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$2-(n^2, 2n, 2n-1)$ Designs Obtained from Affine Planes*

Andrea CAGGEGI

Dipartimento di Metodi e Modelli Matematici,
Facoltà di Ingegneria Università di Palermo,
viale delle Scienze I-90128, Palermo, Italy
e-mail: caggegi@unipa.it

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Abstract

The simple incidence structure $\mathcal{D}(\mathcal{A}, 2)$ formed by points and unordered pairs of distinct parallel lines of a finite affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ of order $n > 2$ is a $2-(n^2, 2n, 2n-1)$ design. If $n = 3$, $\mathcal{D}(\mathcal{A}, 2)$ is the complementary design of $\mathcal{A}$. If $n = 4$, $\mathcal{D}(\mathcal{A}, 2)$ is isomorphic to the geometric design $AG_3(4, 2)$ (see [2; Theorem 1.2]). In this paper we give necessary and sufficient conditions for a $2-(n^2, 2n, 2n-1)$ design to be of the form $\mathcal{D}(\mathcal{A}, 2)$ for some finite affine plane $\mathcal{A}$ of order $n > 4$. As a consequence we obtain a characterization of small designs $\mathcal{D}(\mathcal{A}, 2)$.

Key words: $2-(n^2, 2n, 2n-1)$ designs; incidence structure; affine planes.

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By a $2-(v, k, \lambda)$ *design* we mean a pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a collection of distinguished subsets of $\mathcal{P}$ called blocks such that each block contains k points and any two distinct points are contained in exactly λ common blocks[1]. Our main result is the following

Theorem 1 *Let n be an integer with $n > 4$ and let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a $2-(n^2, 2n, 2n-1)$ design. Then $\mathcal{D}$ is of the form $\mathcal{D}(\mathcal{A}, 2)$ if and only if the following two conditions are satisfied: (c_1) any three distinct points of $\mathcal{D}$*

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[1]For further definitions (and basic results) about 2-designs see [1].

are contained in exactly 3 or $n-1$ common blocks; (c_2) if $X_1, X_2, \ldots, X_{n-1}$ are $n-1$ distinct blocks of $\mathcal{D}$ such that $|X_1 \cap X_2 \cap \cdots \cap X_{n-1}| > 2$, then $X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j$ whenever $i \neq j$.

Before proving the theorem we need some preliminary results about $2-(n^2, 2n, 2n-1)$ designs.

Lemma 1 *Suppose $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ is a finite affine plane of order $n > 4$ and let $\mathcal{D}(\mathcal{A}, 2)$ be the system of points and unordered pairs of distinct parallel lines of $\mathcal{A}$. Then $\mathcal{D}(\mathcal{A}, 2)$ is a $2-(n^2, 2n, 2n-1)$ design satisfying the following properties:*

(1) any three distinct collinear points of $\mathcal{A}$ are contained in exactly $n-1$ blocks of $\mathcal{D}(\mathcal{A}, 2)$;

(2) any three distinct non-collinear points of $\mathcal{A}$ are joined by precisely 3 blocks of $\mathcal{D}(\mathcal{A}, 2)$;

(3) if $X_1, X_2, \ldots, X_{n-1}$ are $n-1$ distinct blocks of $\mathcal{D}(\mathcal{A}, 2)$ such that $|X_1 \cap X_2 \cap \cdots \cap X_{n-1}| > 2$, then $X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j$ whenever $i \neq j$.

Proof This follows directly from the definition of $\mathcal{D}(\mathcal{A}, 2)$. □

Lemma 2 *Let n be an integer greater than 4 and let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a $2-(n^2, 2n, 2n-1)$ design any three distinct points of which are contained in exactly 3 or $n-1$ blocks. Then for any choice of two distinct points x, y in $\mathcal{D}$ there are precisely $n-2$ points $z \in \mathcal{P} \setminus \{x, y\}$ with the property that x, y, z are joined by $n-1$ distinct blocks of $\mathcal{D}$.*

Proof Let x, y be any two distinct points of $\mathcal{D}$ and denote by c the number of points $z \in \mathcal{P} \setminus \{x, y\}$ with the property that x, y, z are joined by $n-1$ blocks of $\mathcal{D}$. Then $0 \leq c \leq n^2 - 2$ and $n^2 - 2 - c$ is the number of points $w \in \mathcal{P} \setminus \{x, y\}$ with the property that x, y, w are joined by exactly 3 blocks of $\mathcal{D}$. Thus, counting the point block pairs (p, C) with $x \neq p \neq y$ and $\{x, y, p\} \subset C$, we find $3(n^2 - 2 - c) + (n-1)c = (2n-2)(2n-1)$ which can be written as $(n-4)c = (n-4)(n-2)$. Hence, since $n - 4 \neq 0$, $c = n-2$ and the lemma is proved. □

Lemma 3 *Let n be an integer with $n > 4$ and let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a $2-(n^2, 2n, 2n-1)$ design. If $X_1, X_2, \ldots, X_{n-1}$ are $n-1$ distinct blocks of $\mathcal{D}$ such that $X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j$ whenever $i \neq j$, then $|X_1 \cap X_2 \cap \cdots \cap X_{n-1}| \geq n$ with equality if and only if $X_1 \cup X_2 \cup \ldots X_{n-1} = \mathcal{P}$.*

Proof Write $X_1 \cup X_2 \cup \cdots \cup X_{n-1} = l \cup (X_1 \setminus l) \cup (X_2 \setminus l) \cup \cdots \cup (X_{n-1} \setminus l)$, where $l = X_1 \cap X_2 \cap \cdots \cap X_{n-1}$. Then $|X_1 \cup X_2 \cup \cdots \cup X_{n-1}| = a + (n-1)(2n-a) = n^2 + (n-2)(n-a)$ with $a = |l|$. Thus, since $\mathcal{D}$ has n^2 points, we obtain $n^2 \geq n^2 + (n-2)(n-a)$ which, since $n > 4$, gives $n \leq a$. Moreover $n = a$ is equivalent to ask $|X_1 \cup X_2 \cup \cdots \cup X_{n-1}| = n^2$, i.e. $X_1 \cup X_2 \cup \cdots \cup X_{n-1} = \mathcal{P}$, and the lemma is proved. □

Proof of Theorem 1 In view of Lemma 1, we have only to prove that $\mathcal{D} = \mathcal{D}(\mathcal{A}, 2)$ for some affine plane $\mathcal{A}$ (of order n), provided conditions (c_1) and (c_2) hold. Define $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ by taking $\mathcal{P}$ as the set of points and the set $\mathcal{L} = \{l \subset P : |l| > 2, l = L_1 \cap L_2 \cap \cdots \cap L_{n-1}$ with $L_1, L_2, \dots, L_{n-1}$ distinct blocks of $\mathcal{D}\}$ as the set of lines. By Lemma 2, $\mathcal{L}$ is non empty. Let $l \in \mathcal{L}$ and let $L_1, L_2, \dots, L_{n-1}$ be the $n-1$ distinct blocks of $\mathcal{D}$ such that $l = L_1 \cap L_2 \cap \cdots \cap L_{n-1}$. Then condition (c_2) gives $l = L_i \cap L_j$ whenever $i \neq j$ so that, by Lemma 3, l contains at least n points. On the other hand, as any three distinct points of l are joined by the $n-1$ blocks L_i $(i = 1, 2, \dots, n-1)$, it follows from Lemma 2 that l contains at most $2 + (n-2) = n$ points. Thus we must have $n \leq |l| \leq n$ and consequently $|l| = n$. Let x, y be any two distinct points of $\mathcal{D}$. By Lemma 2 we may choose a point $z \in \mathcal{P} \setminus \{x, y\}$ and $n-1$ distinct blocks $Z_1, Z_2, \dots, Z_{n-1} \in \mathcal{B}$ such that $\{x, y, z\} \subseteq Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}$. Therefore $h = Z_1 \cap Z_2 \cap \dots Z_{n-1}$ belongs to $\mathcal{L}$ and passes through both x and y. Assume that $\{x, y\} \subseteq k$ for some $k \in \mathcal{L}$ with $k \neq h$. Writing k as the intersection $k = W_1 \cap W_2 \cap \dots W_{n-1}$ of $n-1$ distinct blocks $W_1, W_2, \dots, W_{n-1} \in \mathcal{B}$ we obtain $\{x, y, p\} \subseteq Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}$ or $\{x, y, p\} \subseteq W_1 \cap W_2 \cap \cdots \cap W_{n-1}$ whenever $p \in h \cup k$ is a point such that $x \neq p \neq y$. Then from Lemma 2 we deduce $|h \cup k| \leq 2 + (n-2) = n$ which contradicts our assumption $k \neq h$ and shows that h is the unique element in $\mathcal{L}$ containing $\{x, y\}$. Thus each $l \in \mathcal{L}$ has n points and each pair of points is on exactly one common point set $m \in \mathcal{L}$: this is sufficient to conclude that $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ is a finite affine plane of order n. Note that such a plane $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ has the properties: (i) for any line $l \in \mathcal{L}$ and any point $x \in \mathcal{P}, x \notin l$, there is just one block of $\mathcal{D}$ containing both l and x; (ii) if a block $C \in \mathcal{B}$ contains a line $h \in \mathcal{L}$ and if $y \in C$ is a point not on h, then $C = h \cup k$ where $k \in \mathcal{L}$ is the only line of $\mathcal{A}$ through y not intersecting h. Property (i) follows from the fact that (by condition (c_2) and Lemma 3) the point set $\mathcal{P}$ can be written as disjoint union $P = l \cup (L_1 \setminus l) \cup (L_2 \setminus l) \cup \cdots \cup (L_{n-1} \setminus l)$, if $L_1, L_2, \dots, L_{n-1}$ are the $n-1$ distinct blocks of $\mathcal{D}$ through the line $l \in \mathcal{L}$. To show (ii) we proceed as follows. Denote by k the line of $\mathcal{A}$ through y parallel to h. Let $z \in C \setminus h$ be a point distinct from y and denote by l the line of $\mathcal{A}$ joining y to z. We claim that $l = k$. In fact $l \neq h$ and $l = W_1 \cap W_2 \cap \cdots \cap W_{n-1}$ for suitable $n-1$ distinct blocks $W_1, W_2, \dots, W_{n-1} \in \mathcal{B}$. Suppose there is a point $w \in h \cap l$. Then y, z, w are three distinct points belonging to l and, by condition (c_1), there is no block in $\mathcal{D}$ containing $\{y, z, w\}$, apart from the blocks W_i. But $h \subset C$ forces $w \in C$ and consequently $\{y, z, w\} \subset C$. Thus we have $C = W_i$ for some $i \in \{1, 2, \dots, n-1\}$ so that $l \subset C$. Then $l \cup h \subseteq C$ and there is just one point $p \in C$ such that $p \notin l \cup h$, since $|C| = 2n = 1 + |l \cup h|$. As p belongs to $n+1$ lines of $\mathcal{A}$, we may choose a line $s \in \mathcal{L}$ through p such that $w \notin s$ and s meets both l and h. Since $C = \{p\} \cup l \cup h$, we have that s intersects C in exactly three points, namely $p, l \cap s$ and $h \cap s$. On the other hand, if $S_1, S_2, \dots, S_{n-1}$ are the $n-1$ distinct blocks of $\mathcal{D}$ such that $s = S_1 \cap S_2 \cap \cdots \cap S_{n-1}$, we infer from condition (c_1) that $S_1, S_2, \dots, S_{n-1}$ are the only blocks of $\mathcal{D}$ containing $p, l \cap s, h \cap s$. Since

$\{p, l \cap s, h \cap s\} \subset C$, we obtain $C = S_j$ for some $j \in \{1, 2, \ldots, n-1\}$ and hence $s \subset C$. Therefore $s = s \cap C$ consists of three points, a contradiction. Thus l and h do not intersect and l is the unique line of $\mathcal{A}$ through y not intersecting h, i.e. $l = k$. Therefore $z \in k$. As this is true for every point $z \in C \setminus h$ distinct from y and $|C \setminus h| = n = |k|$, we may conclude that $C \setminus h = k$. So $C = h \cup k$ and (ii) holds.

As any parallel class of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ consists of n lines and $\mathcal{A}$ has $n+1$ parallel classes, we infer from (i) and (ii) that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ contains exactly $(n+1)\frac{n(n-1)}{2}$ blocks X of the form $X = l \cup m$ with l, m distinct parallel lines of $\mathcal{A}$. But any $2-(n^2, 2n, 2n-1)$ design has precisely $b = (n+1)\frac{n(n-1)}{2}$ blocks. Then we must have

$$\mathcal{B} = \{X \subset \mathcal{P} : X = l \cup m \text{ with } l, m \text{ distinct parallel lines of } \mathcal{A}\}$$

and hence $\mathcal{D} = \mathcal{D}(\mathcal{A}, 2)$. The theorem is proved. □

Since up to isomorphism there is just one affine plane of order $5, 7$ or 8 we have the following characterization of small designs $\mathcal{D}(\mathcal{A}, 2)$.

Corollary 1 *Suppose n is one of the numbers $5, 7, 8$ and let $\mathcal{A}(n)$ be the desarguesian affine plane of order n. There exists up to isomorphisms exactly one $2-(n^2, 2n, 2n-1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ satisfying conditions (c_1), (c_2) of Theorem 1, namely the 2-design $\mathcal{D}(\mathcal{A}(n), 2)$.*

We end our investigation with a few remarks

Remark 1 If $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ is a finite affine plane of order $n > 4$, then $0, 4, n$ are the intersection numbers of the $2-(n^2, 2n, 2n-1)$ design $\mathcal{D}(\mathcal{A}, 2)$: i.e. $\{0, 4, n\} = \{|X \cap Y| : X, Y \text{ are two distinct blocks of } \mathcal{D}(\mathcal{A}, 2)\}$.

Remark 2 There is no plane of order $n = 6$, but there is an example of a $2-(36, 12, 11)$ design produced by H. Hanany [3], Table 5.23, p. 343. The $2-(25, 10, 9)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ exhibited by H. Hanany, loc. cit. Table 5.23, p. 334 is not of the form $\mathcal{D}(\mathcal{A}, 2)$: since $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ admits 8 as an intersection number (i.e. $|X \cap Y| = 8$ for suitable distinct blocks $X, Y \in \mathcal{B}$).

References

[1] Beth, T., Jungnickel, D, Lenz, H.: Designs Theory. *Bibliographisches Institut, Mannheim–Wien,* 1985.

[2] Caggegi, A.: *Uniqueness of* $AG_3(4, 2)$. Italian Journal of Pure and Applied Mathematics **15** (2004), 9–16.

[3] Hanani, H.: *Balanced incomplete block designs and related designs.* Discrete Math. **11** (1975), 255–369.

[4] Hughes, D. R., Piper, F. C.: Projective Planes. *Springer-Verlag, Berlin–Heidelberg–New York,* 1982, second printing.