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ZEROS OF CONTINUOUS FUNCTIONS AND THE STRUCTURE OF TWO FUNCTION SPACES

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ABSTRACT. In this paper, the structure of two spaces of continuous functions is studied from the point of view of metric and topological properties of the sets $Z(f) = f^{-1}(0) = \{x : f(x) = 0\}$ and porosity of some sets. Some results of the paper [BENAVIDES, T. D.: *How many zeros does a continuous function have?* Amer. Math. Monthly **93** (1986), 464–466] are here extended and deepened.

Introduction

The author of the paper [1] studies the structure of the space $C_0(a, b)$ of all continuous real-valued functions on the interval $[a, b]$ having at least one zero. Denote by $Z(f)$ the set of zeros of f. It is shown, that in $C_0(a, b)$, there are typical those functions having $\operatorname{card} Z(f) = c$ (cardinality of continuum) and $\lambda(Z(f)) = 0$, where λ denotes Lebesgue measure. Note that if F is a space of functions and $F_1 \subseteq F$ is residual in F, then each function $f \in F_1$ is said to be typical in the space F.

In the first part of the paper we investigate the position of $C_0(a, b)$ as the subset of the space $C(a, b)$ of all continuous real valued functions on the interval $[a, b]$ with the metric ϱ, $\varrho(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$, the metric in $C_0(a, b)$ being $\varrho|_{C_0(a,b) \times C_0(a,b)}$.

In Section 2 we describe the structure of the spaces $C(a, b)$, $C_0(a, b)$ from the point of view of topological properties of the sets $Z(f)$.

In Section 3 we extend the result in [1] and show that in $C(a, b)$ and $C_0(a, b)$, there are typical functions having $\dim Z(f) = 0$. The symbol $\dim M$ denotes Hausdorff dimension of the set M (see [3; p. 50–78]).

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Throughout the paper, for better distinguishing between $C(a,b)$ and $C_0(a,b)$, we will denote the ball in $C(a,b)$ by the symbol $B(g,\delta)$ ($g \in C(a,b)$, $\delta > 0$) and the ball in $C_0(a,b)$ by $B_0(g,\delta)$ ($g \in C_0(a,b)$, $\delta > 0$).

The notion of porosity in a metric space is introduced in agreement with the definition of porosity on line (see [5; p. 183–190]) as follows. Let (X,d) be a metric space and $Y \subseteq X$, $x \in X$, $\delta > 0$, then symbol $\gamma(x,\delta,Y)$ denotes the supremum of the set of all $t > 0$ for which there exists $y \in X$ such that $B(y,t) \subseteq B(x,\delta)\backslash Y$. If there exist no such $t > 0$, then $\gamma(x,\delta,Y) = 0$.

The numbers $\underline{p}(x,Y) = \liminf\limits_{\delta\to 0+} \frac{\gamma(x,\delta,Y)}{\delta}$ and $\overline{p}(x,Y) = \limsup\limits_{\delta\to 0+} \frac{\gamma(x,\delta,Y)}{\delta}$ are called the *lower* and *upper porosity of* Y *at* x respectively. We say that Y is *porous* or *very porous at* x if $\overline{p}(x,Y) > 0$ or $\underline{p}(x,Y) > 0$ respectively. If the number $p(x,Y) = \lim\limits_{\delta\to 0+} \frac{\gamma(x,\delta,Y)}{\delta}$ exists, it is called the *porosity of* Y *at* x. If $\overline{p}(x,Y) \geq c$ or $\underline{p}(x,Y) \geq c$ and $c > 0$, then Y is called *c-porous* or *very-c-porous at* x respectively. The set Y is called *σ-porous*, *σ-c-porous*, *σ-very-porous* or *σ-very-c-porous at* x if $Y = \bigcup\limits_{n=1}^{\infty} Y_n$ and every set Y_n for $n = 1,2,3,\dots$ is porous, c-porous, very porous or very-c-porous at x, respectively.

§1. The position of the subset $C_0(a,b)$ in the space $C(a,b)$

It is well known that $C(a,b)$ is a complete metric space, therefore it is a Baire space. It is easy to see that the set $C_0(a,b)$ is a closed subspace of the space $C(a,b)$. For this suffices to show that the set $C(a,b)\backslash C_0(a,b)$ is an open set in the space $C(a,b)$. Let $g \in C(a,b)\backslash C_0(a,b)$. Then g has no zero on $[a,b]$ therefore for each $x \in [a,b]$ we have $g(x) > 0$ or $g(x) < 0$ for all $x \in [a,b]$. In the first case we put $\delta = \min g(x)$ and for the second case $\delta = |\max g(x)|$. It can be easily verified that $B(g,\delta) \subseteq C(a,b)\backslash C_0(a,b)$. So we see that $C_0(a,b)$ is also a complete metric space.

According to the previous consideration the set $C(a,b)\backslash C_0(a,b)$ is of the second Baire category in $C(a,b)$.

We show that $\operatorname{Int} C_0(a,b) \neq \emptyset$. It is enough to take a continuous real valued function g on $[a,b]$ such that $g\left(\frac{a+b}{2}\right) = 0$, $g(a) < 0$, $g(b) > 0$ and put $\delta < \min\{|g(a)|, g(b)\}$. Then, evidently, the ball $B(g,\delta)$ in $C(a,b)$ is a subset of $C_0(a,b)$. If $f \in B(g,\delta)$, then $f(a) < 0$, $f(b) > 0$ and therefore f has at least one zero on $[a,b]$.

Since each of the sets $C_0(a,b)$ and $C(a,b)\backslash C_0(a,b)$ has a non-empty interior in the space $C(a,b)$ they are neither dense nor nowhere dense.

We shall investigate their porosity at points of $C(a,b)$. Since $C_0(a,b)$ is a closed subset of $C(a,b)$ its porosity at each point $g \notin C_0(a,b)$ equals 1. Hence it suffices to investigate its porosity only at the points of $C_0(a,b)$.

THEOREM 1.1. *Let $g \in C_0(a,b)$*

(i) *If $g(x) \geq 0$ for all $x \in [a,b]$ or $g(x) \leq 0$ for all $x \in [a,b]$, then $\underline{p}(g, C_0(a,b)) \geq 1/2$.*

(ii) *Otherwise we have $p(g, C_0(a,b)) = 0$.*

Proof.

(i) Let for example $g(x) \geq 0$ for each $x \in [a,b]$ and $B(g,\delta)$ be an arbitrary ball. Define function h, $h(x) = g(x) + \delta/2$. Evidently $B(h,\delta/2) \subseteq B(g,\delta)$ and $B(h,\delta/2) \cap C_0(a,b) = \emptyset$. Therefore $\gamma(g,\delta,C_0(a,b)) \geq \delta/2$ and from this $\underline{p}(g, C_0(a,b)) \geq 1/2$.

(ii) According to the assumption, there exist numbers $t_1, t_2 \in [a,b]$, $t_1 \neq t_2$ such that $g(t_1) < 0 < g(t_2)$. Put $\delta = \min\{|g(t_1)|, g(t_2)\}$ and take $B(g,\eta)$, $0 < \eta < \delta$.

If $f \in B(g,\eta)$, on the basis of the definition of number δ we have $f(t_1) < 0$, $0 < f(t_2)$ and by the continuity of f on $[a,b]$ there exists a zero between t_1 and t_2, thus $f \in C_0(a,b)$. Therefore $B(g,\eta) \subseteq C_0(a,b)$. In this way we have proved that $\gamma(g,\eta,C_0(a,b)) = 0$ for each $\eta \in (0,\delta)$, hence $p(g, C_0(a,b)) = 0$. □

Now we are going to investigate the porosity of the set $C(a,b)\backslash C_0(a,b)$. Since $C(a,b)\backslash C_0(a,b)$ is an open set, it is interesting to investigate its porosity only at points of the set $C_0(a,b)$.

THEOREM 1.2. *Let $g \in C_0(a,b)$.*

(i) *If $g(x) = 0$ for each $x \in [a,b]$, then $\underline{p}(g, C(a,b)\backslash C_0(a,b)) \geq 1/2$.*

(ii) *If for each $x \in [a,b]$ we have $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in [a,b]$, and g is not identically zero, then $\underline{p}(g, C(a,b)\backslash C_0(a,b)) \geq 1/2$.*

(iii) *If $\min\limits_{x\in[a,b]} g(x) < 0 < \max\limits_{x\in[a,b]} g(x)$, then $p(g, C(a,b)\backslash C_0(a,b)) = 1$.*

Proof.

(i) Let $B(g,\delta)$ be an arbitrary ball in $C(a,b)$. Put $h(x) = \left(x - \frac{a+b}{2}\right)\frac{\delta}{b-a}$ for $x \in [a,b]$. Evidently $B(h,\delta/2) \subseteq B(g,\delta)$ and $B(h,\delta/2) \subseteq C_0(a,b)$. Therefore, $B(h,\delta/2) \cap \left[C(a,b)\backslash C_0(a,b)\right] = \emptyset$, so we have $\underline{p}(g, C(a,b)\backslash C_0(a,b)) \geq 1/2$.

(ii) Assume, without loss of generality, that $g(x) \geq 0$ for each $x \in [a,b]$ and $\delta = \max\limits_{x\in[a,b]} g(x) > 0$. Then according to the continuity of g, there exists such a point $t_1 \in [a,b]$ that $g(t_1) = \max g(x) = \delta > 0$. Construct $B(g,\delta)$ and put $h(x) = g(x) - \eta/2$ for η, $0 < \eta < \delta$. Then clearly $B(h,\eta/2) \subseteq B(g,\eta)$. Since $g \in C_0(a,b)$, there exists a point $t_0 \in [a,b]$ such that $g(t_0) = 0$. Now, for $f \in B(h, \eta/2)$ we have $f(t_1) = h(t_1) + \varepsilon$, $0 \le |\varepsilon| < \eta/2$. Hence $f(t_1) = g(t_1) - \eta/2 + \varepsilon > 0$ and

$$f(t_0) = h(t_0) + \varepsilon = g(t_0) - \eta/2 + \varepsilon < 0 .$$

Since f is continuous, there exists a zero of f between t_0 and t_1, thus $f \in C_0(a,b)$. This implies $B(h, \eta/2) \subseteq C_0(a,b)$ for each η, $0 < \eta < \delta$. Therefore $\gamma\big(g, \eta, C(a,b) \setminus C_0(a,b)\big) \ge \eta/2$. Hence $\underline{p}\big(g, C(a,b) \setminus C_0(a,b)\big) \ge 1/2$.

(iii) Let t_1, t_2 be points in the interval $[a,b]$ such that $g(t_1) < 0$, $g(t_2) > 0$. Put $\delta = \min\{|g(t_1)|, g(t_2)\}$. Consider the ball $B(g, \eta)$, $0 < \eta < \delta$. Then for each $f \in B(g, \eta)$ we have $f(t_1) < 0$, $f(t_2) > 0$. Therefore $f \in C_0(a,b)$. Hence $B(g, \eta) \cap \big[C(a,b) \setminus C_0(a,b)\big] = \emptyset$ so we have $\gamma\big(g, \eta, C(a,b) \setminus C_0(a,b)\big) \ge \eta$. This implies $p\big(g, C(a,b) \setminus C_0(a,b)\big) = 1$. □

§2. The structure of spaces $C(a,b)$ and $C_0(a,b)$ from point of view of topological properties of sets $Z(f)$

As it was mentioned earlier, in the paper [1] it is shown that in $C_0(a,b)$, there are typical functions for which $\operatorname{card}\big(Z(f)\big) = c$ and $\lambda\big(Z(f)\big) = 0$ simultaneously. We will investigate the structure of spaces $C(a,b)$ and $C_0(a,b)$ from the point of view of nowhere density and perfectness of sets $Z(f)$.

Denote by $H(a,b)$ or $H_0(a,b)$ the set of those functions f, $f \in C(a,b)$ or $f \in C_0(a,b)$, respectively, for which $Z(f)$ is a perfect and nowhere dense set in $[a,b]$. We will show that in $C(a,b)$ or $C_0(a,b)$ are typical functions for which $Z(f)$ is a perfect and nowhere dense set.

THEOREM 2.1.

(i) *The set $H(a,b)$ is a residual set in $C(a,b)$.*

(ii) *The set $H_0(a,b)$ is a residual set in $C_0(a,b)$.*

P r o o f.

(i) Let $A(a,b)$ be the set of all $f \in C(a,b)$ for which the set $Z(f)$ is not nowhere dense. We claim that $A(a,b)$ is a set of the first Baire category. Denote by $C^*(a,b)$ the set of those functions f, $f \in C(a,b)$, which are not monotone on any subinterval $J \subseteq [a,b]$. In the paper [4] it is shown that $C^*(a,b)$ is a residual set in $C(a,b)$. Let f be a function in $A(a,b)$. Then $Z(f)$ is not a nowhere dense set. Since $Z(f)$ is a closed set, there exists an interval $I \subseteq [a,b]$ such that $I \subset Z(f)$ and so f is monotone on I. Thus f belongs to the set $C(a,b) \setminus C^*(a,b)$ and therefore $A(a,b) \subseteq C(a,b) \setminus C^*(a,b)$. Since $C(a,b) \setminus C^*(a,b)$ is the set of the first Baire category, we have that the set $A(a,b)$ is also the set of the first Baire category. This implies that the set of those functions for which $Z(f)$ is a nowhere dense set is a residual set in $C(a,b)$.

Denote by $D(a,b)$ the set of all $f \in C(a,b)$ for which the set $Z(f)$ is not perfect. Because the empty set is perfect we have that $Z(f) \neq \emptyset$. It means that the set $Z(f)$ has an isolated point. Let $S = \{I_n : n \in \mathbb{N}\}$ be the collection of all closed subintervals of $[a,b]$ whose endpoints are rational numbers from $[a,b]$ or belong to the set $\{a,b\}$. Define $D_n = \{f \in C(a,b) : f \text{ has a unique zero in } I_n\}$.

We claim, that D_n is nowhere dense in $C(a,b)$. Let $B(g,\delta)$ be an arbitrary ball in $C(a,b)$. We will prove that there exists a ball $B_1 \subseteq B(g,\delta)$ disjoint with the set D_n. We will distinguish the following two cases:

a) g has no zero in I_n,
b) g has a zero in I_n.

a) Put $\eta = \min\{\delta, \min\limits_{x\in I_n} |g(x)|\}$ and $B_1 = B(g,\eta)$. Evidently $B(g,\eta) \subseteq B(g,\delta)$. Further, if $f \in B(g,\eta)$, then f has no zero in I_n, hence $B(g,\eta) \cap D_n = \emptyset$.

b) Put $I_n = [a_n, b_n]$. Let $g(x_0) = 0$, $x_0 \in [a_n, b_n]$. At first, assume that $x_0 \neq a_n$ and $x_0 \neq b_n$. Choose $\eta > 0$, $\eta < \min\{|a_n - x_0|, |b_n - x_0|, \delta/4\}$ such that $|g(x)| < \delta/4$ whenever $|x - x_0| < 2\eta$. Note that such η exists because g is a continuous function. Define a continuous function h on $[a,b]$ as follows:

$$h(x) = \begin{cases} \frac{\delta}{2\eta}(x - x_0) - \frac{\delta}{4} & \text{for } x \in [x_0, x_0+\eta), \\ -\frac{\delta}{2\eta}(x - x_0) - \frac{\delta}{4} & \text{for } x \in [x_0-\eta, x_0), \\ g(x) & \text{for } |x - x_0| \geq 2\eta, \\ \text{linear} & \text{for the rest in } [a,b]. \end{cases}$$

For this continuous function h we have $\varrho(g,h) < \delta/2$ and put $B_1 = B(h,\delta/4)$. It is obvious that $B(h,\delta/4) \subseteq B(g,\delta)$. Further, from construction of the function h we get

$$h(x_0 - \eta) = \delta/4, \qquad h(x_0) = -\delta/4, \qquad h(x_0 + \eta) = \delta/4. \tag{1}$$

If $f \in B(h,\delta/4)$, then, according to (1), we have $f(x_0 - \eta) > 0$, $f(x_0) < 0$ and $f(x_0 + \eta) > 0$. Therefore, the function f has in every interval $(x_0-\eta, x_0)$, $(x_0, x_0+\eta)$ at least one zero. Then f has at least two zeros in $[x_0-\eta, x_0+\eta] \subset I_n = [a_n, b_n]$. This implies $B(h,\delta/4) \cap D_n = \emptyset$.

Now, assume that $x_0 = a_n$ (if $x_0 = b_n$ a similar argument can be used). Choose $\eta > 0$, $\eta < \delta/4$, $2\eta < b_n - a_n$ such that $|g(x) - g(a_n)| = |g(x)| < \delta/4$ whenever $|x - a_n| < 2\eta$. Once more, the existence of such η is implied by the continuity of g. Now we can define a continuous function h.

$$h(x) = \begin{cases} \frac{\delta}{\eta}(x - a_n) & \text{for } x \in [a_n, a_n+\eta/4]\,, \\ -\frac{2\delta}{\eta}\left(x - a_n - \frac{\eta}{4}\right) + \frac{\delta}{4} & \text{for } x \in (a_n+\eta/4, a_n+\eta/2)\,, \\ \frac{\delta}{\eta}(x - a_n - \frac{\eta}{2}) - \frac{\delta}{4} & \text{for } x \in [a_n+\eta/2, a_n+\eta]\,, \\ \text{linear} & \text{for } x \in (a_n+\eta, a_n+2\eta)\,, \\ g(x) & \text{for the rest in } [a,b]\,. \end{cases}$$

It is clear that $\varrho(g,h) < \delta/2$. Put $B_1 = B(h,\delta/4)$, then $B(h,\delta/4) \subseteq B(g,\delta)$. If $f \in B(h,\delta/4)$, then according to the construction of h we get $f(a_n + \eta/4) > 0$, $f(a_n + \eta/2) < 0$ and $f(a_n + \eta) > 0$.

Similarly as in the previous case we get $B(h,\delta/4) \cap D_n = \emptyset$.

We will prove that

$$D(a,b) = \bigcup_{n=1}^{\infty} D_n\,. \tag{2}$$

Suppose $f \in D(a,b)$, then $Z(f)$ is not a perfect set in $[a,b]$. Therefore it has an isolated point x_0, for which there exists positive integer n such that $I_n \cap Z(f) = \{x_0\}$. Hence $f \in D_n$. The converse inclusion is obvious. Thus (2) holds.

Since the right-hand side of (2) is a set of the first Baire category we see that the set $D(a,b)$ is also the set of the first Baire category.

According to the previous reasoning we have that the set $A(a,b) \cup D(a,b)$ is a set of the first Baire category in $C(a,b)$.

It is obvious that $C(a,b) \backslash H(a,b) = A(a,b) \cup D(a,b)$. Therefore $H(a,b)$ is a residual set in $C(a,b)$.

(ii) Let $A_0(a,b)$ be the set of all $f \in C_0(a,b)$ for which the set $Z(f)$ is not nowhere dense. We will show that $A_0(a,b)$ is the set of the first Baire category.

Let $S = \{I_n\ :\ n \in \mathbb{N}\}$ have the same meaning as before. Define $A_n = \{f \in C_0(a,b)\ :\ f(x) = 0 \text{ for } x \in I_n\}$. We will prove that A_n is a nowhere dense in $C_0(a,b)$.

Let $B_0(g,\delta)$ be an arbitrary sphere in $C_0(a,b)$. We have to prove that there exists a ball $B \subseteq B_0(g,\delta)$ disjoint with the set A_n.

For this we will distinguish two cases:

a) $g \notin A_n$,
b) $g \in A_n$.

a) Since $g \notin A_n$, there exists a point $x_0 \in I_n$ such that $g(x_0) \neq 0$. Put $\eta = \min\left\{\frac{|g(x_0)|}{2}, \delta\right\}$. Put $B = B_0(g,\eta)$, then $B_0(g,\eta) \subseteq B(g,\delta)$. If $f \in B_0(g,\eta)$, then $f(x_0) \in \big(g(x_0)-\eta,\, g(x_0)+\eta\big)$, therefore $f(x_0) \neq 0$. Thus $f \notin A_n$. From this we get $B_0(g,\eta) \cap A_n = \emptyset$.

b) Assume $I_n = [a_n, b_n]$, $x_0 = \frac{a_n+b_n}{2}$. Define a continuous function h on $[a,b]$ as follows:

$$h(x) = \begin{cases} -\left|\frac{\delta}{b_n-a_n}(x-x_0)\right| + \frac{\delta}{2} & \text{for } x \in [a_n, b_n]\,, \\ g(x) & \text{for the rest of } [a,b]\,. \end{cases}$$

Evidently $h \in C_0(a,b)$. Consider the ball $B = B_0(h, \delta/2)$. According to the definition of h we have $B_0(h,\delta/2) \subseteq B_0(g,\delta)$. If $f \in B_0(h,\delta/2)$, then $f(x_0) \in (0,\delta)$. Hence $f \notin A_n$, which implies $B_0(h,\delta/2) \cap A_n = \emptyset$. Therefore the set A_n is nowhere dense.

Now we will show that

$$A_0(a,b) = \bigcup_{n=1}^{\infty} A_n\,. \tag{3}$$

Indeed, let $f \in A_0(a,b)$. According to the definition of $A_0(a,b)$, the set $Z(f)$ contains a certain interval of the collection S. Then f belongs to A_n for a positive integer n. The converse inclusion is obvious. Thus (3) holds.

Since the right-hand side of (3) is a set of the first Baire category, $A_0(a,b)$ is a set of the first Baire category.

Denote $D_0(a,b)$ the set of all $f \in C_0(a,b)$ for which the set $Z(f)$ is not a perfect set. Similarly as in the proof of (i) we can prove that $D_0(a,b)$ is of the first Baire category.

Again we have $C_0(a,b) \backslash H_0(a,b) = A_0(a,b) \cup D_0(a,b)$. Therefore $H_0(a,b)$ is a residual set in $C_0(a,b)$.

This completes the proof of Theorem 2.1. □

Now we will investigate density of the sets $A_0(a,b)$ and $D_0(a,b)$. In this way we will complete the previous theorem.

THEOREM 2.2. *Sets $A_0(a,b)$ and $D_0(a,b)$ are dense in $C_0(a,b)$.*

Proof. Let $B_0(g,\eta)$ be an arbitrary sphere in $C_0(a,b)$. We will prove that

$$B_0(g,\eta) \cap A_0(a,b) \neq \emptyset\,, \tag{4}$$

$$B_0(g,\eta) \cap D_0(a,b) \neq \emptyset\,. \tag{5}$$

Denote x_0 a zero of $g \in C_0(a,b)$. Since g is a continuous function, for $\eta/2$ there exists $\delta > 0$, such that $|g(x) - g(x_0)| < \eta/2$ for all $x \in [a,b]$ whenever $|x - x_0| < \delta$. Choose two rational numbers a_n, b_n so that $a_n, b_n \in (x_0-\delta, x_0+\delta) \cap [a,b]$ and $a_n < b_n$. Denote $I_n = [a_n, b_n]$ and define a continuous function h in the following way:

$$h(x) = \begin{cases} 0 & \text{for } x \in I_n\,, \\ g(x) & \text{for } [a,b] - (x_0-\delta, x_0+\delta)\,, \\ \text{linear} & \text{for the rest of } [a,b]\,, \end{cases}$$

From the construction of h it is clear that $h \in B_0(g,\eta)$ and $h \in A_n \subset A_0(a,b)$, thus (4) is true.

In order to prove (5) we define the continuous function h as follows:

$$h(x) = \begin{cases} g(x) & \text{for } [a,b] - (x_0-\delta, x_0+\delta)\,, \\ \frac{\eta}{b_n-a_n}(x-b_n)+\frac{\eta}{2} & \text{for } x \in I_n\,, \\ \text{linear} & \text{for the rest of } [a,b]\,. \end{cases}$$

According to the definition of h we have $h \in B_0(g,\eta)$. Further $\frac{a_n+b_n}{2}$ is a unique zero of h in the interval I_n. Therefore $h \in D_n \subset D_0(a,b)$. Hence $h \in B_0(g,\eta) \cap D_0(a,b)$, which implies (5). □

COROLLARY 2.3. *The set $C_0(a,b)\backslash H_0(a,b)$ is dense in $C_0(a,b)$.*

In connection with Theorem 2.2 and Corollary 2.3 the following question arises: Is a similar assertion true in the space $C(a,b)$? The answer is negative.

THEOREM 2.4. *The set $C(a,b)\backslash H(a,b)$ is neither dense nor nowhere dense in $C(a,b)$.*

Proof. In order to show that the set $C(a,b)\backslash H(a,b)$ is not nowhere dense in $C(a,b)$ it is enough to construct such a sphere in $C(a,b)$ that the set $C(a,b)\backslash H(a,b)$ is dense in it. Put $x_0 = \frac{a+b}{2}$. Choose a continuous function g on $[a,b]$ such that $g(x_0)=0$, $g(a)<0$, $g(b)>0$. Let

$$\delta = \min\Big\{\big|\min_{x\in[a,b]} g(x)\big|,\ \max_{x\in[a,b]} g(x)\Big\}\,.$$

Consider a sphere $B(g,\delta)$. Let $B(f,\eta)$ be an arbitrary ball in $B(g,\delta)$. Since $f \in B(g,\delta)$, according to the definition of δ, there exist numbers $x_1, x_2 \in [a,b]$, $x_1 < x_0 < x_2$ such that $f(x_1) < 0$, $f(x_2) > 0$. Therefore there exists a point y_0, $x_1 < y_0 < x_2$ such that $f(y_0) = 0$. We will show that

$$\big[C(a,b)\backslash H(a,b)\big] \cap B(f,\eta) \neq \emptyset\,. \tag{6}$$

According to the continuity of f there exists $\eta_1 > 0$, such that $|f(x)| < \eta/2$ whenever $x \in [y_0-\eta_1, y_0+\eta_1] \cap [a,b]$. Put $h(x) = 0$ for $x \in [y_0-\eta_1/2, y_0+\eta_1/2]$, $h(x) = g(x)$ for $x \in [a,b]\backslash[y_0-\eta_1, y_0+\eta_1]$ and h is linear and continuous on intervals $J_1 = [y_0-\eta_1, y_0-\eta_1/2]\cap[a,b]$, $J_2 = [y_0+\eta_1/2, y_0+\eta_1]\cap[a,b]$. Thus h belongs to $C(a,b)\backslash H(a,b)$. We will show that $h \in B(f,\eta)$. If $x \in [y_0-\eta_1/2, y_0+\eta_1/2]$, according to the continuity of f, we have $|h(x)-f(x)| = |f(x)| < \eta/2$. If $x \in J_1$, then according to linearity of h, the number $h(x)$ is between numbers $f(y_0-\eta_1/2)$ and $f(y_0-\eta_1)$, by the continuity of f, we have $|h(x)| < \eta/2$. Hence for $x \in J_1$, $|f(x)-h(x)| < \eta/2+\eta/2 = \eta$. If $x \in J_2$ a similar argument can be used. Finally, if $x \in [a,b]\backslash[y_0-\eta_1, y_0+\eta_1]$, it is clear that $|h(x)-f(x)| = 0 < \eta$. This completes the proof that $h \in B(f,\eta)$, which implies the validity of (6).

Now we will show that the set $C(a,b)\backslash H(a,b)$ is not dense in $C(a,b)$. Choose $g \in C(a,b)$, such that $g(x) \geq \delta > 0$ for each $x \in [a,b]$. Consider $B(g,\delta/2)$. If $f \in B(g,\delta/2)$, then $Z(f) = \emptyset$ and it is a perfect, nowhere dense set. Hence $B(g,\delta/2) \subseteq H(a,b)$, which implies $\left[C(a,b)\backslash H(a,b)\right] \cap B(g,\delta/2) = \emptyset$. This shows that the set $C(a,b)\backslash H(a,b)$ is not dense in $C(a,b)$. □

Now we will give the estimation of σ-porosity of sets $A_0(a,b)$, $D_0(a,b)$ and $M_0(a,b)$ in $C_0(a,b)$, where $M_0(a,b)$ denotes the set of all $f \in C_0(a,b)$ for which $\operatorname{card} Z(f) \leq \aleph_0$.

THEOREM 2.5. *The set $A_0(a,b)$ is σ-very-$1/2$-porous in $C_0(a,b)$.*

Proof. In the proof of Theorem 2.1.(ii) we showed that $A_0(a,b) = \bigcup\limits_{n=1}^{\infty} A_n$, A_n were nowhere dense in $C_0(a,b)$. We distinguished two cases:

a) $g \notin A_n$,

b) $g \in A_n$ and found a ball B disjoint with the set A_n for all n.

Using this construction of the ball B let us restrict our considerations to $\gamma(g,\delta,A_n)$. Without loss of generality, in the case a) we can confine to such $\delta > 0$, that $\delta < \frac{|g(x_0)|}{2}$ and we get $\gamma(g,\delta,A_n) = \delta$. Thus $p(g,A_n) = 1$. In the case b) we get $\gamma(g,\delta,A_n) \geq \delta/2$ hence $\underline{p}(g,A_n) \geq 1/2$. □

THEOREM 2.6. *The set $D_0(a,b)$ is σ-very-$1/4$-porous in $C_0(a,b)$.*

Proof. In the proof of Theorem 2.1.(i) we showed that $D(a,b) = \bigcup\limits_{n=1}^{\infty} D_n$, D_n were nowhere dense in $C(a,b)$. We again distinguished two cases:

a) g has no zero in I_n,

b) g has a zero in I_n and we found a ball B_1 disjoint with the set D_n for all n.

Similarly we can show that $D_0(a,b) = \bigcup\limits_{n=1}^{\infty} D_n$, where D_n are a nowhere dense in $C_0(a,b)$. Using this and the construction of the ball B_1 in the space $C_0(a,b)$, we can count $\gamma(g,\delta,D_n)$. Again, in the case a) we can restrict our considerations to such $\delta > 0$, $\delta = \min\limits_{x \in I_n} |g(x)|$ and then $\gamma(g,\delta,D_n) = \delta$. Thus $p(g,D_n) = 1$. In the case b) we get $\gamma(g,\delta,D_n) \geq \delta/4$ hence $\underline{p}(g,D_n) \geq 1/4$. □

COROLLARY 2.7. *The set $M_0(a,b)$ is σ-very-$1/4$-porous in $C_0(a,b)$.*

Proof. Let $f \in M_0(a,b)$, then $\operatorname{card} Z(f) \leq \aleph_0$. Every closed countable set has an isolated point. There exist two rational points a_n, b_n such that the interval $I_n = [a_n, b_n]$ contains a unique zero of f. Thus $f \in D_n \subset D_0(a,b)$ and $M_0(a,b) \subset D_0(a,b)$. Using the previous Theorem 2.6 we have that the set $M_0(a,b)$ is σ-very-$1/4$-porous. □

Remark 2.8. Similar assertions as Theorem 2.5, Theorem 2.6 and Corollary 2.7 can be obtained for the space $C(a,b)$.

§3. The structure of the spaces $C(a,b)$ and $C_0(a,b)$ from the point of view of measure of the sets $Z(f)$

In [1; Theorem 1] it was shown that the set of functions $f \in C_0(a,b)$ for which $\lambda(Z(f)) = 0$ is a residual set in $C_0(a,b)$. We extend this result. We will consider only the space $C_0(a,b)$, for the space $C(a,b)$ it can be done analogically.

Throughout this section, we shall assume that μ is a measure defined on certain family F of subsets of the interval $[a,b]$ containing all open and closed subsets of $[a,b]$. The notion of measure we understand in the sense [3; p. 2, Definition 1]. It means $\mu(\emptyset) = 0$, μ is nonnegative, σ-subadditive, monotone set function and moreover we assume that for every $M \in F$ we have $\mu(M) = \inf\limits_{\substack{G \supseteq M \\ G \text{ is an open set}}} \mu(G)$ and $\mu(\{x\}) = 0$ for each $x \in [a,b]$.

THEOREM 3.1. *Let μ be a measure on F possessing the previous properties. Then the set $W_0(a,b)$ of functions f, $f \in C_0(a,b)$ for which $\mu(Z(f)) = 0$ is a residual set in the space $C_0(a,b)$.*

P r o o f . We use the same procedure as in [1]. Define $M_n = \{f \in C_0(a,b) : \mu(Z(f)) < 1/n\}$. We shall show that M_n is an open set in $C_0(a,b)$. Let $f \in M_n$. Since $\mu(Z(f)) < 1/n$ and according to the properties of the measure μ, there exists an open set G, $G \supseteq Z(f)$ such that $\mu(G) \le 1/n$. Put $\eta = \inf\{|f(x)| : x \in [a,b] \setminus G\}$. Since f is continuous and $[a,b]\setminus G$ is a closed set, we have $\eta > 0$. It is enough to show that $B_0(f,\eta) \subseteq M_n$. Let $h \in B_0(f,\eta)$. If $x \in Z(h)$, we have $|f(x)| = |f(x) - h(x)| \le \varrho(f,g) < \eta$. Thus $|f(x)| < \eta$ and according to the definition of η, the point x belongs to G, which implies $Z(h) \subseteq G$. From monotonicity of the measure μ we get $\mu(Z(h)) \le \mu(G) < 1/n$, therefore $h \in M_n$. Put $M = \bigcap\limits_{n=1}^{\infty} M_n$. Then M is a G_δ set and contains all polynomials from $C_0(a,b)$ (see [1; Lemma 1]). On the basis of the Weierstrass approximation theorem we have that M is dense in the complete metric space $C_0(a,b)$. Since a dense G_δ set is a residual set (see [2; p. 49]), we see that M is a residual subset of $C_0(a,b)$. If $h \in M$, then $\mu(Z(h)) < 1/n$ for all n, thus $\mu(Z(h)) = 0$. Hence $M \subseteq W_0(a,b)$, so we get that $W_0(a,b)$ is also a residual subset of $C_0(a,b)$. □

Consider Hausdorff measure defined by means of any function of class H_0 (see [3; pp. 50–51]), where H_0 is a class of all such functions $g\colon [0,\infty] \to [0,\infty]$, that $g(t) > 0$ for $t > 0$ and $\lim\limits_{t \to 0+} g(t) = 0 = g(0)$.

Hausdorff measures fulfill all conditions imposed on measure μ above, before Theorem 3.1. Using this and Theorem 3.1 we obtain two following corollaries.

COROLLARY 3.2. *If $g \in H_0$ and μ^g is a Hausdorff measure defined by g, then in the space $C_0(a,b)$ functions with $\mu^g(Z(f)) = 0$ are typical.*

In the special case, if $g(t) = t^\alpha$, $0 < \alpha < 1$ we have the following corollary:

COROLLARY 3.3. *In the space $C_0(a,b)$ functions with $\dim Z(f) = 0$ are typical.*

Remark 3.4. Results similar to Theorem 3.1 and Corollaries 3.2, 3.3 hold also for the space $C(a,b)$ (it suffices to write $C(a,b)$ instead of $C_0(a,b)$ in the theorem and the corollaries and it is possible to prove them by the same way that we used in the proof of Theorem 3.1).

Combining results of paper [1] with our results (Theorem 2.1 and Theorem 3.1) we get Theorem 3.5 and Theorem 3.7.

THEOREM 3.5. *In the space $C_0(a,b)$, those functions f are typical for which $Z(f)$ is a perfect and nowhere dense set, $\operatorname{card} Z(f) = c$, and $\dim Z(f) = 0$.*

Remark 3.6. According to the fact, that the set $C(a,b)\backslash C_0(a,b)$ is of the second Baire category, all results which hold for the space $C_0(a,b)$ cannot be transformed directly for the space $C(a,b)$. However we have the following theorem.

THEOREM 3.7. *In the space $C(a,b)$, functions f for which $Z(f)$ is a perfect and nowhere dense set and $\dim Z(f) = 0$ are typical.*

The above mentioned results can be transformed to describe the structure of spaces $C_P(a,b)$ and $C(a,b)$ by means of fixed points of functions, where $C_P(a,b)$ is the class of all continuous functions on the interval $[a,b]$ having at least one fixed point. Let $P(f)$ be a set of fixed points x of a function f, i.e. such $x \in [a,b]$ that $f(x) = x$. We give details only for $C(a,b)$.

We define a mapping F, $F\colon C(a,b) \to C(a,b)$ in the following way $F(f) = f - f_0$ for $f \in C(a,b)$, where $f_0(x) = x$ (identity function for $[a,b]$). If $x_0 \in [a,b]$ is a fixed point for a function f, then x_0 is zero for the function $F(f)$. Obviously that F is isometry on the space $C(a,b)$. For the space $C_P(a,b)$ we define a mapping F, $F\colon C_P(a,b) \to C_0(a,b)$ in the same way. From this we immediately have the validity of the following theorems.

THEOREM 3.8. *In the space $C_P(a,b)$, functions f for which $P(f)$ is a perfect and nowhere dense set, $\operatorname{card} P(f) = c$, and $\dim P(f) = 0$ are typical.*

THEOREM 3.9. *In the space $C(a,b)$, those functions f are typical for which $P(f)$ is a perfect and nowhere dense set and $\dim P(f) = 0$.*

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