Jaroslav Ivančo

The Hadwiger number of complements of some graphs

Mathematica Slovaca, Vol. 47 (1997), No. 4, 393--403

Persistent URL: http://dml.cz/dmlcz/133144

Math. Slovaca, **47** (1997), No. 4, 393–403

Mathematica Slovaca

THE HADWIGER NUMBER OF COMPLEMENTS OF SOME GRAPHS

JAROSLAV IVANČO

(*Communicated by Martin Škoviera*)

ABSTRACT. The Hadwiger number of a graph G is the maximum size of a complete graph to which G can be contracted. We investigate the Hadwiger number of some graphs by the special structure of their complements. Determined here are the Hadwiger number for the Zykov sum of graphs, and the Hadwiger number for the complement of graphs without short circuits.

1. Introduction

In the present paper, we consider only finite undirected graphs without loops or multiple edges. Concepts and notation not defined in this paper will be used as in standard texts, for example [1].

Let G be a connected graph. A decomposition $\{V_1, \dots, V_m\}$ of its vertex set $V(G)$ into nonempty subsets with the following properties

(i) V_i induces a connected subgraph of G for all $i = 1, \dots, m$,
(ii) $V_i \cup V_j$ induces a connected subgraph of G for all $i = 1, \dots, m$ and all $j = 1, \dots, m$,

is called an *H-decomposition* of G.

The *Hadwiger number* $\eta(G)$ of a connected graph G is the maximum positive integer m such that there exists an H-decomposition of G into m subsets. The Hadwiger number of a disconnected graph is the maximum Hadwiger number of its components.

In [2], there are established bounds of $\eta(G)$ depending on $\omega(G)$ (i.e., the maximal number of vertices in a clique of G) and $\alpha_0(G)$ (i.e., the vertex covering number of G).

AMS Subject Classification (1991): Primary 05C35.
Key words: Hadwiger number, H-decomposition, complement of graph.

THEOREM 1. ([2]) *Let G be a graph. Then*

$$\omega(G) \le \eta(G) \le \min\left\{1+\alpha_0(G), \left\lfloor \frac{\omega(G)+|V(G)|}{2} \right\rfloor\right\}.$$

The vertices of an independent set of any graph G induce a clique of $\overline{G}$ (i.e., the complement of G), therefore $\omega(\overline{G})$ is equal to the vertex independence number $\beta_0(G)$ of the graph G. By Theorem 1 and the Gallai theorem (i.e., $\alpha_0(G)+\beta_0(G)=|V(G)|$), we get:

COROLLARY 1. *Let $\overline{G}$ be the complement of a graph G. Then*

$$|V(G)|-\alpha_0(G) \le \eta(\overline{G}) \le \min\left\{1+|V(G)|-\omega(G), |V(G)|-\left\lceil \frac{\alpha_0(G)}{2} \right\rceil\right\}.$$

Let K_n (P_{n+1}) denote the complete graph of order n (the path of length n), then we get following Corollary.

COROLLARY 2. *Suppose G is a graph such that $\alpha_0(G) \le 2$. Then the Hadwiger number of the complement of G satisfies:*

If $\alpha_0(G)=0$, then $\eta(\overline{G})=|V(G)|$.
If $\alpha_0(G)=1$, then $\eta(\overline{G})=|V(G)|-1$.
If $\alpha_0(G)=2$ and G contains no subgraph isomorphic to K_3 or P_4, then $\eta(\overline{G})=|V(G)|-1$.
If $\alpha_0(G)=2$ and G contains a subgraph isomorphic to K_3 or P_4, then $\eta(\overline{G})=|V(G)|-2$.

Proof. By Corollary 1, the assertions are evident for $\alpha_0(G)=0$ and $\alpha_0(G)=1$. Therefore, let us assume that $\alpha_0(G)=2$. Let $\{u,v\}$ denote a minimal vertex covering set of G. Now we consider the following three cases.

Case 1. If G contains no subgraph isomorphic to K_3 or P_4, then $uv \notin E(G)$, and each vertex of the independent set $V(G)-\{u,v\}=\{w_1,\dots,w_t\}$ is adjacent to at most one vertex of $\{u,v\}$. Thus $\{\{u,v\},\{w_1\},\dots,\{w_t\}\}$ is clearly an H-decomposition of $\overline{G}$, and so $\eta(\overline{G}) \ge |V(G)|-1$. The opposite inequality follows from Corollary 1.

Case 2. If G contains a subgraph isomorphic to K_3, then by Theorem 1, $\omega(G)=3$ (because $\alpha_0(G)=2$), and by Corollary 1, we get $\eta(\overline{G})=|V(G)|-2$.

Case 3. If G contains a subgraph isomorphic to P_4, then there exist edges xu, uv, vy in the graph G. Let G_1 be the subgraph of G such that $V(G_1)=V(G)$ and $E(G_1)=\{xu,uv,vy\}$. If a partition $\{V_1,\dots,V_r\}$ is an H-decomposition of $\overline{G_1}$, then the sets $\{u\}$, $\{v\}$ cannot both belong to this partition. Thus, without loss of generality, let $u \in V_1$ and $|V_1| \ge 2$. It can be easily seen that $\{V_1-\{u\}, V_2,\dots,V_r\}$ is an H-decomposition of $\overline{G_1}-u$, and so $\eta(\overline{G_1})=\eta(\overline{G_1}-u)$.

Since $\overline{G}$ is a subgraph of $\overline{G_1}$ and $\alpha_0(G_1 - u) = 1$, we have $\eta(\overline{G}) \le \eta(\overline{G_1}) = \eta(\overline{G_1} - u) = |V(G) - \{u\}| - 1 = |V(G)| - 2$. The opposite inequality again follows from Corollary 1. $\square$

2. The Hadwiger number of the Zykov sum of graphs

In general, the problem to determine $\eta(\overline{G})$ is difficult for a graph G when $\alpha_0(G) > 2$. Therefore, next we will study only graphs of some special types.

Let G be a disconnected graph. If $U \subseteq V(G)$ contains vertices from at least two components of G, then it can be easily seen that for each vertex $w \in V(G)$ the set $U \cup \{w\}$ induces a connected subgraph of $\overline{G}$. This fact can be useful for the determination of $\eta(\overline{G})$. In this section, we determine the Hadwiger number for the complement of a disconnected graph.

The *Zykov sum* (or *join*) $G_1 + G_2$ of disjoint graphs G_1 and G_2 is a graph obtained from G_1 and G_2 by joining each vertex of G_1 with every vertex of G_2 by an edge. It is clear that G is disconnected if and only if $\overline{G}$ is the Zykov sum of some of its disjoint subgraphs. In [7], B. Zelinka proved that

$$\eta(G_1 + G_2) \ge \eta(G_1) + \eta(G_2) \quad \text{and} \quad \eta(K_n + G) = n + \eta(G)\,.$$

The following assertion implies these results. However, first we remark that for a set U of vertices in a graph G, we denote by $\langle U \rangle$ the subgraph of G induced by U.

THEOREM 2. *Let G_1 and G_2 be two disjoint graphs, and suppose that $|V(G_1)| - \omega(G_1) \le |V(G_2)| - \omega(G_2)$. Then*

$$\eta(G_1 + G_2) = |V(G_1)| + \max\{\eta(\langle U \rangle) : U \subseteq V(G_2),\ |U| = |V(G_2)| - |V(G_1)| + \omega(G_1)\}\,.$$

Proof. Let P be a subset of $V(G_2)$ such that $|P| = |V(G_2)| - |V(G_1)| + \omega(G_1)$, $\eta(\langle P \rangle) = k = \max\{\eta(\langle T \rangle) : T \subseteq V(G_2),\ |T| = |P|\}$. Then there exists an H-decomposition $\{P_1, \dots, P_k\}$ of $\langle P \rangle$. Let $U = \{u_1, \dots, u_t\}$ be a subset of $V(G_1) = \{u_1, \dots, u_p\}$ such that $t = \omega(G_1)$ and $\langle U \rangle$ is a clique of G_1. Since $|V(G_2) - P| = |V(G_2)| - |P| = |V(G_1)| - \omega(G_1) = |V(G_1)| - |U| = |V(G_1) - U|$, there exists a bijective mapping $f \colon (V(G_1) - U) \to (V(G_2) - P)$. If $u \in (V(G_1) - U)$, then $\{u, f(u)\}$ induces a connected subgraph of $G_1 + G_2$. Now, it is clear that $\{\{u_1\}, \dots, \{u_t\}, \{u_{t+1}, f(u_{t+1})\}, \dots, \{u_p, f(u_p)\}, P_1, \dots, P_k\}$ is an H-decomposition of $G_1 + G_2$, and hence $\eta(G_1 + G_2) \ge p + k$. Thus $\eta(G_1 + G_2) \ge |V(G_1)| + \max\{\eta(\langle T \rangle) : T \subseteq V(G_2),\ |T| = |V(G_2)| - |V(G_1)| + \omega(G_1)\}$. Moreover, $\eta(G_1 + G_2) \ge |V(G_1)| + \omega(G_2)$ because $|P| \ge \omega(G_2)$, which implies that $\eta(\langle P \rangle)$ is at least $\omega(G_2)$.

For the proof of the opposite inequality, we define some invariants of an H-decomposition of $G_1 + G$ into r subsets, where $r = \eta(G_1 + G_2)$ and G is a subgraph of G_2 (note that $\eta(G_1 + G) = \eta(G_1 + G_2)$). For an H-decomposition $\mathcal{U} = \{U_1, \dots, U_r\}$ of $G_1 + G$ we put

$$\begin{aligned} p(\mathcal{U}) &= \left|\{i \in \{1, \dots, r\} : \ U_i \cap V(G_1) \neq \emptyset\}\right|, \\ q(\mathcal{U}) &= \left|\{i \in \{1, \dots, r\} : \ U_i \subseteq V(G_1)\}\right|, \\ s(\mathcal{U}) &= \left|\{v \in U_i \cap V(G_2) : \ U_i \cap V(G_1) \neq \emptyset\}\right|. \end{aligned}$$

Now, let us assume that $\mathcal{S} = \{S_1, \dots, S_r\}$ is an H-decomposition of $G_1 + G$ such that any H-decomposition $\mathcal{U} = \{U_1, \dots, U_r\}$ of $G_1 + G'$, where G' is any subgraph of G_2, satisfies exactly one of the following conditions

(1) $p(\mathcal{S}) > p(\mathcal{U})$,
(2) $p(\mathcal{S}) = p(\mathcal{U})$ and $q(\mathcal{S}) > q(\mathcal{U})$,
(3) $p(\mathcal{S}) = p(\mathcal{U})$, $q(\mathcal{S}) = q(\mathcal{U})$ and $s(\mathcal{S}) \leq s(\mathcal{U})$.

Suppose some class of the H-decomposition $\mathcal{S}$ (e.g., S_1) contains at least two vertices of $V(G_1)$. Then at least $1 + \omega(G_2)$ classes of $\mathcal{S}$ contain no vertex of G_1 (because $r = \eta(G_1 + G_2) \geq |V(G_1)| + \omega(G_2)$). Since the union of one element classes of an H-decomposition induces a clique, there exists a class of $\mathcal{S}$ (e.g., S_2) which contains at least two vertices of G_2 and no vertex of G_1. If $u \in S_1 \cap V(G_1)$ and $v \in S_2$, then it can be easily seen that $\mathcal{S}_1 = \{\{v\} \cup (S_1 - \{u\}), \{u\} \cup (S_2 - \{v\}), S_3, \dots, S_r\}$ is an H-decomposition of $G_1 + G$. However $p(\mathcal{S}_1) = p(\mathcal{S}) + 1$, which contradicts our assumptions. Hence we conclude that any class of $\mathcal{S}$ contains at most one vertex of G_1, and so $p(\mathcal{S}) = |V(G_1)|$. Therefore, without loss of generality we may assume that $|S_i \cap V(G_1)| = 1$ for $1 \leq i \leq p = p(\mathcal{S})$, $S_i \subseteq V(G_1)$ for $1 \leq i \leq q = q(\mathcal{S})$, and $S_i \subseteq V(G_2)$ for $p < i \leq r$.

Since $S_1 \cup S_2 \cup \dots \cup S_q$ induces a clique of G_1 of size q, then $q \leq \omega(G_1)$. If $q < \omega(G_1) = t$, and the set $U = \{u_1, \dots, u_t\} \subseteq V(G_1) = \{u_1, \dots, u_p\}$ induces a clique of G_1, then $\mathcal{S}_2 = \{\{u_1\}, \dots, \{u_t\}, \{u_{t+1}\} \cup (S_{t+1} \cap V(G_2)), \dots, \{u_p\} \cup (S_p \cap V(G_2)), S_{p+1}, \dots, S_r\}$ is an H-decomposition of $G_1 + (G - ((S_{q+1} \cup \dots \cup S_t) \cap V(G_2)))$. However, $q(\mathcal{S}_2) = \omega(G_1) > q(\mathcal{S})$ and $p(\mathcal{S}_2) = p(\mathcal{S})$, which is a contradiction. Thus we deduce that $q = \omega(G_1)$.

Suppose some class of the collection $S_{q+1}, \dots, S_p$ (e.g., S_p) contains at least two vertices of G_2, and $x \in S_p \cap V(G_2)$. Then $\mathcal{S}_3 = \{S_1, \dots, S_{p-1}, S_p - \{x\}, S_{p+1}, \dots, S_r\}$ is evidently an H-decomposition of $G_1 + (G - x)$. This is again a contradiction because $p(\mathcal{S}_3) = p(\mathcal{S})$, $q(\mathcal{S}_3) = q(\mathcal{S})$, and $s(\mathcal{S}_3) = s(\mathcal{S}) - 1$. Therefore $|S_i \cap V(G_2)| = 1$ for each $i = q+1, \dots, p$, and $|S_{p+1} \cup \dots \cup S_r| \leq |V(G_2)| - (p - q) = |V(G_2)| - |V(G_1)| + \omega(G_1)$. $\{S_{p+1}, \dots, S_r\}$ is an H-decomposition of the subgraph $\langle S_{p+1} \cup \dots \cup S_r \rangle$ of G_2, and so $r - p \leq \max\{\eta(\langle T \rangle) : T \subseteq V(G_2),\ |T| = |V(G_2)| - |V(G_1)| + \omega(G_1)\}$. Hence $\eta(G_1 + G_2) \leq |V(G_1)| +$

$\max\{\eta(\langle T\rangle) : T \subseteq V(G_2),\ |T| = |V(G_2)| - |V(G_1)| + \omega(G_1)\}$, which completes the proof. $\square$

The *complete k-partite graph* is a graph whose vertices can be partitioned into k classes $U_1, \dots, U_k$ such that two vertices are adjacent if and only if they belong to distinct classes. If $|U_i| = n_i$ for all $i = 1, \dots, k$, then the complete k-partite graph is denoted by $K(n_1, \dots, n_k)$. It can easily be seen that $\overline{K(n_1, \dots, n_k)} = \overline{K_{n_1}} + \overline{K_{n_2}} + \dots + \overline{K_{n_k}}$, and so the Hadwiger number of the complete multipartite graph can be determined by Theorem 2. Details are left to the reader (see also [2], where it is proved by Theorem 1).

COROLLARY 3. *Let $k \geq 2$, and $1 \leq n_1 \leq \dots \leq n_k$ be integers. Then the complete k-partite graph $K(n_1, \dots, n_k)$ satisfies:*

$$\eta(K(n_1, \dots, n_k)) = \min\left\{1 + n_1 + \dots + n_{k-1},\ \left\lfloor \frac{k + n_1 + \dots + n_k}{2} \right\rfloor\right\}.$$

V. G. Vizing [5] suggested the study of the function $\lambda_k(n)$ which denotes the maximal possible number of edges of a graph with n vertices and with the Hadwiger number k. A. A. Zykov [8] and B. Zelinka [6] proved that $\lambda_k(n) = (k-1)n - \binom{k}{2}$ for $k \leq 4$, $n \geq k$. The following theorem extends this result.

THEOREM 3. *Let n, k be two positive integers such that $\frac{3n-2}{4} \leq k < n$. Then*

$$\lambda_k(n) = 1 + 2k + \frac{n(n-5)}{2}.$$

Proof. The assumption $\frac{3n-2}{4} \leq k < n$ implies $2\big(2(n-k) - 1\big) \leq n$ and $n - k \geq 1$. Let K denote the complete $(2k-n+1)$-partite graph with $2(n-k)-1$ classes of cardinality 2 and $n - 2\big(2(n-k) - 1\big)$ classes of cardinality 1. By Corollary 3, we have $\eta(K) = k$. Also, $|E(K)| = 1 + 2k + \frac{n(n-5)}{2}$, and hence $\lambda_k(n) \geq 1 + 2k + \frac{n(n-5)}{2}$.

On the other hand, we will show that there does not exist a graph G with more than $1 + 2\eta(G) + \frac{|V(G)|(|V(G)|-5)}{2}$ edges. Proceeding by contradiction, suppose G to be a graph with the minimal possible number of vertices and simultaneously the maximal possible number of edges which has more than $1 + 2\eta(G) + \frac{|V(G)|(|V(G)|-5)}{2}$ edges. It is clear that G is not a complete graph, and so the minimum degree $\delta(G)$ is less than $|V(G)| - 1$. Let x, y be two non-adjacent vertices of G, where x has the degree $\delta(G)$. The graph G has the maximal number of edges, then the Hadwiger number of the graph G_1, which we obtain from G by adding the edge xy, is greater than $\eta(G) = k$. Therefore there exists an H-decomposition $\{V_1, \dots, V_{k+1}\}$ of some component of G_1. Without loss of generality, suppose $x \in V_{k+1}$. Evidently, $\{V_1, \dots, V_k\}$ is an H-decomposition of a subgraph of G which does not contain the vertex x. Hence the Hadwiger number of $G_2 = G - x$ is at least k. G_2 is a subgraph of G, and so $\eta(G_2) = \eta(G) = k$. The graph G_2 has fewer vertices than G, thus $|E(G)| - \delta(G) = |E(G_2)| \le 1 + 2k + \frac{|V(G_2)|(|V(G_2)|-5)}{2} = 1 + 2k + \frac{|V(G)|(|V(G)|-5)}{2} - (|V(G)| - 3)$. Therefore $\delta(G)$ is at least $|V(G)| - 2$, and since G is not a complete graph, $\delta(G) = |V(G)| - 2$. This means that G is a complete multipartite graph every part of which has at most two vertices. Let r denote the number of parts of cardinality two. Then by Corollary 3, we have $k = \eta(G) = |V(G)| - \lceil \frac{r}{2} \rceil$ and $|E(G)| = \frac{|V(G)|(|V(G)|-1)}{2} - r \le \frac{|V(G)|(|V(G)|-1)}{2} - 2\lceil \frac{r}{2} \rceil + 1 = \frac{|V(G)|(|V(G)|-5)}{2} + 2|V(G)| - 2\lceil \frac{r}{2} \rceil + 1 = 1 + 2k + \frac{|V(G)|(|V(G)|-5)}{2}$. This is a contradiction to our assumption, which completes the proof. □

3. The Hadwiger number of complements of graphs without short circuits

For conciseness, we will denote by $\mathcal{G}_7$ the family of graphs which contain no circuit of length less than 7. In this section, we determine the Hadwiger number of complements of graphs which belong to $\mathcal{G}_7$. First, we prove the following assertion, which we shall use in the next.

PROPOSITION 1. *Suppose $G \in \mathcal{G}_7$. Let P be a set of vertices of G such that the distance between any pair of vertices of P is at most two. Then there exists a vertex $w \in V(G)$ which is adjacent to each vertex of $P - \{w\}$.*

Proof. We prove the assertion by induction on the cardinality of P. For $|P| = 1$ and $|P| = 2$ the assertion is obvious. Assume $|P| \ge 3$. Let x, u, v be distinct vertices of P. By the induction hypothesis, there exists a vertex $w \in V(G)$ which is adjacent to each vertex of $(P - \{x\}) - \{w\}$. As the distance between x and u (v) is at most two, there exists a vertex a (b) ($a = u$ ($b = v$) is also allowed) such that the set $\{x, a, u\}$ ($\{x, b, v\}$, respectively) induces a path. If $x \ne w$ and x is not adjacent to w, then the subgraph of G induced by $\{w, u, a, x, b, v\}$ contains a circuit, which contradicts $G \in \mathcal{G}_7$. Therefore, either $x = w$ or x is adjacent to w, which completes the proof. □

LEMMA 1. *Suppose $G \in \mathcal{G}_7$. Then $\eta(\overline{G}) \ge |V(G)| - \left\lceil \frac{\alpha_0(G)}{2} \right\rceil - 1$.*

Proof. Proceeding by contradiction, let us assume that $G \in \mathcal{G}_7$ is a graph with the minimal possible vertex covering number, and the Hadwiger number of its complement less than $|V(G)| - \lceil \frac{\alpha_0(G)}{2} \rceil - 1$. Then $\alpha_0(G) > 3$ because $\alpha_0(G) \le \lceil \frac{\alpha_0(G)}{2} \rceil + 1$, and by Corollary 1, we get $\eta(\overline{G}) \ge |V(G)| - \alpha_0(G) \ge$

$|V(G)| - \lceil\frac{\alpha_0(G)}{2}\rceil - 1$, otherwise. Let P be a minimal vertex covering set of G, i.e., $|P| = \alpha_0(G) > 3$. If the distance between some pair of vertices $u, v \in P$ is at least 3 in G, then $\{u, v\}$ induces a connected subgraph of $\overline{G}$, and for every vertex $w \in V(G) - \{u, v\}$ there exists an edge of $\overline{G}$ which joins w with u or v. Thus $\{\{u, v\}, U_1, \dots, U_k\}$ is an H-decomposition of $\overline{G}$ if $\{U_1, \dots, U_k\}$ is any H-decomposition of $\overline{G} - \{u, v\}$. This implies $\eta(\overline{G}) \geq 1 + \eta(\overline{G} - \{u, v\})$. Evidently, the graph $G_1 = G - \{u, v\} \in \mathcal{G}_7$, $\alpha_0(G_1) = \alpha_0(G) - 2$ and $\overline{G}_1 = \overline{G} - \{u, v\}$. Since G has the minimal vertex covering number, $\eta(\overline{G}_1) \geq |V(G_1)| - \lceil\frac{\alpha_0(G_1)}{2}\rceil - 1 = |V(G)| - \lceil\frac{\alpha_0(G)}{2}\rceil - 2$, and so $\eta(\overline{G}) \geq 1 + \eta(\overline{G}_1) \geq |V(G)| - \lceil\frac{\alpha_0(G)}{2}\rceil - 1$. This contradicts our assumption, and thus the distance between any pair of vertices of P is at most two in G.

Now, by Proposition 1, there exists a vertex $w \in V(G)$ which is adjacent to each vertex of $P - \{w\}$. Each vertex of $V(G) - (P \cup \{w\})$ is adjacent to at most one vertex of P, and so every nontrivial component of $G_2 = G - w$ is a star whose central vertex belongs to P. If $P - \{w\} = P_1 \cup \dots \cup P_k$, where $k = \lfloor\frac{|P - \{w\}|}{2}\rfloor$ and $|P_i| \geq 2$ for all $i = 1, \dots, k$, (evidently, such sets exist) and $V(G) - (P \cup \{w\}) = \{v_1, \dots, v_t\}$, then it is clear that $\{P_1, \dots, P_k, \{v_1\}, \dots, \{v_t\}\}$ is an H-decomposition of $\overline{G}_2 = \overline{G} - w$. Therefore $\eta(\overline{G}) \geq \eta(\overline{G}_2) \geq t + k = |V(G) - (P \cup \{w\})| + \lfloor\frac{|P - \{w\}|}{2}\rfloor$. If w belongs to P, then $|V(G) - (P \cup \{w\})| = |V(G)| - |P| = |V(G)| - \alpha_0(G)$ and $\lfloor\frac{|P - \{w\}|}{2}\rfloor = \lfloor\frac{(\alpha_0(G) - 1)}{2}\rfloor \geq \lfloor\frac{\alpha_0(G)}{2}\rfloor - 1$. If w is not an element of P, then $|V(G) - (P \cup \{w\})| = |V(G)| - |P| - 1 = |V(G)| - \alpha_0(G) - 1$ and $\lfloor\frac{|P - \{w\}|}{2}\rfloor = \lfloor\frac{|P|}{2}\rfloor = \lfloor\frac{\alpha_0(G)}{2}\rfloor$. Thus $\eta(\overline{G}) \geq |V(G)| - \alpha_0(G) + \lfloor\frac{\alpha_0(G)}{2}\rfloor - 1 = |V(G)| - \lceil\frac{\alpha_0(G)}{2}\rceil - 1$. This is again a contradiction to our assumption, which completes the proof. □

Let k be a positive integer. We denote by T_k a graph (tree) with the vertex set $\{w, u_1, \dots, u_k, v_1, \dots, v_k\}$ and the edge set $\{wu_1, \dots, wu_k, u_1v_1, \dots, u_kv_k\}$. The vertex w is called a *central* vertex of T_k. It can be easily seen that $|V(T_k)| = 2k+1$, $\alpha_0(T_k) = k$, and thus, by Corollary 1 and Lemma 1, we have $2k - \lceil\frac{k}{2}\rceil \leq \eta(\overline{T}_k) \leq 2k - \lceil\frac{k}{2}\rceil + 1$.

LEMMA 2. *Let $k \geq 3$ be an integer. Then $\eta(\overline{T}_k) = 2k - \lceil\frac{k}{2}\rceil$.*

Proof. Let $\mathcal{U} = \{U_1, \dots, U_t\}$ (where $t = \eta(\overline{T}_k)$) be an H-decomposition of the complement of T_k. Without loss of generality, we may assume that $w \in U_1$. If $|U_1| = 1$, then $U_i - \{u_1, \dots, u_k\} \neq \emptyset$ for all $i = 1, \dots, t$, and so $t \leq k+1$. This implies $t < 2k + 1 - \lceil\frac{k}{2}\rceil$ because $k \geq 3$. If $|U_1| > 1$, then it can be easily seen that $\{U_1 - \{w\}, U_2, \dots, U_t\}$ is an H-decomposition of $\overline{T}_k - w$. By Corollary 1, we get $t \leq \eta(\overline{T}_k - w) \leq 2k - \lceil\frac{k}{2}\rceil$.

The opposite inequality follows from Lemma 1. □

THEOREM 4. *Suppose $G \in \mathcal{G}_7$ and $k = \alpha_0(G)$. If k is odd, then*

$$\eta(\overline{G}) = \begin{cases} |V(G)| - \lceil \frac{k}{2} \rceil - 1 & \text{if } 3 \le k \text{ and } T_k \text{ is a subgraph of } G, \\ |V(G)| - \lceil \frac{k}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof. For $k = 1$, the assertion follows from Corollary 2.

Suppose $k \ge 3$ and T_k is a subgraph of G. Then $\overline{G}$ is a subgraph of $K_n + \overline{T}_k$, where $n = |V(G)| - 2k - 1$. By Lemmas 1, 2 and Theorem 2, we get $|V(G)| - \lceil \frac{k}{2} \rceil - 1 \le \eta(\overline{G}) \le \eta(K_n + \overline{T}_k) = n + \eta(\overline{T}_k) = |V(G)| - \lceil \frac{k}{2} \rceil - 1$, and so $\eta(\overline{G}) = |V(G)| - \lceil \frac{k}{2} \rceil - 1$.

Suppose $k \ge 3$ and G contains no subgraph isomorphic to T_k. By Corollary 1 and Lemma 1, we have $|V(G)| - \lceil \frac{k}{2} \rceil - 1 \le \eta(\overline{G}) \le |V(G)| - \lceil \frac{k}{2} \rceil$. Next, let us assume by way of contradiction that G is a graph with the minimal possible vertex covering number k, satisfying: $G \in \mathcal{G}_7$, k is odd, $k \ge 3$, $\eta(\overline{G}) = |V(G)| - \lceil \frac{k}{2} \rceil - 1$, and G contains no subgraph isomorphic to T_k. Let P be a minimal vertex covering set of G, i.e., $|P| = k$. Without loss of generality, we may assume that P contains a vertex x of degree one only if x is adjacent to a vertex also of degree one. Now suppose that A is a maximal subset of P such that the distance between any pair of its vertices is at most two in the graph G. By Proposition 1, there exists a vertex $w \in V(G)$ which is adjacent to each vertex of $A - \{w\}$. Moreover, each vertex of A is adjacent to some vertex from the set $V(G) - (P \cup \{w\})$ because $A \subseteq P$ cannot contain a vertex of degree one by our assumption. Therefore G contains a subgraph isomorphic to T_t, where either $t = |A|$ (if $w \notin A$) or $t = |A| - 1$ (if $w \in A$).

If $|A| = k$, then $t = |A| - 1 = k - 1$ because G contains no subgraph isomorphic to T_k. Thus $w \in A = P$ and each nontrivial component of $G - w$ is a star. As in the proof of Lemma 1, the graph $\overline{G} - w$ has an H-decomposition into $|V(G - w)| - \lceil \frac{\alpha_0(G-w)}{2} \rceil$ subsets. Since k is odd, $\lceil \frac{k}{2} \rceil = 1 + \lceil \frac{k-1}{2} \rceil = 1 + \lceil \frac{\alpha_0(G-w)}{2} \rceil$. Therefore $\eta(\overline{G}) \ge \eta(\overline{G} - w) \ge |V(G)| - 1 - \lceil \frac{\alpha_0(G-w)}{2} \rceil = |V(G)| - \lceil \frac{k}{2} \rceil$, which contradicts our choice of G.

If $|A| < k$, then there exist vertices $x \in P - A$ and $y \in A$ such that their distance is at least 3. As in the proof of Lemma 1, we get $\eta(\overline{G}) \ge 1 + \eta(\overline{G}_1)$, where $G_1 = G - \{x, y\}$ and $\alpha_0(G_1) = k - 2$. If $\eta(\overline{G}_1) = |V(G_1)| - \lceil \frac{\alpha_0(G_1)}{2} \rceil$, then $\eta(\overline{G}) \ge 1 + |V(G_1)| - \lceil \frac{\alpha_0(G_1)}{2} \rceil = 1 + |V(G)| - 2 - \lceil \frac{k-2}{2} \rceil = |V(G)| - \lceil \frac{k}{2} \rceil$, a contradiction. If $\eta(\overline{G}_1) = |V(G_1)| - \lceil \frac{\alpha_0(G_1)}{2} \rceil - 1$, then G_1 contains a subgraph isomorphic to T_{k-2} ($k-2 \ge 3$) because G is a counter-example with the minimal vertex covering number. Hence $|A| = k-1$ and $w \notin A$. However, each nontrivial component of $G_2 = G - \{x, w\}$ is a star, and as in the proof of Lemma 1, there exists an H-decomposition $\{U_1, \dots, U_r\}$ of $\overline{G}_2$, where $r = |V(G_2)| - \lceil \frac{\alpha_0(G_2)}{2} \rceil$ and $|U_i \cap P|$ is equal to either 0 or 2 for all $i = 1, \dots, r$. Since $xw \notin E(G)$ and the vertex x is adjacent to at most one vertex of A, $\{\{x,w\},U_1,\dots,U_r\}$ is an H-decomposition of $\overline{G}$. Therefore $\eta(\overline{G}) \ge 1+r = 1+|V(G_2)| - \lceil\frac{\alpha_0(G_2)}{2}\rceil = |V(G)|-1-\lceil\frac{k-1}{2}\rceil = |V(G)|-\lceil\frac{k}{2}\rceil$. This again contradicts our assumption. The proof is complete. □

Let G be a graph, and P be a subset of its vertex set. By $M(G,P)$ we will denote a graph with the vertex set P, and two vertices u, v are adjacent in $M(G,P)$ if and only if either u, v are adjacent in G, or there exists a vertex $w \in V(G)-P$ such that uw and wv are edges of G. Note that K_s $(s \ge 3)$ in $M(G,P)$ enforces a star $K_{1,s}$ with central vertex in $V(G)-P$ or a circuit of length less than 7 in G.

LEMMA 3. *Let G be a graph without 4-circuits whose vertex covering number $\alpha_0(G)$ is even. Then $\eta(\overline{G}) = |V(G)| - \frac{\alpha_0(G)}{2}$ if and only if there exists a vertex covering set P such that $|P| = \alpha_0(G)$, and the complement of $M(G,P)$ has a matching.*

Proof. Suppose $\eta(\overline{G}) = |V(G)| - \frac{\alpha_0(G)}{2}$, and let $\{U_1,\dots,U_t\}$ denote an H-decomposition of $\overline{G}$, where $t = |V(G)| - \frac{\alpha_0(G)}{2}$. Without loss of generality, we may assume that $|U_i| \ge 2$ for $i=1,\dots,r$, and $|U_j| = 1$ for $j = r+1,\dots,t$, $1 \le r \le t$. As $U_{r+1}\cup\dots\cup U_t = Q$ induces a complete subgraph of $\overline{G}$, $U_1 \cup \dots \cup U_r = P$ is a vertex covering set of G, and so $|P| \ge \alpha_0(G)$. Evidently $r \le \frac{|P|}{2}$, and the equality is true only if $|U_i| = 2$ for all $i=1,\dots,r$. Therefore $|P|+|Q|-\frac{\alpha_0(G)}{2} = |V(G)| - \frac{\alpha_0(G)}{2} = t = r+(t-r) \le \frac{|P|}{2}+|Q|$. This implies $|P| \le \alpha_0(G)$. Since the opposite inequality is also true, we have $|P| = \alpha_0(G)$ and $|U_i| = 2$ (i.e. $U_i = \{u_i,v_i\}$) for all $i=1,\dots,r$. As U_i and $U_i\cup\{w\}$, for every vertex $w \in Q = V(G)-P$, induce connected subgraphs of $\overline{G}$, u_iv_i is an edge in $\overline{M(G,P)}$. Thus $\{u_1v_1,\dots,u_rv_r\}$ is a matching of $\overline{M(G,P)}$.

On the other hand, let P be a vertex covering set of G such that $|P| = \alpha_0(G)$ and $\overline{M(G,P)}$ has a matching $\{u_1v_1,\dots,u_kv_k\}$. By the definition of $M(G,P)$, it is clear that the sets $\{u_i,v_i\}$ and $\{u_i,v_i,w_j\}$ (where $\{w_1,\dots,w_t\} = V(G)-P$) induce connected subgraphs of $\overline{G}$ for all $i=1,\dots,k$ and all $j=1,\dots,t$. Similarly, the set $\{u_i,v_i,u_j,v_j\}$ induces a connected subgraph of $\overline{G}$ for all $i,j = 1,\dots,k$, $i \ne j$, because G contains no 4-circuit. Thus $\{\{u_1,v_1\},\dots,\{u_k,v_k\},\{w_1\},\dots,\{w_t\}\}$ is an H-decomposition of $\overline{G}$, and so $\eta(\overline{G}) \ge k+t = \frac{|P|}{2} + |V(G)-P| = |V(G)| - \frac{|P|}{2} = |V(G)| - \frac{\alpha_0(G)}{2}$. The opposite inequality follows from Corollary 1. □

Let $k \ge 4$ be an even integer. Denote by:

$\mathcal{A}_k$ the family of trees T, which contain a vertex w such that $1+\alpha_0(T-w) = \alpha_0(T) = k$, and each component of $T-w$ is isomorphic to either K_2 or T_t, and moreover, in the second case, w is adjacent to the central vertex of T_t in T;

$\mathcal{B}_k$ the family of graphs which can be constructed from the complete bipartite graph $K_{t,k-t}$ (where t is an odd integer, $3 \le t \le k-3$) by replacing at least $t(k-t)-1$ of its edges by paths of length two;

$\mathcal{C}_k$ the family of graphs which are the union of $T_{1+\frac{k}{2}}$ and a graph with the vertex covering number $\frac{k}{2}-1$ and no circuit of length less than 7.

Finally, let $\mathcal{D}_k = \mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{C}_k \cup \{T_{k-1}\}$. By Lemma 3, it can be easily seen that if $G \in \mathcal{D}_k$, then $\eta(\overline{G}) = |V(G)| - 1 - \frac{k}{2}$.

THEOREM 5. *Suppose $G \in \mathcal{G}_7$ and $k = \alpha_0(G)$. If k is even, then*

$$\eta(\overline{G}) = \begin{cases} |V(G)| - \frac{k}{2} - 1 & \text{if } k=2 \text{ and } P_4 \text{ is a subgraph of } G, \\ & \text{or } k \ge 4 \text{ and } G \text{ contains a subgraph} \\ & \text{which belongs to } \mathcal{D}_k, \\ |V(G)| - \frac{k}{2} & \text{otherwise.} \end{cases}$$

Proof. Since $G \in \mathcal{G}_7$ is a graph with vertex covering number k, then, by Corollary 1 and Lemma 1, we have $|V(G)| - 1 - \frac{k}{2} \le \eta(\overline{G}) \le |V(G)| - \frac{k}{2}$. Let us assume that $\eta(\overline{G}) = |V(G)| - 1 - \frac{k}{2}$, $k \ge 4$ and $|P| = k$, where $P \subseteq V(G)$ is a vertex covering set of G such that P contains a vertex x of degree one only if x is adjacent to a vertex also of degree one. By Lemma 3, the complement of $M(G,P)$ has no matching, and, by a well-known result of Tutte's [4] (also [1]), there is a set $S \subseteq P$ such that the number of odd components of $\overline{M(G,P)} - S$ exceeds $|S|$.

If $\overline{M(G,P)} - S$ has at least 3 components, then they have the cardinality one (because G contains no short circuits), and there exists a vertex $w \in V(G) - P$ which is adjacent to each vertex of $P - S$. Now, it can be easily seen that G contains a subgraph G' isomorphic to $T_{1+\frac{k}{2}}$, and a subgraph of G induced by $V(G) - V(G')$ has the vertex covering number $\frac{k}{2} - 1$, i.e., G contains a subgraph which belongs to $\mathcal{C}_k$.

If $\overline{M(G,P)} - S$ has less than 3 components, then $S = \emptyset$ and $\overline{M(G,P)}$ consists of two components with odd cardinalities t and $|P| - t$. Evidently, if $3 \le t \le |P| - t$, then G contains a subgraph which belongs to $\mathcal{B}_k$, and if $t = 1$, then G contains a subgraph isomorphic to T_{k-1} or a subgraph which belongs to $\mathcal{A}_k$.

Other cases are obvious. □

In [3], M. Stiebitz studied the function $f(n,k)$ which denotes the minimal possible Hadwiger number of the complement of a graph with n vertices and with Hadwiger number k. It was proved that $f(n,k) \ge \frac{k+1}{2k} n - \frac{k+1}{2}$ for $2 \le k \le 3$. By Theorems 4 and 5, we easily get:

COROLLARY 4. *Let n, t, r be integers such that $4 \le r \le 7$, $t \ge 0$ and $n = 4t + r$. Then*

$$f(n,2) = 3t + \left\lfloor \tfrac{2r}{3} \right\rfloor \text{ and}$$
$$f(n,k) = 2, \text{ for each } k,\ n > k \ge 3t + \left\lfloor \tfrac{2r}{3} \right\rfloor .$$

REFERENCES

[1] HARARY, F.: *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.

[2] IVANČO, J.: *Some results on the Hadwiger numbers of graphs*, Math. Slovaca **38** (1988), 221–226.

[3] STIEBITZ, M.: *On Hadwiger's number–A problem of the Nordhaus-Gaddum type*, Discrete Math. **101** (1992), 307–317.

[4] TUTTE, W. T.: *The factorizations of linear graphs*, J. London Math. Soc. **22** (1947), 107–111.

[5] VIZING, V. G.: *Some unsolved problems of graph theory*, Uspekhi Mat. Nauk **23** (1968), 117–134. (Russian)

[6] ZELINKA, B.: *On some graph-theoretical problems of V. G. Vizing*, Časopis Pěst. Mat. **98** (1973), 56–66.

[7] ZELINKA, B.: *Hadwiger numbers of finite graphs*, Math. Slovaca **26** (1976), 23–30.

[8] ZYKOV, A. A.: *On the number of edges of a graph with Hadwiger number at most* 4, Prikl. Mat. i Programmirovanie **7** (1972), 52–55. (Russian)

Received June 28, 1993
Revised April 25, 1995

Department of Geometry and Algebra
P. J. Šafárik University
Jesenná 5
SK-041 54 Košice
SLOVAKIA

E-mail: ivanco@duro.upjs.sk