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Peter Vassilev Danchev

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Direct Decompositions and Basic Subgroups in Commutative Group Rings

P. V. DANCHEV

13, General Kutuzov Street, block 7, floor 2, apartment 4,
4003 Plovdiv, Bulgaria
e-mail: pvdanchev@yahoo.com

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Abstract

An attractive interplay between the direct decompositions and the explicit form of basic subgroups in group rings of abelian groups over a commutative unitary ring are established. In particular, as a consequence, we give a simpler confirmation of a more general version of our recent result in this aspect published in Czechoslovak Math. J. (2006).

Key words: Direct decompositions; basic subgroups; normed units; group rings.

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1 Introduction

Throughout the text of this brief paper, let G be an abelian group with p-component G_p, written by multiplicative record, and R a commutative ring with identity (of prime characteristic, for instance p, for applications). As usual, RG denotes the group ring of G over R with group of normalized units $V(RG)$, abbreviated for facilitating of the exposition via $V(G)$. For a subgroup A of G, we define by the same reason $I(G; A)$ as the relative augmentation ideal of RG with respect to A. All other notation and terminology from the abelian group theory, not expressly given here, are either standard or follow the classical book of Fuchs [5].

Nachev has demonstrated in [6] that there is a transversal between the basic subgroups of G and $V(G)$, provided that G is p-primary and R is p-perfect with prime characteristic p. Specifically, Nachev established in an explicit form a series of basic subgroups of $V(G)$ by the usage of the basic subgroups of a p-group G under the limitation on R to be p-perfect of prime characteristic p. In the proof, he uses intensively a relationship between the properties of a fixed basic subgroup B of G and the decomposition of $(1 + I(G; B)) \cap V(G)_p$ into appropriate direct factors.

This approach of finding such a connection also captures the spirit of the present short note. We thus identify and focus on certain suitable decompositions in $V(G)_p$ by developing the Nachev's method to the mixed case. Thereby we state and prove many of the results in the more general setting of arbitrary groups. We hereafter will also accent on the kind of a basic subgroup of $V(G)_p$ over a coefficient ring larger than the corresponding one in [4].

2 Main Results

We first hasten to the following key technicality (see [1] and [3] too). It is worthwhile noticing that it encompasses Lemma 7 from [6].

Lemma 1 ([2], Lemma 6) *Assume that $G = C \times A$ is an abelian group and R is any commutative unitary ring. Then $V(G)_p = V(C)_p \times [(1+I(G;A)) \cap V(G)_p]$.*

We are now ready to proceed by proving the following formula which is a non-trivial strengthening of formula (11) of the scheme for proof in [6].

Theorem 1 (Decomposition) *Suppose G is an abelian group with a p-basic subgroup B and suppose R is a commutative ring with identity element. Then the following decomposition holds:*

$$(1 + I(G; B)) \cap V(G)_p = \left(1 + I(G; B_0) + \coprod_{n=1}^{\infty} (1 + I(G_{n-1}; B_n))\right) \cap V(G)_p,$$

where $B = \coprod_{n=0}^{\infty} B_n$; $B_n \cong \coprod_{\alpha_n} \langle p^n \rangle$, $\forall n \geq 1$; $B_0 \cong \coprod_{\alpha_0} \mathrm{Z}$ (where α_n is a cardinal $\forall\ n \geq 0$ and Z is an infinite cyclic group) and $G = \coprod_{1 \leq i \leq n} B_i \times G_n$ with $G_n = B_{n+1} \times G_{n+1}$ and $G_n = (B_0 \times \coprod_{i=n+1}^{\infty} B_i) G^{p^n}$, $\forall n \geq 1$; $G_0 = G$.

Proof In accordance with [5, p. 163, Theorem 32.4] we subsequently write down $B = \coprod_{n=0}^{\infty} B_n$, $G = \coprod_{1 \leq i \leq n} B_i \times G_n$, $G_n = G_{n+1} \times B_{n+1}$ and $G_n = B_n^* G^{p^n}$, where $B_n^* = B_0 \times \coprod_{i=n+1}^{\infty} B_i$. It is worth to noting that B_0 is not in general a direct factor of G.

Employing the foregoing Lemma 1 for $n = 1$ we derive that

$$V(G)_p = V(G_1)_p \times [(1 + I(G; B_1)) \cap V(G)_p].$$

Consequently, by induction on n we obtain

$$V(G)_p = V(G_n)_p \times \Big[\coprod_{1 \le i \le n} (1 + I(G_{i-1}; B_i)) \cap V(G)_p \Big],$$

where $G_0 = G$. Furthermore, we observe that

$$\Big[\coprod_{1 \le i \le n} (1 + I(G_{i-1}; B_i)) \Big] \cap V(G)_p \subseteq \Big(1 + I\Big(G; \coprod_{1 \le i \le n} B_i\Big)\Big) \cap V(G)_p \le V(G)_p.$$

Therefore, the previous decomposition implies that

$$\Big(1 + I\Big(G; \coprod_{1 \le i \le n} B_i\Big)\Big) \cap V(G)_p =$$

$$= \Big[\coprod_{1 \le i \le n} (1 + I(G_{i-1}; B_i)) \cap V(G)_p \Big] \times \Big(V(G_n)_p \cap \Big(1 + I\Big(G; \coprod_{1 \le i \le n} B_i\Big)\Big)\Big).$$

But $G_n \cap \coprod_{1 \le i \le n} B_i = 1$, so the latter intersection is equal to 1 (e.g. [1]). Thus the last decomposition transforms to the following:

$$\Big(1 + I\Big(G; \coprod_{1 \le i \le n} B_i\Big)\Big) \cap V(G)_p = \Big(\coprod_{1 \le i \le n} (1 + I(G_{i-1}; B_i))\Big) \cap V(G)_p, \quad \forall n \ge 1.$$

Finally, since B is the union of an ascending chain of subgroups $B_0 \times \coprod_{1 \le i \le n} B_i$ ($n = 1, 2, \ldots$), whence because of the finite support $(1 + I(G; B)) \cap V(G)_p$ is the union of

$$\Big(1 + I\Big(G; B_0 \times \coprod_{1 \le i \le n} B_i\Big)\Big) \cap V(G)_p = \Big(1 + I(G; B_0) + I\Big(G; \coprod_{1 \le i \le n} B_i\Big)\Big) \cap V(G)_p,$$

by taking in both sides of the last identity the limit operation $n \to \infty$, we deduce that

$$\Big(1 + I\Big(G; \coprod_{i=0}^{\infty} B_i\Big)\Big) \cap V(G)_p = \Big(1 + I(G; B_0) + \coprod_{i=1}^{\infty} (1 + I(G_{i-1}; B_i))\Big) \cap V(G)_p,$$

which is precisely the desired equality. □

We are now in a position to give an alternative verification of the following assertion (see, for instance, [4, Theorem 2]).

Theorem 2 (Basis) *Let G be an abelian group with a p-basic subgroup B and let R be a perfect commutative ring with 1 of prime characteristic p. Then $[1 + I(G; B)] \cap V(G)_p$ is a basic subgroup of $V(G)_p$.*

Proof In order to show the truthfulness of this claim, it is enough to check only the validity of three conditions from the definition of a basic subgroup (see, for example, [5]).

1) The fact that $[1+I(G;B)]\cap V(G)_p$ is a coproduct of cyclic groups follows at once by the method described in [4] and applied to R and B_0 as well as by the preceding Theorem, because $B_n^{p^n}=1$ forces that

$$(1+I(G_{n-1};B_n))^{p^n}=1+I^{p^n}(G_{n-1};B_n)=1+I(G_{n-1}^{p^n};B_n^{p^n})=1.$$

2) The property of $[1+I(G;B)]\cap V(G)_p$ to be a pure subgroup of $V(G)_p$ follows like this: For each $n\geq 1$ we calculate with the aid of [1] that

$$\begin{aligned}[(1+I(G;B))\cap V(G)_p]\cap V(G)_p^{p^n} &= (1+I(G;B))\cap V(G^{p^n})_p \\ = (1+I(G^{p^n};B^{p^n}))\cap V(G^{p^n})_p &= (1+I(G;B)^{p^n})\cap V(G)_p^{p^n} \\ = (1+I(G;B))^{p^n}\cap V(G)_p^{p^n} &= [(1+I(G;B))\cap V(G)_p]^{p^n}.\end{aligned}$$

3) The divisibility of the quotient group $V(G)_p/[(1+I(G;B))\cap V(G)_p]$ can be verified as follows: Writing $G=BG^p$, and taking into account that $G_p=B_pG_p^p$, we conclude by application of the main proposition in [3] that

$$V(G)_p=V(G^p)_p[(1+I(G;B))\cap V(G)_p].$$

Since $V(G^p)_p=V(G)_p^p$, we are done. □

Remark 1 In [4] the same affirmation as alluded to above was proved under the more restrictive assumption on R to be a field. The foregoing theorem extends this result to an arbitrary commutative unitary ring R. Besides, the idea used here is at all different to that in [4].

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