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THE DISTRIBUTIVITY PROPERTY OF FINITE INTERSECTIONS OF VALUATION RINGS

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In this paper we assume that the reader is familiar with the basic facts of the valuation theory. (See e.g. [1], [2] or [3].)

Let K be a field, A, B, C be subrings of the field K. In [2] we have proved that if A, B, C are valuation rings, then both the distributive identities

$$C\cap(A\vee B)=(C\cap A)\vee(C\cap B) \tag{1}$$

$$C\vee(A\cap B)=(C\vee A)\cap(C\vee B) \tag{2}$$

hold. (Here as usual $A\vee B$ means the subring of the field K generated by the set $A\cup B$.)

In the present paper we shall show that this remains true also if A, B, C are finite intersections of valuation rings of the field K. (The present proof gives also a different proof of a more special theorem in [2].)

Theorem. *Let K be a field and A, B, C finite intersections of valuation rings of the field K. Then both the distributive properties* (1), (2) *hold.*

In the proof of the theorem we shall need the following well-known statements.

(A) Every overring of a finite intersection of valuation rings of the field K is itself a finite intersection of valuation rings of the field K. (See e.g. [1], [3] or [2], a final remark.)

(B) Every valuation ring E which contains a finite intersection $\bigcap_{i\in I} E_i$ of valuation rings contains some of the valuation rings E_i. (See [3], Chapter E, Corollary 2c.)

Proof of the theorem. First we shall prove the identity (1). Let $A=A_1\cap A_2\cap\ldots\cap A_m$, $B=B_1\cap B_2\cap\ldots\cap B_n$, $C=C_1\cap C_2\cap\ldots\cap C_l$ be intersections of valuation rings $A_1, A_2, \ldots, A_m$, $B_1, B_2, \ldots, B_n$, $C_1, C_2, \ldots, C_l$ of the field K. Let us consider the ring $(A\cap C)\vee(B\cap C)=D$. Since D is an overring of the finite intersection of valuation rings of the field K (e.g., $A\cap C$) we have by (A) that D is an intersection of valuation rings. Let $D=\bigcap_{i\in I} D_i$, where D_i are valuation rings of the field K. Then for each $i\in I$ we have $D_i\supset A\cap C$ and $D_i\supset B\cap C$. In both cases D_i

can be considered as an overring of the finite intersection of valuation rings of the field K. By (B) we get

(i) $D_i \supset C_k$ for some $k \in \{1, \dots, l\}$ or $D_i \supset A_j$ for some $j \in \{1, \dots, m\}$,

(ii) $D_i \supset C_r$ for some $r \in \{1, \dots, l\}$ or $D_i \supset B_s$ for some $s \in \{1, \dots, n\}$.

From this we have that $D_i \supset C$ or $D_i \supset A \vee B$. Hence, we have $D_i \supset C \cap (A \vee B)$.

Finally, we get $D = \bigcap_{i \in I} D_i \supset (A \vee B) \cap C$.

Since the converse is true in every lattice, the identity (1) is proved.

The proof of the second distributive identity (2) can be done exactly in the same way as in [2]. [Or it is possible to give an analogous proof to the proof of the identity (1).] In this way we can prove the identity (2) under a slightly weaker hypothesis: A, B are finite intersections of valuation rings of the field K. No assumption is made about the ring C. See also the remark.

The theorem is proved.

Remark. Example 1 in [2] shows that the theorem cannot be extended to arbitrary intersections of valuation rings (i.e. integrally closed subrings in the field K, see [1], § 10, (10.9), Corollary). Indeed, in this example there are constructed three subrings A, B, C of the field $K = L(x, y)$ (L is a field, x, y are independent indeterminates) with the following properties:

A, C are valuation rings of the field K.

$B = L[x^{-1}]$. Hence B is an integrally closed subring of K.

The rings A, B, C do not satisfy the identity (1). If A, B are finite intersections of valuation rings, then the distributivity identity (2) holds.

In any other case for the fulfilment of the distributivity law it is not sufficient to assume that only two rings from the triple $\{A, B, C\}$ are finite intersections of valuation rings. (Examples are in [2].)

REFERENCES

[1] ENDLER, O.: Valuation theory, Springer Verlag, Berlin—Heidelberg—New York 1972.

[2] MINÁČ, J.: The distributivity property of valuation rings, Math. Slovaca 31, 1981, 187—192.

[3] RIBENBOIM, P.: Théorie des valuations, Les presses de l'Université de Montreal 1965.

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СВОЙСТВО ДИСТРИБУТИВНОСТИ КОНЕЧНЫХ ПЕРЕСЕЧЕНИЙ КОЛЕЦ НОРМИРОВАНИЯ

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Резюме

В статье показывается, что все семейства, состоящие из трех конечных пересечений колец нормирования, удовлетворяют обом дистрибутивным тождествам. Это не всегда верно, если только два элемента из этого семейства являются даже кольцами нормирования.