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L^p-APPROXIMATION OF GENERALIZED BIAXIALLY SYMMETRIC POTENTIALS OVER CARATHÉODORY DOMAINS

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ABSTRACT. Let $F^{\alpha,\beta}$ be a real generalized biaxially symmetric potentials (GBASP) defined on the Carathéodory domain and let $L^p(D)$ be the class of functions $F^{\alpha,\beta}$ holomorphic in D such that $\|F^{\alpha,\beta}\|_{D,p} = \left(A^{-1}\iint\limits_D |F^{\alpha,\beta}|\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$, A is the area of the domain D. For $F^{\alpha,\beta} \in L^p(D)$, set $E_n^p(F^{\alpha,\beta}) = \inf\{\|F^{\alpha,\beta} - P^{\alpha,\beta}\|_{D,p} : P^{\alpha,\beta} \in H_n\}$, H_n consists of all even biaxially symmetric harmonic polynomials of degree at most $2n$. This paper deals with the growth of entire function GBASP in terms of approximation error in L^p-norm on D. The analysis utilizes the Bergman and Gilbert integral operator method to extend results from classical function theory on the best polynomial approximation of analytic functions of a complex variable.

1. Introduction

Let $F^{\alpha,\beta}$ be a real valued regular solution of the generalized biaxisymmetric potential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + \frac{2\beta+1}{y}\frac{\partial}{\partial y}\right)F^{\alpha,\beta} = 0\,, \qquad \alpha > \beta > -\frac{1}{2}\,, \tag{1.1}$$

where α, β are fixed in a neighbourhood of the origin and the analytic Cauchy data

$$F_x^{\alpha,\beta}(0,y) - F_y^{\alpha,\beta}(x,0) = 0$$

are satisfied along the singular lines in the open hypersphere $\Sigma_r^{\alpha,\beta}\colon x^2+y^2 < r^2$.

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Such functions with even harmonic extensions are referred to as generalized biaxisymmetric potentials (GBASP) having local expansion of the form

$$F^{\alpha,\beta}(x,y) = \sum_{n=0}^{\infty} a_n R_n^{\alpha,\beta}(x,y)$$

such that

$$R_n^{\alpha,\beta}(x,y) = \frac{(x^2+y^2)^n}{P_n^{\alpha,\beta}(1)} P_n^{\alpha,\beta}\left(\tfrac{x^2-y^2}{x^2+y^2}\right), \qquad n = 0,1,2,\dots, \tag{1.2}$$

where $P_n^{\alpha,\beta}$ are the Jacobi polynomials ([1], [12]). Suitable limits of the parameters (α,β), after quadratic transformation from [1] as necessary, produce various special functions from the $R_n^{\alpha,\beta}$. For example, $\alpha = \beta = 0$ gives the zonal harmonics so that $F^{\alpha,\beta}$ is interpreted as an axisymmetric potential on $\mathbb{R}^2$, and $\alpha = \beta = -1/2$ gives the even circular harmonics on $\mathbb{R}^2$, where the interpretation is $F^{\alpha,\beta} = \operatorname{Re} f$, f being real analytic. The Euler-Poisson-Darboux equation arising in gas dynamics is viewed in terms of equation (1.1) after a transformation [4; p. 223]. Thus, global properties characterizing solutions to this partial differential equation that are determined by local properties are of special interest.

The invertible integral operators $\kappa_{\alpha,\beta}$ and $\kappa_{\alpha,\beta}^{-1}$ developed in [9] are fundamental to such type of studies. These operators locally associate regular GBASP $F^{\alpha,\beta}$, equation (1.2) and the unique analytic function f: $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ as follows

$$F^{\alpha,\beta}(x,y) = \kappa_{\alpha,\beta}(f) = \int_0^1 \int_0^\pi f(\zeta)\mu_{\alpha,\beta}(t,s)\, \mathrm{d}t\, \mathrm{d}s\,,$$

$$\zeta^2 = x^2 - y^2t^2 - 2\,\mathrm{i}xyt\cos s\,,$$

$$\mu_{\alpha,\beta}(t,s) = \gamma_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin s)^{2\alpha}\,,$$

$$\gamma_{\alpha,\beta} = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}\,.$$

The inverse operator applies orthogonality of the *Jacobi polynomials* ([1]) and the *Poisson kernel* ([1]) to uniquely define the transform

$$f(z) = \kappa_{\alpha,\beta}^{-1}(F^{\alpha,\beta}) = \int_{-1}^{1} F^{\alpha,\beta}\big(r\xi, r\sqrt{1-\xi^2}\,\big)\nu_{\alpha,\beta}\big((z/r)^2,\xi\big)\, \mathrm{d}\xi\,,$$

$$\nu_{\alpha,\beta}(\tau,\xi) = S_{\alpha,\beta}(\tau,\xi)(1-\xi)^\alpha(1+\xi)^\beta\,,$$

where the kernel is written with the help of [1] in closed form as

$$S_{\alpha,\beta}(\tau,\xi) = \eta_{\alpha,\beta}\frac{1-\tau}{(1+\tau)^{\alpha+\beta+2}}F\left(\tfrac{\alpha+\beta+2}{2};\tfrac{\alpha+\beta+3}{2};\beta+1;\tfrac{2\tau(1+\xi)}{(1+\tau)^2}\right),$$
$$\eta_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

The normalizations $\kappa_{\alpha,\beta}(1) = \kappa_{\alpha,\beta}^{-1}(1)$ are taken. The kernel $S_{\alpha,\beta}(\tau,\xi)$ is analytic in $\|\tau\| < 1$ for $-1 \le \xi \le 1$. The local function elements $F^{\alpha,\beta}$ and f are continued harmonically and analytically by contour deformation using the envelope method ([3]).

Let B denote a Carathéodory domain, that is bounded simply connected domain, such that the boundary of B coincides with the boundary of the domain lying in the complement of the closure of B and containing the point ∞. In particular, a domain bounded by a Jordan curve is Carathéodory domain. Let $L^p(B)$ and $\ell^p(B)$, $1 \le p \le \infty$, denote the class of GBASP $F^{\alpha,\beta}$ and associate f holomorphic in B such that

$$\|F^{\alpha,\beta}\|_{B,p} = \left[\frac{1}{A}\iint_B |F^{\alpha,\beta}(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y\right]^{1/p} < \infty,$$
$$\|f\|_{B,p} = \left[\frac{1}{A}\iint_B |f(z)|^p \,\mathrm{d}x\,\mathrm{d}y\right]^{1/p} < \infty,$$

where these norms are understood to be $\sup_{(x,y)\in B}|F^{\alpha,\beta}(x,y)|$, $\sup_{z\in B}|f(z)|$ for $p=\infty$, and $\|\cdot\|_{B,p}$ denotes the L^p-norm and ℓ^p-norm for $F^{\alpha,\beta}$ and f, respectively and A is the area of domain B. For $f \in \ell^p(B)$, define b_n, $n = 0,1,2,\dots$, the Fourier coefficients as

$$b_n = \iint_B f(z)\overline{p_n(z)}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.3}$$

Also,

$$\delta_m^n = \iint_B p_n(z)\overline{p_m(z)}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\delta_m^n = 1$ for $m = n$ and $\delta_m^n = 0$ otherwise, and $\{p_n\}_{n=1}^{\infty}$ forms a complete orthonormal sequence of polynomials in $\ell^p(B)$, p_n being even polynomial of degree at most $2n$. It is known ([11; p. 273]) that $f \in \ell^p(B)$ is entire if and only if $\lim_{n\to\infty}|b_n|^{1/n} = 0$. Moreover, $f(z) = \sum_{n=0}^{\infty} b_n p_n(z)$ holds in the whole complex plane.

For $p=\infty$, the best polynomial approximation error for GBASP and its associate (see [5]) is defined as

$$e_n(f) \equiv e_n(f,B) = \inf\{\|f-\pi\| :\ \pi \in h_n\}\,, \qquad n = 0,1,\dots\,,$$
$$\|f-\pi\| = \sup_{z\in B} |f(z)-\pi(z)|\,.$$

Here, we define

$$E_n^p(F^{\alpha,\beta}) \equiv E_n^p(F^{\alpha,\beta},B) = \inf\left\{\|F^{\alpha,\beta}-P^{\alpha,\beta}\|_{B,p} :\ P^{\alpha,\beta}\in H_n\right\}, \qquad p>0\,,$$

and for $p=\infty$,

$$E_n^\infty(F^{\alpha,\beta}) = \|F^{\alpha,\beta}-P^{\alpha,\beta}\| = \sup_{(x,y)\in B} \left|F^{\alpha,\beta}(x,y)-P^{\alpha,\beta}(x,y)\right|.$$

The set h_n contains all even polynomials of degree at most $2n$, and the set H_n contains all even biaxisymmetric harmonic polynomials of degree $2n$. The operators $\kappa_{\alpha,\beta}$ and $\kappa_{\alpha,\beta}^{-1}$ establish one-one equivalence of the sets h_n and H_n.

Let L^0 denote the class of functions $\phi(x)$ defined on $[a,\infty)$, satisfying the conditions H(i) and H(ii):

H(i) $\phi(x)$ is positive, strictly increasing, differentiable, and $\phi(x)\to\infty$ as $x\to\infty$.

H(ii) $\lim\limits_{x\to\infty} \frac{\phi(x(1+\varphi(x))}{\phi(x)} = 1$ for every $\varphi(x)$ such that $\varphi(x)\to 0$ as $x\to\infty$.

Let Δ be the class of functions $\phi(x)$ satisfying conditions H(i) and H(iii):

H(iii) $\lim\limits_{x\to\infty} \frac{\phi(cx)}{\phi(x)} = 1$ for every $0<c<\infty$.

Let Ω be the class of functions $\phi(x)$ satisfying H(i) and H(iv):

H(iv) There exists a $\delta(x)\in\Delta$ and x_0, K_1, K_2 such that

$$0 < K_1 \le \frac{\mathrm{d}(\phi(x))}{\mathrm{d}(\delta(\ln x))} \le K_2 < \infty \qquad \text{for all} \quad x > x_0\,.$$

Also, let $\overline{\Omega}$ be the class of functions $\phi(x)$ satisfying H(i) and H(v):

H(v) $\lim\limits_{x\to\infty} \frac{\mathrm{d}(\phi(cx))}{\mathrm{d}(\ln x)} = K\,, \qquad 0<K<\infty$.

The generalized growth parameters of an entire function GBASP are defined as

$$\limsup_{r\to\infty} \frac{\alpha(\ln M(r,F^{\alpha,\beta}))}{\alpha(\ln r)} =: \rho(\alpha,\alpha,F^{\alpha,\beta})\,,$$
$$\liminf_{r\to\infty} \frac{\alpha(\ln M(r,F^{\alpha,\beta}))}{\alpha(\ln r)} =: \lambda(\alpha,\alpha,F^{\alpha,\beta})\,,$$

where $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$ and $M(r,F^{\alpha,\beta}) = \sup\limits_{(x,y)\in B} |F^{\alpha,\beta}(x,y)|$.

DEFINITION. An entire GBASP is said to be of *regular growth* if $1 < \lambda(\alpha, \alpha, F^{\alpha,\beta}) = \rho(\alpha, \alpha, F^{\alpha,\beta}) < \infty$.

Following the reasoning of McCoy [9], it can be shown that generalized orders of entire GBASP and its associate are the same. McCoy [9], [10] has characterized classical order and type of an entire GBASP in terms of approximation error in L^p-norm on $[-1,1]$.

In this paper, we extend the results of McCoy to arbitrary domains and generalized growth parameters. We identify those GBASP $F^{\alpha,\beta} \in L^p(B)$ that harmonically continue as an entire function GBASP. The characteristic feature follows from the rate of convergence of a sequence of best GBASP polynomial approximates to $F^{\alpha,\beta}$ in $L^p(B)$ and sup norms. The generalized growth parameters of an entire GBASP have been characterized in terms of the approximation error $E_n^p(F^{\alpha,\beta})$ in L^p and sup norms on Carathéodory domains.

The following notations will be used throughout the paper

$$\vartheta_\eta(\nu) = \begin{cases} \max\{1, \nu\} & \text{if } \alpha(x) \in \Omega\,, \\ \eta + \nu & \text{if } \alpha(x) \in \overline{\Omega}\,. \end{cases}$$

We shall write $\vartheta(\nu)$ for $\vartheta_1(\nu)$.

2. Auxiliary results

Let B^* be the component of the complement of the closure of the Carathéodory domain that contains the point ∞. Set $B_r = \{z : |\psi(z)| = r\}$, $r > 1$, where the function $w = \psi(z)$ maps B^* conformally onto $|w| > 1$ such that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$.

LEMMA 1. *Suppose $F^{\alpha,\beta}$ is an entire GBASP having generalized growth parameters $\rho(\alpha, \alpha, F^{\alpha,\beta})$ and $\lambda(\alpha, \alpha, F^{\alpha,\beta})$. Then*

$$\limsup_{r\to\infty} \frac{\alpha(\ln \overline{M}(r, F^{\alpha,\beta}))}{\alpha(\ln r)} = \rho(\alpha, \alpha, F^{\alpha,\beta})\,,$$
$$\liminf_{r\to\infty} \frac{\alpha(\ln \overline{M}(r, F^{\alpha,\beta}))}{\alpha(\ln r)} = \lambda(\alpha, \alpha, F^{\alpha,\beta})\,,$$

where $\overline{M}(r, F^{\alpha,\beta}) = \max_{z\in B_r} |F^{\alpha,\beta}(z,0)|$.

Proof. The proof follows on the lines of [6; Lemma 1] and taking the definition of generalized growth parameters into account. □

Lemma 2. *Let $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function GBASP having generalized growth parameters $\rho(\alpha, \alpha, F^{\alpha,\beta})$ and $\lambda(\alpha, \alpha, F^{\alpha,\beta})$. Then $g(z) = \sum_{n=0}^{\infty} b_n z^{2n}$, b_n are given by (1.3), is also an entire function satisfying*

$$\rho(\alpha, \alpha, F^{\alpha,\beta}) = \rho(\alpha, \alpha, g) \qquad \text{and} \qquad \lambda(\alpha, \alpha, F^{\alpha,\beta}) = \lambda(\alpha, \alpha, g)\,.$$

Proof. In view of $|b_n|^{1/n} \to 0$ as $n \to \infty$, g is an entire function. From [11], we have $\max_{z \in B_r} |p_n(z)| \le Cr^n$, $n = 0, 1, \dots$, where C is a constant independent of n and r $(r > 1)$. Thus, applying the Bernstein inequality ([2]) for each term of the series $\sum_{n=0}^{\infty} b_n p_n(z)$, we get

$$|f(z)| \le |b_0| + C \sum_{n=1}^{\infty} |b_n| (rr')^n\,, \qquad z \in B_r\,,$$

$$\overline{M}(r, f) \le |b_0| + CM(rr', g)\,, \qquad r > 1\,. \tag{2.1}$$

Let us consider the relation $F^{\alpha,\beta}(x, y) = \kappa_{\alpha,\beta}(f)$ defined globally in [10]. The nonnegativity and normalization of the measure lead directly to the bound

$$\overline{M}(r, F^{\alpha,\beta}) \le \overline{M}(r, f)\,. \tag{2.2}$$

In view of (2.1) and (2.2), we have

$$\overline{M}(r, F^{\alpha,\beta}) \le |b_0| + CM(rr', g)\,, \qquad r > 1\,. \tag{2.3}$$

Thus, using Lemma 1 and the fact that either $\alpha \in \Omega$ or $\alpha \in \overline{\Omega}$, (2.3) gives

$$\rho(\alpha, \alpha, F^{\alpha,\beta}) \le \rho(\alpha, \alpha, g) \qquad \text{and} \qquad \lambda(\alpha, \alpha, F^{\alpha,\beta}) \le \lambda(\alpha, \alpha, g)\,. \tag{2.4}$$

Fix $r^* > 1$. Since f is entire, it follows that ([9]) there exists a sequence of polynomials $\{Q_n\}_{n=1}^{\infty}$, Q_n being of degree at most $2n$ such that

$$|f(z) - Q_n(z)| < \frac{2}{3} \overline{M}(r, f) \frac{(r^*/r)^{n+1}}{(1 - r^*/r)}\,, \qquad z \in \overline{B}\,,$$

for all sufficiently large n and all $r > r^*$. Also,

$$b_n = \iint_B f(z) \overline{p_n(z)}\, \mathrm{d}x\, \mathrm{d}y = \iint_B [f(z) - Q_{n-1}(z)] \overline{p_n(z)}\, \mathrm{d}x\, \mathrm{d}y\,.$$

Since p_n is orthogonal to any polynomial of degree $\le 2n$, using the Schwarz inequality, we get

$$|b_n| \le \|f - Q_n\|_{B,p} \le A^{1/p} \max_{z \in \overline{B}} |f(z) - Q_n(z)|\,, \qquad 1 \le p < \infty\,,$$

where A is the area of B. Using (2.2) in above, it follows that

$$|b_n| \le \gamma \overline{M}(r,f) \left(\frac{r^*}{r}\right)^n \tag{2.5}$$

for large values of n and $r > 2r^*$, γ is a constant independent of n and r. Moreover, (2.5) gives

$$\mu(r/r^*; g) \le \gamma \overline{M}(r,f) \,. \tag{2.6}$$

The inverse relation $f(z) = \kappa_{\alpha,\beta}^{-1}(F^{\alpha,\beta})$, valid globally in view of [9; Theorem 1], leads to

$$|f(z)| \le M(r, F^{\alpha,\beta}) N_{\alpha,\beta}(\tau)\,, \qquad \tau = \left(\frac{z}{r}\right)^n ,$$

$$N_{\alpha,\beta}(\tau) = \max_{-1\le\xi\le1} \eta_{\alpha,\beta}^{-1} |S_{\alpha,\beta}(\tau,\xi)| \,.$$

However, for $z = \varepsilon r \,\mathrm{e}^{\mathrm{i}\theta}$ (ε real), $M(\varepsilon r, f) \le M(r, F^{\alpha,\beta}) N_{\alpha,\beta}$ implies

$$\overline{M}(r,f) \le \overline{M}(r/\varepsilon, F^{\alpha,\beta}) N_{\alpha,\beta}(\varepsilon^2)\,, \qquad z \in B_r \,. \tag{2.7}$$

Using [3; Theorem 3], Lemma 1, (2.6) and (2.7) and the fact that either $\alpha \in \Omega$ or $\alpha \in \overline{\Omega}$, we obtain

$$\rho(\alpha,\alpha,g) \le \rho(\alpha,\alpha,F^{\alpha,\beta}) \qquad \text{and} \qquad \lambda(\alpha,\alpha,g) \le \lambda(\alpha,\alpha,F^{\alpha,\beta})\,. \tag{2.8}$$

Combining (2.4) and (2.8), the desired results are available. For $p = \infty$, just proceed on the lines of [13]. □

Lemma 3. *Let $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function GBASP having generalized growth parameters $\rho(\alpha,\alpha,F^{\alpha,\beta})$ and $\lambda(\alpha,\alpha,F^{\alpha,\beta})$. Then $\widetilde{g}(z) = \sum\limits_{n=0}^{\infty} E_n^p(F^{\alpha,\beta}) z^{2n}$ is also an entire function. Further, we have*

$$\rho(\alpha,\alpha,F^{\alpha,\beta}) = \rho(\alpha,\alpha,\widetilde{g}) \qquad \textit{and} \qquad \lambda(\alpha,\alpha,F^{\alpha,\beta}) = \lambda(\alpha,\alpha,\widetilde{g})\,.$$

Proof. This is a direct consequence of [7; Lemma 3], and (2.2) and (2.8) for an even function. □

3. Main results

Theorem 1. *Let $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function GBASP having generalized order $\rho(\alpha,\alpha,F^{\alpha,\beta})$ and generalized lower order $\lambda(\alpha,\alpha,F^{\alpha,\beta})$. Then*

(i) $\rho(\alpha,\alpha,F^{\alpha,\beta}) = \vartheta(L)$,

(ii) $\rho(\alpha,\alpha,F^{\alpha,\beta}) = \vartheta(L^*)$, *where*

$$L = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left(-\frac{1}{n}\ln E_n^p(F^{\alpha,\beta})\right)}, \quad L^* = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left(\ln\left(E_{n-1}^p(F^{\alpha,\beta})/E_n^p(F^{\alpha,\beta})\right)\right)}.$$

(iii) $\lambda(\alpha,\alpha,F^{\alpha,\beta}) \ge \vartheta(\widetilde{\ell}\,)$,

$$\widetilde{\ell} = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\left(-\frac{1}{n}\ln E_n^p(F^{\alpha,\beta})\right)}.$$

(iv) *If we take* $\alpha(x) = \alpha(a)$ *on* $(-\infty,a)$, *then* $\lambda(\alpha,\alpha,F^{\alpha,\beta}) \ge \vartheta(\ell^*)$,

$$\ell^* = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\left(\ln\left(E_{n-1}^p(F^{\alpha,\beta})/E_n^p(F^{\alpha,\beta})\right)\right)}.$$

THEOREM 2. *Let* $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, *be the restriction to* B *of an entire function GBASP having generalized order* $\rho(\alpha,\alpha,F^{\alpha,\beta})$ *and generalized lower order* $\lambda(\alpha,\alpha,F^{\alpha,\beta})$. *If* $E_n^p(F^{\alpha,\beta})/E_{n+1}^p(F^{\alpha,\beta})$ *is nondecreasing, then*

(i) $\rho(\alpha,\alpha,F^{\alpha,\beta}) = \vartheta(L) = \vartheta(L^*)$,

(ii) $\lambda(\alpha,\alpha,F^{\alpha,\beta}) = \vartheta(\widetilde{\ell}\,) = \vartheta(\ell^*)$, *where* $\widetilde{\ell}$ *and* ℓ^* *have the same meaning as in Theorem* 1.

THEOREM 3. *Let* $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, *be the restriction to* B *of an entire function GBASP having generalized lower order* $\lambda(\alpha,\alpha,F^{\alpha,\beta})$. *Then*

(i) *if* $\alpha(x) \in \Omega$,

$$\lambda(\alpha,\alpha,F^{\alpha,\beta}) = \max_{\{n_k\}_{k=1}^\infty} \{\vartheta_\xi(\ell')\}, \tag{3.1}$$

(ii) *further, if* $\alpha(x) = \alpha(a)$ *on* $(-\infty,a)$,

$$\lambda(\alpha,\alpha,F^{\alpha,\beta}) = \max_{\{n_k\}_{k=1}^\infty} \{\vartheta_\xi(\ell'^*)\}, \tag{3.2}$$

where

$$\xi \equiv \xi(n_k) = \liminf_{k\to\infty} \alpha(n_{k-1})/\alpha(n_k),$$

$$\ell' \equiv \ell'(n_k) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\alpha\left(-\frac{1}{n_k}\ln E_{n_k}^p(F^{\alpha,\beta})\right)},$$

$$\ell'^* = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\alpha\left(-\frac{1}{n_k-n_{k-1}}\ln\left(E_{n_{k-1}}^p(F^{\alpha,\beta})/E_{n_k}^p(F^{\alpha,\beta})\right)\right)}.$$

The maximum in (3.1) *and* (3.2) *is taken over all increasing sequences* $\{n_k\}_{k=1}^\infty$ *of the positive integers.*

Also, if $\{n_m\}_{m=1}^{\infty}$ *is the sequence of principal indices of* $\widetilde{g}(z) = \sum_{n-0}^{\infty} E_n^p(F^{\alpha,\beta}) z^{2n}$ *and* $\alpha(n_m) \simeq \alpha(n_{m-1})$ *as* $m \to \infty$, *then* (3.1) *and* (3.2) *hold for* $\alpha(x) \in \overline{\Omega}$.

P r o o f o f T h e o r e m s 1, 2, 3 . These theorems follow easily from [5; Theorems 4 6, Lemma 1] and Lemma 3 of this paper. □

For $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, let $\{n_k\}_{k=1}^{\infty}$, $n_0 = 0$, be a sequence of positive integers such that $E_{n_{k-1}}^p(F^{\alpha,\beta}) > E_{n_k}^p(F^{\alpha,\beta})$ and

$$E_n^p(F^{\alpha,\beta}) = E_{n_{k-1}}^p(F^{\alpha,\beta}) \qquad \text{for} \quad n_{k-1} \le n \le n_k\,, \quad k = 1, 2, \dots . \tag{3.3}$$

Finally, we derive a result that shows how this sequence influences the growth of an entire GBASP.

THEOREM 4. *Let* $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, *be the restriction to* B *of an entire function GBASP having generalized order* $\rho(\alpha, \alpha, F^{\alpha,\beta})$ *and generalized lower order* $\lambda(\alpha, \alpha, F^{\alpha,\beta})$. *Then*

$$\lambda(\alpha, \alpha, F^{\alpha,\beta}) \ge \rho(\alpha, \alpha, F^{\alpha,\beta}) \lim_{k\to\infty} \frac{\alpha(n_k)}{\alpha(n_{k+1})}\,,$$

where $\{n_k\}_{k=1}^{\infty}$ *is defined by* (3.3).

P r o o f . Define the function

$$h(z) = \sum_{n=1}^{\infty} \left[E_{n-1}^p(F^{\alpha,\beta}) - E_n^p(F^{\alpha,\beta})\right] z^{2n} = \sum_{k=1}^{\infty} b_k z^{2k}\,,$$

where $b_k = E_{k-1}^p(F^{\alpha,\beta}) - E_k^p(F^{\alpha,\beta})$. Clearly, $h(z)$ has the generalized order $\rho(\alpha, \alpha, F^{\alpha,\beta})$ and generalized lower order $\lambda(\alpha, \alpha, F^{\alpha,\beta})$. Now, the application of [5; Theorem 4] to $h(z)$ yields the desired inequality. □

COROLLARY. *Let* $F^{\alpha,\beta} \in L^p(B)$, $1 \le p \le \infty$, *be the restriction to* B *of an entire GBASP with generalized regular growth. Further, if* $\alpha \in \Omega$ *or* $\alpha \in \overline{\Omega}$, *then*

$$\alpha(n_k) \simeq \alpha(n_{k-1}) \qquad \textit{as} \quad k \to \infty\,.$$

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