

Acta Universitatis Palackianae Olomucensis. Facultas Rerum
Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 47 (2008), No. 1, 37--46

Persistent URL: <http://dml.cz/dmlcz/133402>

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On Structure Space of Γ -Semigroups

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(Received August 20, 2007)

Abstract

In this paper we introduce the notion of the structure space of Γ -semigroups formed by the class of uniformly strongly prime ideals. We also study separation axioms and compactness property in this structure space.

Key words: Γ -semigroup; uniformly strongly prime ideal; Noetherian Γ -semigroup, hull-kernel topology, structure space.

2000 Mathematics Subject Classification: 20M17

1 Introduction

In [4], L. Gillman studied “Rings with Hausdorff structure space” and in [7], C. W. Kohls studied “The space of prime ideals of a ring”. In [1], M. R. Adhikari and M. K. Das studied ‘Structure spaces of semirings’.

In [9], M. K. Sen and N. K. Saha introduced the notion of Γ -Semigroup. Some works on Γ -Semigroups may be found in [10], [8], [5], [6], [2] and [3].

In this paper we introduce and study the structure space of Γ -Semigroups. For this we consider the collection \mathcal{A} of all proper uniformly strongly prime ideals of a Γ -Semigroup S and we give a topology $\tau_{\mathcal{A}}$ on \mathcal{A} by means of closure operator defined in terms of intersection and inclusion relation among these ideals of the Γ -Semigroup S . We call the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ —the structure space of the Γ -Semigroup S . We study separation axioms, compactness and connectedness in this structure space.

2 Preliminaries

Definition 2.1 Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ -semigroup if

- (i) $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and
- (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

S is said to be Γ -semigroup with zero if there exists an element $0 \in S$ such that $0\alpha a = a\alpha 0 = 0$ for all $\alpha \in \Gamma$.

Example 2.2 Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let

$$\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}.$$

Let $a, b, c \in S$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b , then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ -semigroup.

Definition 2.3 Let S be a Γ -semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α -idempotent if $e\alpha e = e$. The set of all α -idempotents is denoted by E_α and we denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S .

Definition 2.4 A nonempty subset I of a Γ -semigroup S is called an ideal if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$ where for subsets U, V of S and Δ of Γ , $U\Delta V = \{u\alpha v : u \in U, v \in V, \alpha \in \Delta\}$.

Definition 2.5 A nonempty subset I of a Γ -semigroup S is called an ideal if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$ where for subsets U, V of S and Δ of Γ , $U\Delta V = \{u\alpha v : u \in U, v \in V, \alpha \in \Delta\}$. An ideal I of S is called a proper ideal if $I \neq S$.

Definition 2.6 A proper ideal P of a Γ -Semigroup S is called a prime ideal of S if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any two ideals A, B of S .

Definition 2.7 An ideal I of a Γ -semigroup S is said to be full if $E(S) \subseteq I$.

An ideal I of a Γ -semigroup S is said to be a prime full ideal if it is both prime and full.

Theorem 2.8 Let S be a Γ -semigroup. For an ideal P of S , the following are equivalent.

- (i) If A and B are ideals of S such that $A\Gamma B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.
- (ii) If $a\Gamma S\Gamma b \subseteq P$ then either $a \in P$ or $b \in P$ ($a, b \in S$)
- (iii) If I_1 and I_2 are two right ideals of S such that $I_1\Gamma I_2 \subseteq P$ then either $I_1 \subseteq P$ or $I_2 \subseteq P$.
- (iv) If J_1 and J_2 are two left ideals of S such that $J_1\Gamma J_2 \subseteq P$ then either $J_1 \subseteq P$ or $J_2 \subseteq P$.

Proof (i) \Rightarrow (ii): Suppose $a\Gamma S\Gamma b \subseteq P$. Then $\langle a \rangle \Gamma \langle a \rangle \Gamma \langle b \rangle \Gamma \langle b \rangle \subseteq P$. Since $\langle a \rangle \Gamma \langle a \rangle$, $\langle b \rangle \Gamma \langle b \rangle$ are ideals of S , so by (i) we have either $\langle a \rangle \Gamma \langle a \rangle \subseteq P$ or $\langle b \rangle \Gamma \langle b \rangle \subseteq P$. By repeated uses of (i) we get $a \in \langle a \rangle \subseteq P$ or $b \in \langle b \rangle \subseteq P$.

(ii) \Rightarrow (iii): Let $I_1 \Gamma I_2 \subseteq P$. Let $I_1 \not\subseteq P$. Then there exists an element $a_1 \in I_1$ such that $a_1 \notin P$. Then for every $a_2 \in I_2$ we have $a_1 \Gamma S \Gamma a_2 \subseteq I_1 \Gamma I_2 \subseteq P$. Hence from (ii) $a_2 \in P$. Thus $I_2 \subseteq P$. Similarly (ii) implies (iv).

The proof is completed by observing that (i) is implied obviously either by (iii) or by (iv). \square

Definition 2.9 An ideal P of a Γ -Semigroup S is called a uniformly strongly prime ideal(usp ideal) if S and Γ contain finite subsets F and Δ respectively such that $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in S$.

Theorem 2.10 Let S be a Γ -semigroup. Then every uniformly strongly prime ideal is a prime ideal.

Proof Let P be a uniformly strongly prime ideal of S . Then S and Γ contain finite subsets F and Δ respectively such that $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in S$. Now let $a\Gamma S\Gamma b \subseteq P$. Thus we have $a\Delta F\Delta b \subseteq a\Gamma S\Gamma b \subseteq P$ and hence we have $a \in P$ or $b \in P$. Hence P is prime ideal by Theorem 2.8. \square

Throughout this paper S will always denote a Γ -Semigroup with zero and unless otherwise stated a Γ -Semigroup means a Γ -Semigroup with zero.

3 Structure space of Γ -semigroups

Suppose \mathcal{A} is the collection of all uniformly strongly prime ideals of a Γ -Semigroup S . For any subset A of \mathcal{A} , we define

$$\overline{A} = \{I \in \mathcal{A} : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}.$$

It is easy to see that $\overline{\emptyset} = \emptyset$.

Theorem 3.1 Let A, B be any two subsets of \mathcal{A} . Then

- (i) $A \subseteq \overline{A}$
- (ii) $\overline{\overline{A}} = \overline{A}$
- (iii) $A \subseteq B \implies \overline{A} \subseteq \overline{B}$
- (iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof (i): Clearly, $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ for each α and hence $A \subseteq \overline{A}$.

(ii): By (i), we have $\overline{A} \subseteq \overline{\overline{A}}$. For converse part, let $I_\beta \in \overline{\overline{A}}$. Then $\bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. Now $I_\alpha \in \overline{A}$ implies that $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$ for all $\alpha \in \Lambda$. Thus

$$\bigcap_{I_\gamma \in A} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta \quad \text{i.e.} \quad \bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\beta.$$

So $I_\beta \in \overline{A}$ and hence $\overline{\overline{A}} \subseteq \overline{A}$. Consequently, $\overline{\overline{A}} = \overline{A}$.

(iii): Suppose that $A \subseteq B$. Let $I_\alpha \in \overline{A}$. Then $\bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha$. Since $A \subseteq B$, it follows that

$$\bigcap_{I_\beta \in B} I_\beta \subseteq \bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha.$$

This implies that $I_\alpha \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(iv): Clearly, $\overline{A \cup B} \subseteq \overline{A \cup B}$.

For the reverse part, let $I_\alpha \in \overline{A \cup B}$. Then $\bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha$.

It is easy to see that

$$\bigcap_{I_\beta \in A \cup B} I_\beta = \left(\bigcap_{I_\beta \in A} I_\beta \right) \cap \left(\bigcap_{I_\beta \in B} I_\beta \right).$$

Since $\bigcap_{I_\beta \in A} I_\beta$ and $\bigcap_{I_\beta \in B} I_\beta$ are ideals of S , we have

$$\left(\bigcap_{I_\beta \in A} I_\beta \right) \Gamma \left(\bigcap_{I_\beta \in B} I_\beta \right) \subseteq \left(\bigcap_{I_\beta \in A} I_\beta \right) \cap \left(\bigcap_{I_\beta \in B} I_\beta \right) = \bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha$$

Since every uniformly strongly prime ideal is prime, I_α is a prime ideal of S and hence either $\bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha$ or $\bigcap_{I_\beta \in B} I_\beta \subseteq I_\alpha$ i.e. either $I_\alpha \in \overline{A}$ or $I_\alpha \in \overline{B}$ i.e. $I_\alpha \in \overline{A \cup B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A \cup B}$ and hence $\overline{A \cup B} = \overline{A \cup B}$. \square

Definition 3.2 The closure operator $A \longrightarrow \overline{A}$ gives a topology $\tau_{\mathcal{A}}$ on \mathcal{A} . This topology $\tau_{\mathcal{A}}$ is called the hull-kernel topology and the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called the structure space of the Γ -Semigroup S .

Let I be an ideal of a Γ -Semigroup S . We define

$$\Delta(I) = \{I' \in \mathcal{A} : I \subseteq I'\} \quad \text{and} \quad C\Delta(I) = \mathcal{A} \setminus \Delta(I) = \{I' \in \mathcal{A} : I \not\subseteq I'\}.$$

Now we have the following result:

Proposition 3.3 Any closed set in \mathcal{A} is of the form $\Delta(I)$, where I is an ideal of a Γ -Semigroup S .

Proof Let \overline{A} be any closed set in \mathcal{A} , where $A \subseteq \mathcal{A}$. Let $A = \{I_\alpha : \alpha \in \Lambda\}$ and $I = \bigcap_{I_\alpha \in A} I_\alpha$. Then I is an ideal of S . Let $I' \in \overline{A}$. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I'$. This implies that $I \subseteq I'$. Consequently, $I' \in \Delta(I)$. So $\overline{A} \subseteq \Delta(I)$.

Conversely, let $I' \in \Delta(I)$. Then $I \subseteq I'$ i.e. $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I'$. Consequently, $I' \in \overline{A}$ and hence $\Delta(I) \subseteq \overline{A}$. Thus $\overline{A} = \Delta(I)$. \square

Corollary 3.4 Any open set in \mathcal{A} is of the form $C\Delta(I)$, where I is an ideal of S .

Let S be a Γ -Semigroup and $a \in S$. We define

$$\Delta(a) = \{I \in \mathcal{A} : a \in I\} \quad \text{and} \quad C\Delta(a) = \mathcal{A} \setminus \Delta(a) = \{I \in \mathcal{A} : a \notin I\}.$$

Then we have the following result:

Proposition 3.5 $\{C\Delta(a) : a \in S\}$ forms an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on \mathcal{A} .

Proof Let $U \in \tau_{\mathcal{A}}$. Then $U = C\Delta(I)$, where I is an ideal of S . Let $J \in U = C\Delta(I)$. Then $I \not\subseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Thus $J \in C\Delta(a)$. Now it remains to show that $C\Delta(a) \subset U$. Let $K \in C\Delta(a)$. Then $a \notin K$. This implies that $I \not\subseteq K$. Consequently, $K \in U$ and hence $C\Delta(a) \subset U$. So we find that $J \in C\Delta(a) \subset U$. Thus $\{C\Delta(a) : a \in S\}$ is an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on \mathcal{A} .

Theorem 3.6 The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0 -space.

Proof Let I_1 and I_2 be two distinct elements of \mathcal{A} . Then there is an element a either in $I_1 \setminus I_2$ or in $I_2 \setminus I_1$. Suppose that $a \in I_1 \setminus I_2$. Then $C\Delta(a)$ is a neighbourhood of I_2 not containing I_1 . Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0 -space. \square

Theorem 3.7 $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 -space if and only if no element of \mathcal{A} is contained in any other element of \mathcal{A} .

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a T_1 -space. Suppose that I_1 and I_2 be any two distinct elements of \mathcal{A} . Then each of I_1 and I_2 has a neighbourhood not containing the other. Since I_1 and I_2 are arbitrary elements of \mathcal{A} , it follows that no element of \mathcal{A} is contained in any other element of \mathcal{A} .

Conversely, suppose that no element of \mathcal{A} is contained in any other element of \mathcal{A} . Let I_1 and I_2 be any two distinct elements of \mathcal{A} . Then by hypothesis, $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. This implies that there exist $a, b \in S$ such that $a \in I_1$ but $a \notin I_2$ and $b \in I_2$ but $b \notin I_1$. Consequently, we have $I_1 \in C\Delta(b)$ but $I_1 \notin C\Delta(a)$ and $I_2 \in C\Delta(a)$ but $I_2 \notin C\Delta(b)$ i.e. each of I_1 and I_2 has a neighbourhood not containing the other. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 -space. \square

Corollary 3.8 Let \mathcal{M} be the set of all proper maximal ideals of a Γ -Semigroup S with unities. Then $(\mathcal{M}, \tau_{\mathcal{M}})$ is a T_1 -space, where $\tau_{\mathcal{M}}$ is the induced topology on \mathcal{M} from $(\mathcal{A}, \tau_{\mathcal{A}})$.

Theorem 3.9 $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space if and only if for any two distinct pair of elements I, J of \mathcal{A} , there exist $a, b \in S$ such that $a \notin I, b \notin J$ and there does not exist any element K of \mathcal{A} such that $a \notin K$ and $b \notin K$.

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a Hausdorff space. Then for any two distinct elements I, J of \mathcal{A} , there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $I \in C\Delta(a), J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. Now $I \in C\Delta(a)$ and $J \in C\Delta(b)$ imply

that $a \notin I$ and $b \notin J$. If possible, let K be any element of \mathcal{A} such that $a \notin K$ and $b \notin K$. Then $K \in C\Delta(a)$, $K \in C\Delta(b)$ and hence $K \in C\Delta(a) \cap C\Delta(b)$, a contradiction, since $C\Delta(a) \cap C\Delta(b) = \emptyset$. Thus there does not exist any element K of \mathcal{A} such that $a \notin K$ and $b \notin K$.

Conversely, suppose that the given condition holds and $I, J \in \mathcal{A}$ such that $I \neq J$. Let $a, b \in S$ be such that $a \notin I$, $b \notin J$ and there does not exist any K of \mathcal{A} such that $a \notin K$ and $b \notin K$. Then $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. This implies that $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space. \square

Corollary 3.10 *If $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space, then no proper uniformly strongly prime ideal contains any other proper uniformly strongly prime ideal. If $(\mathcal{A}, \tau_{\mathcal{A}})$ contains more than one element, then there exist $a, b \in S$ such that $\mathcal{A} = C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$, where I is the ideal generated by a, b .*

Proof Suppose that $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space. Since every Hausdorff space is a T_1 -space, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 -space. Hence by Theorem 3.7, it follows that no proper uniformly strongly prime ideal contains any other proper uniformly strongly prime ideal. Now let $J, K \in \mathcal{A}$ be such that $J \neq K$. Since $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space, there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $J \in C\Delta(a)$, $K \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. Let I be the ideal generated by a, b . Then I is the smallest ideal containing a and b . Let $K \in \mathcal{A}$. Then either $a \in K$, $b \notin K$ or $a \notin K$, $b \in K$ or $a, b \in K$. The case $a \notin K$, $b \notin K$ is not possible, since $C\Delta(a) \cap C\Delta(b) = \emptyset$. Now in the first case, $K \in C\Delta(b)$ and hence $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. In the second case, $K \in C\Delta(a)$ and hence $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. In the third case, $K \in \Delta(I)$ and hence $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. So we find that $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. Again, clearly $C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \subseteq \mathcal{A}$. Hence $\mathcal{A} = C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. \square

Theorem 3.11 *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space if and only if for any $I \in \mathcal{A}$ and $a \notin I$, $a \in S$, there exist an ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.*

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a regular space. Let $I \in \mathcal{A}$ and $a \notin I$. Then $I \in C\Delta(a)$ and $\mathcal{A} \setminus C\Delta(a)$ is a closed set not containing I . Since $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space, there exist disjoint open sets U and V such that $I \in U$ and $\mathcal{A} \setminus C\Delta(a) \subseteq V$. This implies that $\mathcal{A} \setminus V \subseteq C\Delta(a)$. Since V is open, $\mathcal{A} \setminus V$ is closed and hence there exists an ideal J of S such that $\mathcal{A} \setminus V = \Delta(J)$, by Proposition 3.3. So we find that $\Delta(J) \subseteq C\Delta(a)$. Again, since $U \cap V = \emptyset$, we have $V \subseteq \mathcal{A} \setminus U$. Since U is open, $\mathcal{A} \setminus U$ is closed and hence there exists an ideal K of S such that $\mathcal{A} \setminus U = \Delta(K)$ i.e. $V \subseteq \Delta(K)$. Since $I \in U$, $I \notin \mathcal{A} \setminus U = \Delta(K)$. This implies that $K \not\subseteq I$. Thus there exists $b \in K (\subset S)$ such that $b \notin I$. So $I \in C\Delta(b)$. Now we show that $V \subseteq \Delta(b)$. Let $M \in V \subseteq \Delta(K)$. Then $K \subseteq M$. Since $b \in K$, it follows that $b \in M$ and hence $M \in \Delta(b)$. Consequently, $V \subseteq \Delta(b)$. This implies that $\mathcal{A} \setminus \Delta(b) \subseteq \mathcal{A} \setminus V = \Delta(J) \implies C\Delta(b) \subseteq \Delta(J)$. Thus we find that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.

Conversely, suppose that the given condition holds. Let $I \in \mathcal{A}$ and $\Delta(K)$ be any closed set not containing I . Since $I \notin \Delta(K)$, we have $K \not\subseteq I$. This implies that there exists $a \in K$ such that $a \notin I$. Now by the given condition, there exists an ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$. Since $a \in K$, $C\Delta(a) \cap \Delta(K) = \emptyset$. This implies that $\Delta(K) \subseteq \mathcal{A} \setminus C\Delta(a) \subseteq \mathcal{A} \setminus \Delta(J)$. Since $\Delta(J)$ is a closed set, $\mathcal{A} \setminus \Delta(J)$ is an open set containing the closed set $\Delta(K)$. Clearly, $C\Delta(b) \cap (\mathcal{A} \setminus \Delta(J)) = \emptyset$. So we find that $C\Delta(b)$ and $\mathcal{A} \setminus \Delta(J)$ are two disjoint open sets containing I and $\Delta(K)$ respectively. Consequently, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space. \square

Theorem 3.12 *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space if and only if for any collection $\{a_{\alpha}\}_{\alpha \in \Lambda} \subset S$ there exists a finite subcollection $\{a_i: i = 1, 2, \dots, n\}$ in S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.*

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a compact space. Then the open cover $\{C\Delta(a_{\alpha}): a_{\alpha} \in S\}$ of $(\mathcal{A}, \tau_{\mathcal{A}})$ has a finite subcover $\{C\Delta(a_i): i = 1, 2, \dots, n\}$. Let I be any element of \mathcal{A} . Then $I \in C\Delta(a_i)$ for some $a_i \in S$. This implies that $a_i \notin I$. Hence $\{a_i: i = 1, 2, \dots, n\}$ is the required finite subcollection of elements of S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.

Conversely, suppose that the given condition holds. Let $\{C\Delta(a_{\alpha}): a_{\alpha} \in S\}$ be an open cover of \mathcal{A} . Suppose to the contrary that no finite subcollection of $\{C\Delta(a_{\alpha}): a_{\alpha} \in S\}$ covers \mathcal{A} . This means that for any finite set $\{a_1, a_2, \dots, a_n\}$ of elements of S ,

$$\begin{aligned} & C\Delta(a_1) \cup C\Delta(a_2) \cup \dots \cup C\Delta(a_n) \neq \mathcal{A} \\ \implies & \Delta(a_1) \cap \Delta(a_2) \cap \dots \cap \Delta(a_n) \neq \emptyset \\ \implies & \text{there exists } I \in \mathcal{A} \text{ such that } I \in \Delta(a_1) \cap \Delta(a_2) \cap \dots \cap \Delta(a_n) \\ \implies & a_1, a_2, \dots, a_n \in I, \text{ which contradicts our hypothesis.} \end{aligned}$$

So the open cover $\{C\Delta(a_{\alpha}): a_{\alpha} \in S\}$ has a finite subcover and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.

Corollary 3.13 *If S is finitely generated, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space.*

Proof Let $\{a_i: i = 1, 2, \dots, n\}$ be a finite set of generators of S . Then for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$, since I is a proper uniformly strongly prime ideal of S . Hence by Theorem 3.12, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space. \square

Definition 3.14 A Γ -Semigroup S is called a Noetherian Γ -Semigroup if it satisfies the ascending chain condition on ideals of S i.e. if $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots$ is an ascending chain of ideals of S , then there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$.

Theorem 3.15 *If S is a Noetherian Γ -Semigroup, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact.*

Proof Let $\{\Delta(I_n)\}_{n=1}^{\infty}$ be a countable collection of closed sets in \mathcal{A} with finite intersection property (FIP). Let us consider the following ascending chain of prime ideals of S : $\langle I_1 \rangle \subseteq \langle I_1 \cup I_2 \rangle \subseteq \langle I_1 \cup I_2 \cup I_3 \rangle \subseteq \dots$

Since S is a Noetherian Γ -Semigroup, there exists a positive integer m such that $\langle I_1 \cup I_2 \cup \dots \cup I_m \rangle = \langle I_1 \cup I_2 \cup \dots \cup I_{m+1} \rangle = \dots$

Thus it follows that $\langle I_1 \cup I_2 \cup \dots \cup I_m \rangle \in \bigcap_{n=1}^{\infty} \Delta(I_n)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(I_n) \neq \emptyset$ and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact. \square

Corollary 3.16 *If S is a Noetherian Γ -Semigroup and $(\mathcal{A}, \tau_{\mathcal{A}})$ is second countable, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.*

Proof Proof follows from Theorem 3.15 and the fact that a second countable space is compact if it is countably compact. \square

Remark 3.17 Let $\{I_{\alpha}\}$ be a collection of prime ideals of a Γ -semigroup S . Then $\bigcap I_{\alpha}$ is an ideal of S but it may not be a prime ideal of S , in general.

However; in particular, we have the following result:

Proposition 3.18 *Let $\{I_{\alpha}\}$ be a collection of prime ideals of a Γ -semigroup S such that $\{I_{\alpha}\}$ forms a chain. Then $\bigcap I_{\alpha}$ is a prime ideal of S .*

Proof Clearly, $\bigcap I_{\alpha}$ is an ideal of S . Let $A\Gamma B \subseteq \bigcap I_{\alpha}$ for any two ideals A, B of S . If possible, let $A, B \not\subseteq \bigcap I_{\alpha}$. Then there exist α and β such that $A \not\subseteq I_{\alpha}$ and $B \not\subseteq I_{\beta}$. Since I_{α} is a chain, let $I_{\alpha} \subseteq I_{\beta}$. This implies that $B \not\subseteq I_{\alpha}$. Since $A\Gamma B \subseteq I_{\alpha}$ and I_{α} is prime, we must have either $A \subseteq I_{\alpha}$ or $B \subseteq I_{\alpha}$, a contradiction. Therefore, either $A \subseteq \bigcap I_{\alpha}$ or $B \subseteq \bigcap I_{\alpha}$. Consequently, $\bigcap I_{\alpha}$ is a prime ideal of S . \square

Definition 3.19 The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called irreducible if for any decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where \mathcal{A}_1 and \mathcal{A}_2 are closed subsets of \mathcal{A} , we have either $\mathcal{A} = \mathcal{A}_1$ or $\mathcal{A} = \mathcal{A}_2$.

Theorem 3.20 *Let A be a closed subset of \mathcal{A} . Then A is irreducible if and only if $\bigcap_{I_{\alpha} \in A} I_{\alpha}$ is a prime ideal of S .*

Proof Let A be irreducible. Let P and Q be two ideals of S such that $P\Gamma Q \subseteq \bigcap_{I_{\alpha} \in A} I_{\alpha}$. Then $P\Gamma Q \subseteq I_{\alpha}$ for all α . Since I_{α} is prime, either $P \subseteq I_{\alpha}$ or $Q \subseteq I_{\alpha}$ which implies for $I_{\alpha} \in A$ either $I_{\alpha} \in \{\overline{P}\}$ or $I_{\alpha} \in \{\overline{Q}\}$. Hence $A = (A \cap \overline{P}) \cup (A \cap \overline{Q})$. Since A is irreducible and $(A \cap \overline{P})$, $(A \cap \overline{Q})$ are closed, it follows that $A = A \cap \overline{P}$ or $A = A \cap \overline{Q}$ and hence $A \subseteq \overline{P}$ or $A \subseteq \overline{Q}$. This implies that $P \subseteq \bigcap_{I_{\alpha} \in A} I_{\alpha}$ or $Q \subseteq \bigcap_{I_{\alpha} \in A} I_{\alpha}$. Consequently, $\bigcap_{I_{\alpha} \in A} I_{\alpha}$ is a prime ideal of S .

Conversely, suppose that $\bigcap_{I_{\alpha} \in A} I_{\alpha}$ is a prime ideal of S . Let $A = A_1 \cup A_2$, where A_1 and A_2 are closed subsets of A . Then $\bigcap_{I_{\alpha} \in A} I_{\alpha} \subseteq \bigcap_{I_{\alpha} \in A_1} I_{\alpha}$ and $\bigcap_{I_{\alpha} \in A} I_{\alpha} \subseteq \bigcap_{I_{\alpha} \in A_2} I_{\alpha}$. Also

$$\bigcap_{I_{\alpha} \in A} I_{\alpha} = \bigcap_{I_{\alpha} \in A_1 \cup A_2} I_{\alpha} = \left(\bigcap_{I_{\alpha} \in A_1} I_{\alpha} \right) \cap \left(\bigcap_{I_{\alpha} \in A_2} I_{\alpha} \right).$$

Now

$$\left(\bigcap_{I_\alpha \in A_1} I_\alpha \right) \Gamma \left(\bigcap_{I_\alpha \in A_2} I_\alpha \right) \subseteq \left(\bigcap_{I_\alpha \in A_1} I_\alpha \right) \quad \text{and} \quad \left(\bigcap_{I_\alpha \in A_1} I_\alpha \right) \Gamma \left(\bigcap_{I_\alpha \in A_2} I_\alpha \right) \subseteq \left(\bigcap_{I_\alpha \in A_2} I_\alpha \right).$$

Thus we have

$$\left(\bigcap_{I_\alpha \in A_1} I_\alpha \right) \Gamma \left(\bigcap_{I_\alpha \in A_2} I_\alpha \right) \subseteq \left(\bigcap_{I_\alpha \in A_1} I_\alpha \right) \cap \left(\bigcap_{I_\alpha \in A_2} I_\alpha \right).$$

Since $\bigcap_{I_\alpha \in A} I_\alpha$ is prime, it follows that either

$$\bigcap_{I_\alpha \in A_1} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha \quad \text{or} \quad \bigcap_{I_\alpha \in A_2} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha.$$

So we find that

$$\bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_1} I_\alpha \quad \text{or} \quad \bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_2} I_\alpha.$$

Let $I_\beta \in A$. Then we have

$$\bigcap_{I_\alpha \in A_1} I_\alpha \subseteq I_\beta \quad \text{or} \quad \bigcap_{I_\alpha \in A_2} I_\alpha \subseteq I_\beta.$$

Since $A_1, A_2 \subseteq A$, so either $I_\alpha \subseteq I_\beta$ for all $I_\alpha \in A_1$ or $I_\alpha \subseteq I_\beta$ for all $I_\alpha \in A_2$. Thus $I_\beta \in \overline{A_1} = A_1$ or $I_\beta \in \overline{A_2} = A_2$, since A_1 and A_2 are closed. i.e. $A = A_1$ or $A = A_2$. \square

Let \mathcal{C} be the collection of all uniformly strongly prime full ideals of a Γ -semi-group S . Then we see that \mathcal{C} is a subset of \mathcal{A} and hence $(\mathcal{C}, \tau_{\mathcal{C}})$ is a topological space, where $\tau_{\mathcal{C}}$ is the subspace topology.

In general, $(\mathcal{A}, \tau_{\mathcal{A}})$ is not compact and connected. But in particular, for the topological space $(\mathcal{C}, \tau_{\mathcal{C}})$, we have the following results:

Theorem 3.21 $(\mathcal{C}, \tau_{\mathcal{C}})$ is a compact space.

Proof Let $\{\Delta(I_\alpha) : \alpha \in \Lambda\}$ be any collection of closed sets in \mathcal{C} with finite intersection property. Let I be the uniformly strongly prime full ideal generated by $E(S)$. Since any uniformly strongly prime full ideal J contains $E(S)$, J contains I . Hence $I \in \bigcap_{\alpha \in \Lambda} \Delta(I_\alpha) \neq \emptyset$. Consequently, $(\mathcal{C}, \tau_{\mathcal{C}})$ is a compact space. \square

Theorem 3.22 $(\mathcal{C}, \tau_{\mathcal{C}})$ is a connected space.

Proof Let I be the uniformly strongly prime ideal generated by $E(S)$. Since any uniformly strongly prime full ideal J contains $E(S)$, J contains I . Hence I belongs to any closed set $\Delta(I')$ of \mathcal{C} . Consequently, any two closed sets of \mathcal{C} are not disjoint. Hence $(\mathcal{C}, \tau_{\mathcal{C}})$ is a connected space. \square

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