Jaroslav Janák; Ladislav Skula

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ON THE INTEGERS x_i FOR WHICH $x_i \mid x_1 \ldots x_{i-1}x_{i+1} \ldots x_n + 1$ HOLDS

JAROSLAV JANÁK (PRAHA) AND LADISLAV SKULA (BRNO)

The present paper deals with the question of the existence of positive integers $x_1, \ldots, x_n$ $(n > 1)$ such that the integer x_i divides the integer $x_1 \ldots x_{i-1}x_{i+1} \ldots x_n + 1$ for each $1 \leqslant i \leqslant n$. This question was motivated by a problem of Š. Znám (1972, cf. [1]). where the integers x_i are required to be greater than 1 and the considered divisibility to be proper.

In his paper [1] L. Skula solved the given question for $2 \leqslant n \leqslant 4$. From this it follows that for these n's there are no integers x_i required in the problem of Znám. He showed that J. Janák had found by means of the computer that the integers 2, 3, 11, 23, 31 satisfy the conditions of Znám's problem for $n = 5$.

Clearly, we can restrict this question to the positive integers greater than 1.

In this paper the question of the existence of positive integers $x_1, \ldots, x_n$ $(n > 1)$ greater than 1 with the property $x_i \mid x_1 \ldots x_{i-1}x_{i+1} \ldots x_n + 1$ $(1 \leqslant i \leqslant n)$ is fully solved for $n = 5$ and $n = 6$ by Theorems 2 and 3. Theorem 1 gives bounds of the integers x_i dependent on the integer n. Hence for a fixed n the question is reduced to a finite number of possibilities and a computer can be used. In this way Theorem 3 was proved. The proof of Theorem 2 is shown directly and it is the development of the method used in [1]. Theorem 2 as well as the Theorem from [1] can be easily obtained by means of a computer and Theorem 1 (as well as Theorem 3); here, however, we have given a direct proof without using a computer. Some of the values x_i for $n = 7$ are given in the table 3.

Throughout this paper n will denote an integer greater than 1 *and* $1 < x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ *will be integers such that for each* $1 \leqslant i \leqslant n$

$$x_i \mid x_1 \ldots x_{i-1}x_{i-1} \ldots x_n + 1 \tag{1}$$

holds. We shall denote the integer $\dfrac{x_1 \ldots x_{i-1}x_{i+1} \ldots x_n + 1}{x_i}$ by y_i.

Clearly, for $1 \leqslant i \neq j \leqslant n$ we have $(x_i, x_j) = 1$, hence

$$1 < x_1 < \ldots < x_n \tag{2}$$

and

$$1 \leqslant y_n < y_{n-1} < \ldots < y_1$$

hold.

By multiplicating the right and the left-hand sides of the relation (1) we have

$$x_1 \ldots x_n \Big/ \sum_{i=1}^{n} \frac{x_1 \ldots x_n}{x_i} + 1 .$$

Hence there exists a positive integer m such that

(3) $$m = \sum_{i=1}^{n} \frac{1}{x_i} + \frac{1}{x_1 \ldots x_n} .$$

Lemma 1. $x_1 \leqslant n$.

Proof. Since $x_1 > 1$ there holds $1 \leqslant m \leqslant \frac{n}{x_1} + \frac{1}{x_1^n} < \frac{n+1}{x_1}$, where $x_1 < n+1$ follows.

Lemma 2. *Let* $1 \leqslant k \leqslant n-1$. *Then*

$$x_{k+1} < x_1 \ldots x_k (n-k+1) .$$

Proof. Let $1 \leqslant k \leqslant n-1$. According to (3) we have $m - \sum_{i=1}^{k} \frac{1}{x_i} > 0$ and because $m x_1 \ldots x_k - \sum_{i=1}^{k} \frac{x_1 \ldots x_k}{x_i}$ is an integer, we have

(4) $$m x_1 \ldots x_k - \sum_{i=1}^{k} \frac{x_1 \ldots x_k}{x_i} \geqslant 1 .$$

According to (2)

$$2 < x_{k+1} < x_{k+2} < \ldots < x_n$$

holds and implies

(5) $$\frac{1}{x_{k+1} \ldots x_n} < \frac{1}{2 \,.\, 3 \ldots (n-k+1)} \leqslant \frac{1}{n-k+1} .$$

From (3), (4) and (5) we obtain

$$\frac{n-k}{x_{k+1}} \geqslant \sum_{i=k+1}^{n} \frac{1}{x_i} = m - \sum_{i=1}^{k} \frac{1}{x_i} - \frac{1}{x_1 \ldots x_n} =$$

$$= \frac{1}{x_1 \ldots x_k} \left(m x_1 \ldots x_k - \sum_{i=1}^{k} \frac{x_1 \ldots x_k}{x_i} \right) - \frac{1}{x_1 \ldots x_n} \geqslant$$

$$\geqslant \frac{1}{x_1 \ldots x_k} \left(1 - \frac{1}{x_{k+1} \ldots x_n} \right) > \frac{1}{x_1 \ldots x_k} \frac{n-k}{n-k+1} .$$

Thus $x_{k+1} < x_1 \dots x_k(n-k+1)$.

From Lemma 1 *and* 2 *we get easily*

Theorem 1. *Let* $C_1 = n$ *and* $C_{k+1} = C_1 \dots C_k(n-k+1) - 1$ *for* $1 \leqslant k \leqslant n-1$. *Then*

$$x_i \leqslant C_i$$

for each $1 \leqslant i \leqslant n$.

Remark. For the constants C_i it obviously holds $C_i \leqslant n^{2^{i-1}}$.

The valuations in Lemma 1 and 2 and hence also in Theorem 1 can be improved. B. Novák, e.g., has communicated that for $n > 3$ we get $x_1 \leqslant n-2$ and in (5) we can use the relation $x_{k+1} \dots x_n \geqslant (k+2) \dots (n+1)$. But the aim of these assertions is only to show the finiteness of possibilities of integers x_i. Using the computer, the bounds $n^{2^{i-1}}$ were applied.

Table 1.

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
2	3	7	43	1 807	815 861	362 605	66 601	1 765	1
2	3	7	47	395	194 933	86 637	15 913	353	5
2	3	11	23	31	11 765	5 229	389	89	49

Theorem 2. *Let* $n = 5$. *Then the following table* 1 *gives all the possibilities of the integers* $x_1, \dots, x_5$.

Proof. By direct calculation we find out that the values given in the table 1 satisfy the given requirements.

For simplicity of notation we put $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$, $x_5 = e$, $y_5 = x$, $y_4 = y$.

By (2) we have

$$2 \leqslant a < b < c < d < e \; .$$

For the integer defined by the relation (3) there holds

$$m = \frac{1}{a} + \dots + \frac{1}{e} + \frac{1}{abcde} \leqslant \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{209}{144} < 2 \, ,$$

thus $m = 1$ and

$$\frac{1}{a} + \dots + \frac{1}{e} + \frac{1}{abcde} = 1 \; . \tag{6}$$

For $A = ab$ we obtain

(7) $$Acd + 1 = ex \ ,$$

(8) $$Ace + 1 = dy$$

Put $D = xy - A^2c^2$. Since $(A, x) = (A, y) = 1$, we have $D \neq 0$ and from (7) and (8) we get

(9) $$d = \frac{Ac + x}{D}, \qquad e = \frac{Ac + y}{D},$$

where from it follows that D is a positive integer. Obviously,

(10) $$x < y \ .$$

By multiplicating (7) and (8) we get $xyde = A^2c^2de + Acd + Ace + 1$, hence $D = \frac{Ac}{e} + \frac{Ac}{d} + \frac{1}{de} = Ac\left(\frac{1}{e} + \frac{1}{d} + \frac{1}{abcde}\right)$, whence and from (6) $D =$ $= Ac\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right)$, therefore

(11) $$D = c(A - a - b) - A \ .$$

The following cases I—V cover all the possibilities of integers a, b.

I. $a = 2$, $b = 3$. Then $A = 6$ and according to (11) $D = c - 6$.

Since $a \nmid c$, $b \nmid c$ and $D > 0$, the following cases (a)—(d) cover all the possibilities of the integer c.

(a) $c = 7$. Then $D = 1$, hence $xy = D + A^2c^2 = 5.353$. Since 353 is a prime, we get from (10) $x = 1, y = 5 \ . \ 353 = 1\ 765$ or $x = 5, y = 353$. According to (9) $d = 43$, $e = 1\ 807$ or $d = 47$, $e = 395$, which are values given in the table 1.

(b) $c = 11$. Then $D = 5$, hence $xy = 7^2.89$ and thus it follows that $x = 1$, $y = 7^2.89$ or $x = 7, y = 7.89$ or $x = 49, y = 89$. From (9) we obtain $d = \frac{67}{5}$ or $d = \frac{73}{5}$ or $d = 23$. Hence $d = 23$, $e = 31$, which are values given in the table 1.

(c) $c = 13$. Then $D = 7$, $xy = 6\ 091$ and, since 6 091 is a prime, it holds $x = 1$, $y = 6\ 091$ and according to (9) $d = \frac{79}{7}$, which is a contradiction.

(d) $c = \geqslant 17$. Then according to (6) $1 = \frac{1}{a} + \ldots + \frac{1}{e} + \frac{1}{a \ldots e} \leqslant \frac{1}{2} + \frac{1}{3} + \frac{1}{17} +$ $+ \frac{1}{19} + \frac{1}{23} + \frac{1}{2 \ . \ 3 \ . \ 17 \ . \ 19 \ . \ 23} = \frac{7\ 342}{7\ 429} < 1$, which is a contradiction.

II. $a = 2$, $b = 5$, $c = 7$. Then $D = 11$, $xy = 3 \ . \ 1\ 637$. Since 1 637 is a prime, we get from (10) $x = 1$, $y = 3 \ . \ 1\ 637$ or $x = 3$, $y = 1\ 637$ whence according to (9) $d = \frac{71}{11}$ or $d = \frac{73}{11}$, which is a contradiction.

Table 2

x_1	x_2	x_3	x_4	x_5	x_6	y_6
2	3	7	43	1 807	3 263 443	1
2	3	7	43	1 823	193 667	17
2	3	7	47	395	779 731	1
2	3	7	47	403	19 403	41
2	3	7	47	415	8 111	101
2	3	7	47	583	1 223	941
2	3	7	55	179	24 323	17
2	3	11	23	31	47 059	1

Table 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_7
2	3	7	43	1 807	3 263 443	10 650 056 950 807	1
2	3	7	43	1 807	3 263 447	2 130 014 000 915	5
2	3	7	43	1 807	3 263 591	71 480 133 827	149
2	3	7	43	1 807	3 264 187	14 298 637 519	745
2	3	7	43	1 823	193 667	637 617 223 447	1
2	3	7	47	395	779 731	607 979 652 631	1
2	3	7	47	395	779 831	6 020 372 531	101
2	3	7	47	403	19 403	15 435 513 367	1
2	3	7	47	415	8 111	6 644 612 311	1
2	3	7	47	583	1 223	1 407 479 767	1
2	3	7	55	179	24 323	10 057 317 271	1
2	3	11	23	31	47 059	2 214 502 423	1
2	3	11	23	31	47 063	442 938 131	5
2	3	11	23	31	47 095	59 897 203	37
2	3	11	23	31	47 131	30 382 063	73
2	3	11	23	31	47 243	12 017 087	185
2	3	11	23	31	47 423	6 114 059	365
2	3	11	31	35	67	369 067	13

III. $a=2, b\geq 5, c\geq 9$. Then according to (6) we have $1 = \frac{1}{a}+\ldots+\frac{1}{e}+\frac{1}{a\ldots e}\leq$ $\leq \frac{1}{2}+\frac{1}{5}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{2.5.9.11.13} = \frac{140}{143}<1$, which is a contradiction.

IV. $a=3, b=4, c=5$. Then $D=13$, $xy=3\,613$. Since 3 613 is a prime, we have, according to (10), $x=1$, $y=3\,613$ and by (9) there holds $d=\frac{61}{13}$, which is a contradiction.

V. $a\geq 3$, $\{a,b,c\}\neq\{3,4,5\}$. Then $b\geq 4, c\geq 7, d\geq 8, e\geq 11$ and according to (6) $1=\frac{1}{a}+\ldots+\frac{1}{e}+\frac{1}{a\ldots e}\leq\frac{1}{3}+\frac{1}{4}+\frac{1}{7}+\frac{1}{8}+\frac{1}{11}+\frac{1}{3.4.7.8.11}=$ $=\frac{995}{1\,056}<1$, which is a contradiction.

The proof of Theorem 2 is now complete.

Theorem 3. *Let $n=6$. Then the following table 2 gives all the possibilities of the integers $x_1, \ldots, x_6$.*

Proof. Theorem 1 and the computer.

Remark. Concluding we introduce table 3 for $n=7$ giving some values of integers $x_1, \ldots, x_7$ obtained by means of a computer. But we do not know if they are complete.

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Výskumní ústav matematických strojů
Stodůlecká 16
150 00 Praha-Jinonice

Katedra algebry a geometrie
Přírodovědecké fakulty UJEP
Janáčkovo nám. 2a
662 95 Brno

О ЦЕЛЫХ ЧИСЛАХ x_i, ДЛЯ КОТОРЫХ СПРАВЕДЛИВО $x_i \mid x_1\ldots x_{i-1}\,.\,x_{i+1}\ldots x_n+1$

Ярослав Янак и Ладислав Скула

Резюме

В работе рассматривается вопрос существования натуральных чисел $x_1, \ldots, x_n$ $(n>1)$, которые удовлетворяют отношению $x_i \mid x_1\ldots x_{i-1}\, x_{i+1}\ldots x_n+1$ для всех $1\leq i\leq n$. Для $2\leq n\leq 4$ этот вопрос решал Л. Цкула. В нашей статье даны все решения для $n=5$ и $n=6$. Для натурального $n>1$ найдена верхняя граница чисел x_i и в этом случае можно использовать вычислительную машину. Таким образом было получено решение для $n=6$. Некоторые величины x_i для $n=7$ тоже даны.