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Bol-loops of Order $3 \cdot 2^n$

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Abstract

In this article we construct proper Bol-loops of order $3 \cdot 2^n$ using a generalisation of the semidirect product of groups defined by Birkenmeier and Xiao. Moreover we classify the obtained loops up to isomorphism.

Key words: Bol-loop; loop; group; semidirect product.

2000 Mathematics Subject Classification: 20N05

1 Introduction

Burn proofs in [3] that the smallest proper Bol-loops are of order 8. But they can not be constructed as a semidirect product defined in [1]. The smallest proper Bol-loops which can be constructed using a semidirect product as defined in this article have order 12. Up to isomorphism these loops can be realised as semidirect product of the cyclic group of order 3 and the elementary abelian groups of order 4. There are no proper Bol-loops of order 9, 10 or 11. It seems that order 12 plays an interesting role in the theory of loops since the smallest proper Moufang-loop has also order 12 (cf. [5]).

A *loop* is a set L with a binary operation $\cdot$, a neutral element 1 and unique solutions of the equations $x \cdot a = b$ and $a \cdot x = b$. The loop L is a *left Bol-loop* if $((x \cdot y)z)y = x((y \cdot z)y)$ for all $x, y, z \in L$ holds. Analogously one defines a right Bol-loop by the identity $x(y(x \cdot z)) = (x(y \cdot x))z$.

In this paper we consider a special case of the semidirect product of loops defined by Birkenmeier and Xiao in [2]. Starting with groups N and Q we obtain a loop L on $N \rtimes Q = \{(a, p)\colon a \in N,\ p \in Q\}$. The multiplication $*$ of L is defined as $(a, p) * (b, q) = (a^{\Phi(q)} \circ b^{\Psi(p)}, p \bullet q)$, where $\circ$ and $\bullet$ are the multiplications of N and Q. The mapping $\Phi(p)$ respectively $\Psi(p)$ from N into N is determined by a mapping Φ respectively Ψ from Q into the set of mappings from N into N. According to [2] we know that $(L,*)$ is a loop with neutral element $(1,1)$ if $\Phi(p)$ and $\Psi(p)$ are bijective, $1^{\Phi(p)} = 1^{\Psi(p)} = 1$ holds for all $p \in Q$ and $\Phi(1) = \Psi(1) = \mathrm{id}_N$. The constructed loops are associative if and only if the mappings Ψ, Φ, $\Phi(p)$ and $\Psi(p)$ are homomorphisms and $\Phi(p)$ and $\Psi(q)$ commute for all $p,q \in Q$.

Although the semidirect product treated by us here is a special case of the semidirect products defined in [1], [2] and [9] the construction presented here yields in general loops with no further identities. For example the 15 non-associative loops $L = C_3 \rtimes C_3$ of order 9, which are the smallest possible examples, are not even power associative and only three of them are commutative.

2 Bol-loops of order $3 \cdot 2^n$

We now construct loops of the form $L = C_3 \rtimes (C_2)^n$. These loops are all power-associative and under certain conditions Bol-loops.

Remark 1 The only two mappings of C_3 into C_3 which are one-to-one and keep the neutral element 1 fixed, are the identity and the inversion. Both mappings are automorphisms of C_3 and commute with each other.

Lemma 1 *All loops $L = C_3 \rtimes (C_2)^n$ are power-associative.*

Proof The restriction of Φ and Ψ to a subloop which is generated by a single element is a homomorphism. Therefore L is power-associative by the preceeding Remark. □

Proposition 1 *A semidirect product $L = C_3 \rtimes (C_2)^n$ is a left respectively right Bol-loop if and only if Φ respectively Ψ is a homomorphism.*

Proof Because of Remark 1 the left Bol-identity yields:

$$\begin{aligned}&\left(a^{\Phi(qpr)}\left(b^{\Phi(pr)}\left(a^{\Phi(r)}c^{\Psi(p)}\right)^{\Psi(q)}\right)^{\Psi(p)}, pqpr\right)\\ &= \left(\left(a^{\Phi(qp)}\left(b^{\Phi(p)}a^{\Psi(q)}\right)^{\Psi(p)}\right)^{\Phi(r)}c^{\Psi(pqp)}, pqpr\right)\end{aligned} \tag{1}$$

If L is a left Bol-loop equation (1) implies for $a = c = 1$ that Φ is a homomorphism.

If Φ is a homomorphism we obtain for the first component of (1):

$$a^{\Phi(qpr)}\left(b^{\Phi(pr)}\right)^{\Psi(p)} = \left(a^{\Phi(qp)}\right)^{\Phi(r)}\left(\left(b^{\Phi(p)}\right)^{\Phi(r)}\right)^{\Psi(p)} \tag{2}$$

which is valid for all $a,b \in C_3$ and all $p,q,r \in (C_2)^n$. Therefore L is a left Bol-loop.

The proof for right Bol-loops is analogous. □

To classify the constructed loops up to isomorphism we now determine the order of the non-trivial elements in the loops.

Lemma 2 *Let* $L = C_3 \rtimes (C_2)^n$ *be a loop,* $a \in C_3 \setminus \{1\}$ *and* $p \in (C_2)^n \setminus \{1\}$. *Then the order of* (a, p) *is* 2 *if and only if* $\Phi(p) \neq \Psi(p)$ *and* 6 *if and only if* $\Phi(p) = \Psi(p)$.

Proof If $\Phi(p) \neq \Psi(p)$ then $(a, p)(a, p) = (1, 1)$ holds because of $\Phi(p), \Psi(p) \in \{\mathrm{id}, \mathrm{inv}\}$. If $\Phi(p) = \Psi(p)$ then the first component of $(a, p)^n$ is a power of a or a^{-1}. The second component alternates between 1 and p. Therefore the order of (a, p) is the least common multiple of 2 and 3.

Conversely if $(a, p)(a, p) = (1, 1)$ then $\Phi(p) \neq \Psi(p)$ because of $\Phi(p), \Psi(p) \in \{\mathrm{id}, \mathrm{inv}\}$. Assume the order of (a, p) to be 6 and $\Phi(p) \neq \Psi(p)$. This is a contradiction to the first part of the proof. □

Proposition 2 **($C_3 \rtimes (C_2)^2$)** *Two proper loops of the form* $C_3 \rtimes (C_2)^2$ *are isomorphic if and only if both loops have the same number of elements with order* 6. *A loop* $L = C_3 \rtimes (C_2)^2$ *is a Bol-loop if and only if it has exactly zero or two elements of order* 6.

Proof Lemma 2 implies that a loop is a Bol-loop if and only if it has exactly zero or two elements of order 6.

Let L_1 and L_2 be loops with the same number of elements with order 6. Then it can be shown that

$$\iota\colon \begin{cases} (a, p) \mapsto (a, p) & \text{if } \Phi_1(p) = \Phi_2(p) \\ (a, p) \mapsto (a^{-1}, p) & \text{if } \Phi_1(p) \neq \Phi_2(p) \end{cases}$$

is an isomorphism between L_1 and L_2. The elements (a, p) with order 6 are assumed to have the same second component $p \in (C_2)^2$ because loops can be transferred in this form by obvious (anti-)isomorphisms. By Lemma 2 this implies that $\Phi_1(p) = \Psi_1(p)$ is equivalent to $\Phi_2(p) = \Psi_2(p)$.

Only the first component has to be analysed to check if ι is an isomorphism. The validity of the equation

$$\iota_{pq}(a^{\Phi_1(q)} b^{\Psi_1(p)}) = (\iota_p(a))^{\Phi_2(q)} (\iota_q(b))^{\Psi_2(p)} \tag{3}$$

is shown by case analysis.

Since $\Phi_1(p)$ in L_1 can be different from $\Phi_2(p)$ in L_2 there are four cases. The mapping Ψ is not considered in the following because it is determined by the order of the elements and the choice of Φ.

First the cases where $\Phi_1(p)$ and $\Phi_2(p)$ are unequal for all p or equal for exactly two elements $p, q \in V_4$: These loops can be trivially antiisomorphic by symmetry of Φ and Ψ. Otherwise they are not (anti-)isomorphic because out of every other pair of loops, which satisfies the preconditions, one and only one loop is a Bol-loop. Therefore in this cases it is not necessary to prove the validity of equation (3).

If there is exactly one element $r \in V_4$ for which $\Phi_1(r) = \Phi_2(r)$, then there are three possibilities, namely $\Phi_1(pq) = \Phi_2(pq)$, $\Phi_1(p) = \Phi_2(p)$ or $\Phi_1(q) = \Phi_2(q)$. In all three cases equation (3) holds for all combinations of $\Phi_1(q)$, $\Psi_1(p)$, $\Phi_2(q)$, $\Psi_2(p) \in \{\mathrm{id}, \mathrm{inv}\}$.

In the last case, which is $\Phi_1(p) = \Phi_2(p)$, $\Phi_1(q) = \Phi_2(q)$ and $\Phi_1(pq) = \Phi_2(pq)$, the validity of equation (3) is obvious. □

Corollary 1 *There are* 32 *Bol-loops of the form* $C_3 \rtimes (C_2)^2$ *which are distributed in two classes of isomorphism.*

Theorem 1 ($C_3 \rtimes (C_2)^n$) *Two proper Bol-loops of the form* $C_3 \rtimes (C_2)^n$ *are isomorphic if and only if they have the same number of elements with order* 6*.*

Proof If L_1 and L_2 are proper Bol-loops of the form $C_3 \rtimes (C_2)^n$ then Φ or Ψ is a homomorphism by Proposition 1. Without loss of gererality we assume both loops to be left Bol-loops. If the loops have the same number of elements with order 6 then the mapping ι as in the proof of Proposition 2 can be shown to be an isomorphism from L_1 onto L_2: Any two elements $\bar{a} = (a, p)$ and $\bar{b} = (b, q)$ of $C_3 \rtimes (C_2)^n$ generate a subloop of $C_3 \rtimes (C_2)^n$ isomorphic to $C_3 \rtimes (C_2)^2$. Therefore ι is an isomorphism by the proof of Proposition 2. □

Corollary 2 *For* $n \geq 3$ *the proper Bol-loops of the form* $C_3 \rtimes (C_2)^n$ *are distributed in* $2^n - 1$ *classes of isomorphism.*

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