_n-L)}$ $\varepsilon$-almost-period of $y^{(k)}(\cdot)$, in the $\bar{G}$-sense. □

# 6 Concluding remarks

First of all, one can readily check that all main theorems remain valid if, instead of the boundedness of $f$, only the existence of a positive constant $D_0 > 0$ is assumed such that (cf. Remark 2)

$$\max_{|y|\leq D_0/|a_n|} |f(y)| + \operatorname*{supess}_{t\in\mathbb{R}} |p(t)| \leq D_0.$$

The same is true for the Lipschitzianity of $f$: it is enough that

$$|f(x) - f(y)| \leq L\, |x - y|$$

holds, with $0 < L < |a_n|$, only for $|x| \leq \frac{D_0}{|a_n|}$, $|y| \leq \frac{D_0}{|a_n|}$.

Therefore, considering the pendulum-type equation

$$y'' + ay' + b \sin y = p(t), \tag{3}$$

where $a, b$ are nonzero constants such that $a^2 \geq 4|b|$ and $p \in \mathrm{L}^1_{loc}(\mathbb{R},\mathbb{R})$ is $\overline{G}$-a.p., and following the arguments in [A1] (cf. [AG, pp. 556–557]), we can easily deduce that eguation (3) *admits at least two $\overline{G}$-a.p. solutions $y_1(\cdot)$ and $y_2(\cdot)$ with $\overline{G}$-a.p. derivatives such that*

$$\sup_{t\in(-\infty,\infty)} |y_1(t)| < \frac{\pi}{2} \qquad \text{and} \qquad \sup_{t\in(-\infty,\infty)} |y_2(t) - \pi| < \frac{\pi}{2},$$

provided only

$$\operatorname*{supess}_{t\in(-\infty,\infty)} |p(t)| < |b|.$$

Furthermore, since multivalued Lipschitz-continuous function with nonempty, convex and compact values $\varphi : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, i.e.

$$\mathrm{d}_H\left(\varphi(x), \varphi(y)\right) \leq L\,|x - y|,$$

where $\mathrm{d}_H$ stands for the Hausdorff metric and $L \in \mathbb{R}$ is a constant, possesses a single-valued Lipschitz continuous selection $f \subset \varphi$ with constant $L_0$ such that $L_0 := L(12\sqrt{3}/5 + 1)$ (see e.g. [HP, pp. 101–103]), and since Stepanov or equi-Weyl a.p. multivalued function with nonempty, convex and compact values $P : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ possesses a single-valued Stepanov or equi-Weyl a.p. selection $p \subset P$, respectively (see [D1], [D2], [DS] resp. [D3]), the existence parts (without uniqueness) of all main theorems can be extended to the differential inclusions

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} \in \varphi(y) + P(t),$$

provided a positive constant $D_0 > 0$ exists such that

$$\max_{|y|\leq D_0/|a_n|} |\varphi(y)| + \operatorname*{supess}_{t\in\mathbb{R}} |P(t)| \leq D_0,$$

$\varphi$ is Lipschitz-continuous, for $|y| \leq \frac{D_0}{|a_n|}$, with a constant $L$ such that

$$L < |a_n|/(12\sqrt{3}/5 + 1),$$

and $P$ is either Stepanov or equi-Weyl almost-periodic in a multivalued sense (for the related definitions and more details, see e.g. [AG, Chapter III.10 and Appendix A.1]).

Finally, all $\overline{G}$-a.p. solutions $y(\cdot)$ and their derivatives, up to the order $(n-1)$, are in fact (see Remark 3) $\overline{G}$-normal (a.p. in the sense of Bochner), i.e. the families $\{y^{(k)}(t + h) \mid h \in \mathbb{R}\}$, $h = 0, 1, \ldots, n - 1$, are $\overline{G}$-precompact, because these solutions and their derivatives are bounded and uniformly continuous; for more details, see [AG, Chapter III.10] and [ABG]. Stepanov a.p. solutions are even uniformly almost-periodic.

Some further remarks are in order.

**Remark 5** Observe the similarity of the estimates for $\varepsilon$-almost-periods with those for bounded solutions and their derivatives in Remark 2.

**Remark 6** Analogous theorems can be obtained when only assuming that the real parts of the roots of the characteristic polynomial (2) are nonzero. On the other hand, the explicit inequality $L < |a_n|$ would be replaced by a rather implicit condition $L < |\alpha_1 \dots \alpha_n|$, where $\alpha_j = \operatorname{Re}\lambda_j$, $j = 1, \dots, n$, denote the real parts of the roots $\lambda_j$ of (2). Moreover, the related $\varepsilon$-almost-periods of a.p. solutions and their derivatives would depend on $\alpha_j$, $j = 1, \dots, n$.

**Remark 7** Similar theorems can be also deduced for a more general equation than (1), namely

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} = \sum_{j=1}^{n} f_j\left(y^{(n-j)}\right) + p(t),$$

or inclusion (without uniqueness)

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} \in \sum_{j=1}^{n} \varphi_j\left(y^{(n-j)}\right) + P(t),$$

but the related calculations would be rather cumbersome. At least in the case of uniformly a.p. solutions, this will be treated by ourselves elsewhere.

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