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František Katrnoška

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ON THE CENTER OF A LEFT JORDAN GROUPOID

FRANTIŠEK KATRNOŠKA

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ABSTRACT. The notions of left Jordan groupoids and homomorphisms of such groupoids are introduced. If R is an associative *ring with identity and if $U(R)$ ($P(R)$, resp.) denotes the set of idempotents (projectors, resp.) of R, then the operations $\circ_1$ and $\circ_2$ defined on R by

$$\begin{aligned} p \circ_1 q &= p - 2pq - 2qp + 4qpq\,, \\ p \circ_2 q &= q - 2pq - 2qp + 4pqp\,, \end{aligned} \qquad \text{for } \; p,q \in U(R) \;\; (p,q \in P(R), \text{ resp.})$$

are operations on $U(R)$ ($P(R)$, resp.) which need not be associative. The author defines the notion of center $C(X)$ of a left Jordan groupoid $(X, \circ, 0, 1, ')$ and establishes various properties of $C(X)$ and of the respective homomorphisms which are defined on the left Jordan groupoid X. These results may find an application in the foundations of quantum theory.

1. Introduction

Let R be an associative *ring with identity and let $U(R)$ ($P(R)$, resp.) be the set of idempotents (projectors, resp.) of the *ring R. Let us define

$$\begin{aligned} e \le f \iff ef &= fe = e\,, \\ e' &= 1 - e \end{aligned}$$

for $e, f \in U(R)$ ($e, f \in P(R)$, resp.). It is well known (see [1], [2]) that the sets $U(R)$ and $P(R)$ form orthomodular orthocomplemented posets which need not be lattices.

Another way of characterizing the set $U(R)$ ($P(R)$, resp.) of idempotents (projectors, resp.) of a *ring R is as the so-called left Jordan groupoid of idempotents $U(R)$ (projectors $P(R)$, resp.) of a *ring R. For $p, q \in U(R)$

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($p, q \in P(R)$, resp.) we define

$$p \circ_1 q = p - 2pq - 2qp + 4qpq\,,$$
$$p \circ_2 q = q - 2pq - 2qp + 4pqp\,.$$

It can be shown that $p \circ_1 q$ and $p \circ_2 q$ belong to $U(R)$ ($P(R)$, resp.) provided $p, q \in U(R)$ ($p, q \in P(R)$, resp.). $\big(U(R), \circ_1, 0, 1,'\big)$ and $\big(P(R), \circ_1, 0, 1,'\big)$ are the left Jordan groupoids of idempotents and projectors of the *ring R. In [2] it is shown that the elements $p, q \in U(R)$ ($p, q \in P(R)$, resp.) are orthogonal (we write $p \perp q$) if $pq = qp = 0$, and the elements $p, q \in U(R)$ ($p, q \in P(R)$, resp.) are compatible if $pq = qp$.

2. The left Jordan groupoid

We can now formalize the whole situation in the following definition.

Definition 1. ([4]) A non-empty set X is called a *left Jordan groupoid* if there is a binary operation $\circ\colon X \times X \to X$ and a unary operation $'\colon X \to X$ (an *orthocomplementation* on X) such that the following conditions are satisfied:

(i) $p \circ p = p, \quad p \in X$,
(ii) $(p \circ q) \circ p = p \circ (q \circ p), \quad p, q \in X$,
(iii) $(p \circ q) \circ q = p, \quad p, q \in X$,
(iv) $(p')' = p, \quad p \in X$,
(v) $(p \circ q)' = p' \circ q', \quad p, q \in X$,
(vi) $p \circ q' = p \circ q, \quad p, q \in X$,
(vii) X contains elements $0, 1 \in X$ such that $p \circ 1 = p \circ 0 = p$, $1 \circ p = 1$, $0 \circ p = 0$ and $0' = 1$.

Remarks.

a) It follows immediately that $p^2 \circ (q \circ p) = \big[(p \circ p) \circ q\big] \circ p$ provided $p, q \in X$.

b) In general, the left Jordan groupoid is non-commutative and non-associative.

3. The center of the left Jordan groupoid

Definition 2. Let $(X, \circ, 0, 1,')$ be a left Jordan groupoid. The *center* $C(X)$ of X is the set of $p \in X$ such that $p \circ q = p$ for each $q \in X$.

Proposition 1. *If* $(X, \circ, 0, 1,')$ *is a left Jordan groupoid then* $0, 1 \in C(X)$ *and* $p' \in C(X)$ *for each* $p \in C(X)$. *Moreover,* $\big(C(X), \circ, 0, 1,'\big)$ *is an associative subgroupoid of* $(X, \circ, 0, 1')$.

Proof. From (vii) of Definition 1 we have that $1 \circ p = 1$ and $0 \circ p = 0$ for each $p \in X$. Therefore $0, 1 \in C(X)$. From (v) and (iv) of Definition 1 it follows that $p' \circ q = (p \circ q')' = p'$ and therefore we have $p' \in C(X)$ for each $p \in C(X)$. If $p \in C(X)$ then $p \circ q = p$ for each $q \in X$. Hence $\circ$ is an operation on $C(X)$. Since $(p \circ q) \circ r = p \circ r = p = p \circ (q \circ r)$ for each $p, q, r \in C(X)$, $C(X)$ is an associative groupoid. □

Let us denote by $Z(U(R))$ the center of the orthoposet $(U(R), \leq, 0, 1, ')$. We will show that $C(U(R)) = Z(U(R))$.

PROPOSITION 2. *Let R be an associative *ring with identity. Let $(U(R), \leq, 0, 1, ')$ be the orthomodular orthoposet of idempotents of R and let $(U(R), \circ_1, 0, 1, ')$ be the left Jordan groupoid of idempotents of R. Then $C(U(R)) = Z(U(R))$.*

Proof. Let $p, q \in U(R)$. If $p \circ_1 q = p$ then $pq + qp = 2qpq$ and therefore $pq = (pq)q = (2qpq - qp)q = qpq = q(2qpq - pq) = q(qp) = qp$ (i.e., $p \leftrightarrow q$). Conversely, if $pq = qp$ then $pq + qp = 2pq = 2(pq)q = 2qpq$ and $p \circ_1 q = p$. □

Now we will show that there are some connections with results of [3].

DEFINITION 3. Let $(P, \leq, 0, 1, ')$ be an orthocomplemented poset. A Boolean subalgebra $B \subset P$ of P is called a *maximal Boolean subalgebra* of P (or a *block* of P) if there is no Boolean subalgebra B_1 such that $B \subset B_1$ and $B \neq B_1$.

LEMMA 1. *Let P be an orthocomplemented poset. Then every Boolean subalgebra B of P is contained in some block of P.*

Proof. Obvious. □

DEFINITION 4. Let $(X, \circ, 0, 1, ')$ be a left Jordan groupoid. If $p, q \in X$ and $p \circ q = p$ then we say that p is *left compatible* with q (we write $p \overset{1}{\leftrightarrow} q$). The set $B_q \subset X$ of left compatible elements with $q \in X$ is called a *left block* of X in q.

PROPOSITION 3. *Let $(X, \circ, 0, 1, ')$ be a left Jordan groupoid. Then the following assertions hold:*

(i) *If $p \in X$ then $p \overset{1}{\leftrightarrow} p$ and $p \overset{1}{\leftrightarrow} p'$.*

(ii) *If $p, q \in X$ and $p \overset{1}{\leftrightarrow} q$ then $q \circ p \overset{1}{\leftrightarrow} q$.*

Proof.

(i) This follows from (i) and (vi) of Definition 1.

(ii) Let $p \overset{1}{\leftrightarrow} q$ $(p, q \in X)$. Then $q \circ (p \circ q) = q \circ p$. By Definition 1, $q \circ (p \circ q) = (q \circ p) \circ q = q \circ p$, i.e. $q \circ p \overset{1}{\leftrightarrow} q$. □

In [3] we presented a characterization of the center of the orthoposet $(P, \leq, 0, 1, ')$. This says that the center $Z(P)$ of the orthoposet P equals the intersection of all the blocks. A similar theorem holds even for a left Jordan groupoid.

PROPOSITION 4. *Let $(P, \circ, 0, 1, ')$ be a left Jordan groupoid. The center $C(P)$ of P is the intersection of all left blocks of P, i.e. $C(P) = \bigcap_{q \in P} B_q$.*

Proof. If B_q $(q \in P)$ is a left block of P in q then $B_q = \{p \in P;\ p \circ q = p\}$ and $\bigcap_{q \in P} B_q = \{p \in P;\ p \circ q = p \text{ for each } q \in P\} = C(P)$. □

4. Example of a left Jordan groupoid and of a homomorphism

EXAMPLE 1. Let us consider a four-element set $X = \{0, p, p', 1\}$ and suppose that X is a left Jordan groupoid. We can show that X has the following Cayley table:

	0	p	p'	1
0	0	0	0	0
p	p	p	p	p
p'	p'	p'	p'	p'
1	1	1	1	1

Since $p \circ q = p$ for each $p, q \in X$, it follows that $p \overset{1}{\leftrightarrow} q$ for each $p, q \in X$. If we define $p \vee p' = 1$, $p \vee 0 = p$, $p \vee 1 = 1$, $p' \vee 0 = p'$, $p' \vee 1 = 1$ and if we introduce similar formulas for $p \wedge q$, then $(X, \leq, 0, 1, ')$ is a Boolean algebra.

DEFINITION 5. Let $(X_1, \circ_1, 0_1, 1_1, ')$ and $(X_2, \circ_2, 0_2, 1_2, ^*)$ be left Jordan groupoids. The mapping $h\colon X_1 \to X_2$ is called a *homomorphism* of X_1 into X_2 if

(i) $h(p_1 \circ_1 p_2) = h(p_1) \circ_2 h(p_2)$, $\quad p_1, p_2 \in X_1$,
(ii) $h(p') = [h(p)]^*$, $\quad p \in X_1$,
(iii) $h(0_1) = 0_2$.

We now exhibit an example of a homomorphism of $(X_1, \circ, 0, 1, ')$ into $(X, \circ, 0, 1, ')$, $X_1 \subset X$.

EXAMPLE 2. Let $(X, \circ, 0, 1, ')$ be a left Jordan groupoid and let $(X_1, \circ, 0, 1, ')$ be an associative subgroupoid of X, i.e. $X_1 \subset X$, and $p \circ q = p$ for each $p, q \in X_1$. We can now define a mapping $h \colon X_1 \to X$ as follows:

$$h(p) = p \qquad \text{for} \quad p \in X_1 .$$

Then h is a homomorphism of X_1 into X.

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Department of Mathematics
Institute of Chemical Technology
Technická 5
CZ–166 28 Praha
CZECH REPUBLIC