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Discriminator Order Algebras*

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Abstract

We prove that an order algebra assigned to a bounded poset with involution is a discriminator algebra.

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In accordance with [1], by an order algebra we mean an algebra defined on an ordered set whose operations are derived by means of the order relation and, conversely, the partial order is determined by these operations.

Let $\mathcal{P} = (P; \leq, 1)$ be an ordered set with the greatest element 1. The following two operations are introduced in [1]:

$$x \to y = \begin{cases} 1 \text{ if } x \leq y \\ \\ y \text{ otherwise} \end{cases} \qquad \text{and} \qquad x \circ y = \begin{cases} y \text{ if } x \leq y \\ \\ 1 \text{ otherwise.} \end{cases}$$

Let us mention that these operations are not independent:

Observation 1 For any ordered set $\mathcal{P} = (P; \leq, 1)$ we have $x \circ y = (x \to y) \to y$.

Moreover, we have $x \to y \in \{y, 1\}$ and hence for any interval $[y, 1]$ of $\mathcal{P}$ it holds $x \to y \in [y, 1]$. Thus, having $a \in [y, 1]$, we can define a unary operation on the interval $[y, 1]$ assigning to a the element $a^y = a \to y$. Evidently, $y^y = 1$ and

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$1^y = y$ thus this operation interchanges the endpoints of the interval $[y, 1]$ and hence it is called a *section switching mapping.* Hence, the operation $\rightarrow$ determines not only the the order $\leq$ but also the family $({}^p)_{p\in P}$ of section switching mappings, i.e. the extended structure $\mathcal{P} = (P; \leq, 1, ({}^p)_{p\in P})$.

We can ask if also conversely the operation $\circ$ can determine the operation $\rightarrow$. Since also $x \circ y \in [y, 1]$, the operation $(x \circ y)^y$ is defined correctly whenever ^{y} denotes the section switching mapping on the interval $[y, 1]$. Hence, we can state

Observation 2 Let $\mathcal{P} = (P; \leq, 1, ({}^p)_{p\in P})$ be an ordered set with 1 and with section switching mappings. Then $x \rightarrow y = (x \circ y)^y$ for the above mention operations $\rightarrow$ and $\circ$.

An ordered set $\mathcal{P}$ is *bounded* if it has a least element 0 and a greatest element 1. This will be expressed by the notation $\mathcal{P} = (P; \leq, 0, 1)$.

By an involution on a set P is meant a mapping of P into itself denoted by $x \mapsto x'$ satisfying $x'' = x$. Every bounded poset $\mathcal{P} = (P; \leq, 0, 1)$ admits some involutions. Let us pick up one of them which satisfies $0' = 1$. Hence, $x'' = x$ gets immediately $1' = 0$ and thus this involution is a switching mapping. Then we can enlarge the type of $\mathcal{P}$ and we will write $\mathcal{P} = (P; \leq, 0, 1,')$ to express the fact that this involution is considered as a basic operation of $\mathcal{P}$. From now on, $\mathcal{P} = (P; \leq, 0, 1,')$ will be called a *poset with involution.*

Let $\mathcal{P} = (P; \leq, 0, 1)$ be a bounded poset. By a *globalization* (frequently called also a Baaz operation named by M. Baaz) is meant a unary operation $\triangle$ on P defined by

$$\triangle(1) = 1 \quad \text{and} \quad \triangle(x) = 0 \quad \text{for } x \neq 1.$$

Observation 3 In every poset with involution $\mathcal{P} = (P; \leq, 0, 1,')$, we can define a globalization $\triangle$ by means of $\rightarrow$, $'$ and 0 as follows

$$\triangle(x) = x' \rightarrow 0.$$

Another binary operation defined on an ordered set $(P; \leq)$ is mentioned in [1]:

$$x \sqcap y = \begin{cases} x & \text{if } x \leq y \\ \\ y & \text{otherwise.} \end{cases}$$

Now, let $\mathcal{P} = (P; \leq, 0, 1,')$ be a poset with involution. Define the assigned order algebra $\mathcal{A}(P) = (P; \rightarrow, \sqcap,', 0)$ of type $(2, 2, 1, 0)$ where $\rightarrow$ and $\sqcap$ are the above mentioned operations and $'$ is the involution of P.

As stated by Observations 1 and 3, the globalization $\triangle$ and the operation $\circ$ (as well as the constant 1) are term operations of $\mathcal{A}(P)$. We can state our main result:

Theorem 1 *Let $\mathcal{P} = (P; \leq, 0, 1,')$ be a poset with involution and $\mathcal{A}(P) = (P; \rightarrow, \sqcap,', 0)$ the assigned algebra. Then $\mathcal{A}(P)$ is a discriminator algebra whose ternary discriminator is*

$$t(x, y, z) = ((\triangle((x \rightarrow y)' \circ (y \rightarrow x)')') \rightarrow z) \sqcap (\triangle(((x \rightarrow y)' \circ (y \rightarrow x)')' \rightarrow 0) \rightarrow x).$$

Proof If $\operatorname{card} P = 1$, the proof is trivial. Suppose $\operatorname{card} P > 1$, i.e. $0 \neq 1$. It is an easy observation that $\sqcap$ satisfies

$$x \sqcap 1 = x = 1 \sqcap x. \tag{1}$$

For the sake of brevity, denote by

$$e(x, y) = ((x \to y)' \circ (y \to x)')'.$$

Due to the previous Observations 1 and 3, $e(x, y)$ is a term operation of $\mathcal{A}(P)$.

Clearly $e(x, x) = (1' \circ 1')' = (0 \circ 0)' = 0' = 1$. Suppose $x \neq y$.

(a) If $x < y$ then $x \neq 1$ and $x \to y = 1$, $y \to x = x$ and hence

$$e(x, y) = (1' \circ x')' = (0 \circ x') = x'' = x \neq 1.$$

(b) If $y < x$ then $y \neq 1$, i.e. $y' \neq 0$ and $x \to y = y$, $y \to x = 1$ thus

$$e(x, y) = (y' \circ 1')' = (y' \circ 0)' = 1' = 0 \neq 1.$$

(c) If $x \parallel y$ then $x \to y = y$, $y \to x = x$ and

$$e(x, y) = (y' \circ x')' = \begin{cases} 1' = 0 & \text{for } y' \leq x' \\ \\ x'' = x & \text{for } y' \not\leq x'. \end{cases}$$

Since $x \parallel y$ we have $x \neq 1$ thus $e(x, y) \neq 1$ for $x \neq y$ in all the cases.

The term $t(x, y, z)$ can be clearly rewritten as follows

$$t(x, y, z) = (\triangle(e(x, y)) \to z) \sqcap (\triangle(e(x, y) \to 0) \to x).$$

Using of (1), we compute

$$t(x, x, z) = (\triangle(1) \to z) \sqcap (\triangle(1 \to 0) \to x) = (1 \to z) \sqcap (0 \to x) = z \sqcap 1 = z$$

and for $x \neq y$

$$t(x, y, z) = (0 \to z) \sqcap (\triangle(0 \to 0) \to x) = 1 \sqcap (\triangle(1) \to x) = 1 \sqcap x = x.$$

Hence, $t(x, y, z)$ is a term function of $\mathcal{A}(P)$ which is the ternary discriminator.

$\square$

References

[1] Berman J., Blok W. J.: *Algebras Defined from Ordered Sets and the Varieties they Generate.* Order **23** (2006), 65–88.