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ON MAGIC GRAPHS

JIŘÍ SEDLÁČEK

In this paper we consider finite undirected graphs without loops and multiple edges only. We write $G=[V, E]$ if the graph G has the vertex set V and the edge set E. In [9] we were inspired by the well-known concept of a magic square to introduce the so called magic graphs. Detailed description of these graphs was given by B. M. Stewart ([13] and [15]) and there also exist further papers dealing with magic graphs — see e.g. [2, 3, 7, 12, 16].

We describe a connected graph G (with $|E|\geqq 2$) as being *pseudo-magic* if and only if there exists a real-valued function α on the edges of G with the property that (i) distinct edges have distinct values assigned, and (ii) the sum of values assigned to all edges incident to a given vertex x is the same for all vertices of G. Let us denote the constant vertex sum by $\sigma(\alpha)$, or shortly by σ. From Appendix 1 in [6] we can conclude that there is no pseudo-magic graph $G=[V, E]$ with $|E|\leqq 4$ and there exist 11 pseudo-magic graphs on five vertices.

We describe a pseudo-magic graph G as being *magic* if and only if there exists an α with $\alpha(e)>0$ for every edge e in G. If α satisfies the property of "positiveness" is also called magic. We say that G is *super-magic* if and only if there exists a magic α so that the set $\{\alpha(e)|e\in E\}$ consists of consecutive integers; we agree that α is also called super-magic. It is easy to see that the classic concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $\langle n, n\rangle$ is super-magic for $n\geqq 3$ (see also [13]).

In the theory of numbers (cf. [11], p. 433—438) magic squares in a more general sense consisting of primes are known. This gave us in [9] an impulse to investigate magic graphs in which each $\alpha(e)$ is a prime number. In Stewart's terminology these graphs are called *prime-magic*. Let us mention that there also exist papers dealing with magic graphs in a slightly different sense. As shown in [8], p. 156—157, the idea of geometric structures with a "magic" valuation have already been studied independently on the graph theory and comprehensive references can be found in [1].

R. L. Guy and F. Harary in [4] and [5] consider the graph M_n (also called the Möbius ladder) and define it as follows: If $n=2m\geqq 6$ then M_n consists of a circuit C_n of length n and the $\frac{1}{2}n$ chords joining opposite pairs of vertices of C_n (see also

[17], p. 165). It can be shown that M_{2m} can be obtained by completing the graph in Fig. 1 by edges u_1v_m, u_mv_1. If $n=2m+1 \geqq 5$ then M_n is defined as the graph consisting of an n-gon C_n together with two chords at each vertex joining it to the two most opposite vertices of C_n. It is easy to see that M_{2m+1} is obtainable from the graph in Fig. 2 by adding the edges u_1v_{m+1}, u_mv_1, v_1v_{m+1}. R. K. Guy and F. Harary point out that M_5 and M_6 are the two Kuratowski graphs $\langle 5 \rangle$ and $\langle 3{,}3 \rangle$ (i.e. the

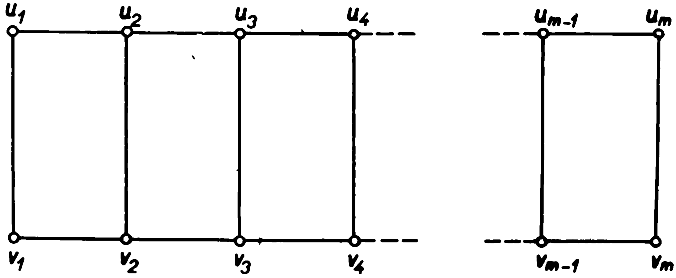

Fig. 1

complete graph with five vertices and the complete bipartite graph on two sets of three vertices each). They also show that every Möbius ladder is minimally nonplanar (i.e. its crossing number is one). In [10] we found that the number of trees spanning the graph M_{2m} can be expressed by a relatively simple formula. In this paper we approach Möbius ladders in a different way.

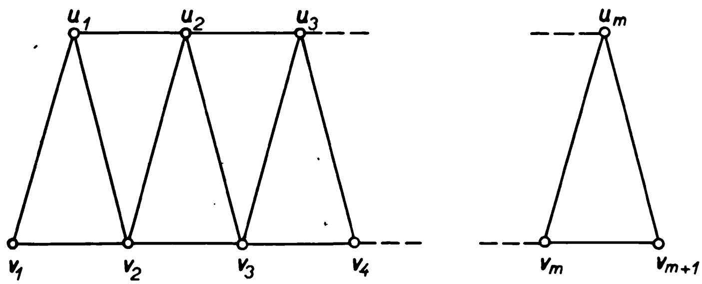

Fig. 2

Theorem 1. *M_{2m+1} is magic.*

Proof. To construct a magic α we discuss two cases.

a) If m is even ($m=2r$) then for $i=1, 2, \ldots, 2r$ we put

$$\alpha(u_iv_i)=i\,, \qquad \alpha(u_iv_{i+1})=2r+i+1\,,$$

for $i=1, 2, \ldots, 2r-1$

$$\alpha(u_iu_{i+1}) = 8r + 2 - i \;,$$

for even $i \leqq 2r$ we have

$$\alpha(v_iv_{i+1}) = 6r + 2 - i \;,$$

for odd $i \leqq 2r - 1$

$$\alpha(v_iv_{i+1}) = 10r + 3 - i \;,$$

and finally

$$\alpha(v_1v_{2r+1}) = 2r + 1 \;, \quad \alpha(v_1u_{2r}) = 6r + 2 \;, \quad \alpha(u_1v_{2r+1}) = 8r + 2 \;.$$

It is easy to see that α is magic with $\sigma = 18r + 6$.

b) If m is odd ($m = 2r + 1$) then α can be constructed as follows: For $i = 1, 2, \ldots, 2r + 1$ put

$$\alpha(u_iv_i) = i \;, \quad \alpha(u_iv_{i+1}) = 2r + i + 2 \;,$$
$$\alpha(v_iv_{i+1}) = 8r + 6 - i \;,$$

for even $i \leqq 2r$ put

$$\alpha(u_iu_{i+1}) = 6r + 4 - i \;,$$

for odd $i \leqq 2r - 1$

$$\alpha(u_iu_{i+1}) = 10r + 7 - i \;,$$

and finally

$$\alpha(v_1v_{2r+2}) = 2r + 2 \;, \quad \alpha(v_1u_{2r+1}) = 8r + 6 \;,$$
$$\alpha(u_1v_{2r+2}) = 6r + 4 \;.$$

We can see again that α is magic with $\sigma = 18r + 14$. Hence the proof.

It is an open problem to decide whether M_{2m+1} is super-magic for some m. The graph M_5 (the complete graph on five vertices) is not super-magic — see [13], p. 1054 and [15], p. 427.

Theorem 2. *If m is odd, $m \geqq 3$, then M_{2m} is super-magic. If m is even, $m \geqq 4$, then M_{2m} is not pseudo-magic.*

Proof. a) Suppose that m is odd, $m = 2r + 1 \geqq 3$. We construct a super-magic α as follows: For $k = 1, 2, \ldots, r$ we put

$$\alpha(u_{2k-1}u_{2k}) = k \;, \qquad \alpha(u_{2k}u_{2k+1}) = 3r + k + 2 \;,$$
$$\alpha(v_{2k-1}v_{2k}) = 2r + k + 1 \;, \qquad \alpha(v_{2k}v_{2k+1}) = r + k + 1 \;,$$

for $i = 1, 2, \ldots, 2r + 1$

$$\alpha(u_iv_i) = 6r + 4 - i$$

and finally

$$\alpha(u_1v_{2r+1})=3r+2\ ,\qquad \alpha(u_{2r+1}v_1)=r+1\ .$$

It is not difficult to check that the values of α are $1, 2, \ldots, 6r+3$ and further that $\sigma=9r+6$.

b) Suppose that m is even and M_{2m} is pseudo-magic under the function α. If we put $\varepsilon=\alpha(v_1v_2)-\alpha(u_1u_2)$ then $\varepsilon\neq 0$. We obtain

$$\alpha(v_iv_{i+1})=\alpha(u_iu_{i+1})-(-1)^i\varepsilon$$

for $i=1, 2, \ldots, m-1$ and further

$$\alpha(v_mu_1)=\alpha(u_mv_1)-\varepsilon\ .$$

From the relation

$$\alpha(v_mu_1)+\alpha(u_1u_2)=\alpha(u_mv_1)+\alpha(v_1v_2)$$

it follows that $\varepsilon=0$, giving the required contradiction. This completes the proof.

The next theorem gives a supplementary result to Theorem 2. Before uttering it we define P_{2m} (for $m\geqq 3$) by adding the edges u_1u_m, v_1v_m to the graph in Fig. 1. The graph P_{2m} may be called the prism (or the untwisted ladder) and was already studied by several authors. For instance N. Biggs in his letter to the present author (April 1971) points out that

$$\frac{m}{2}\left((2+\sqrt{3})^m+(2-\sqrt{3})^m-2\right)$$

is the number of trees spanning the graph P_{2m}.

Theorem 3. *If m is even, $m\geqq 4$, then P_{2m} is magic, but not super-magic. If m is odd, $m\geqq 3$, then P_{2m} is not pseudo-magic.*

Proof. a) If m is even ($m=2r$) then we define α as follows: For $k=1, 2, \ldots,$ r we put

$$\alpha(u_{2k-1}u_{2k})=k\ ,\qquad \alpha(v_{2k-1}v_{2k})=r+k+1\ ,$$

for $k=1, 2, \ldots, r-1$

$$\alpha(u_{2k}u_{2k+1})=3r+k+3\ ,\qquad \alpha(v_{2k}v_{2k+1})=2r+k+2\ ,$$

for $i=2, 3, \ldots, 2r-1$

$$\alpha(u_iv_i)=6_r-i+2\ ,$$

and finally

$$\alpha(u_1u_{2r})=2r+2\ ,\qquad \alpha(v_1v_{2r})=r+1\ ,$$
$$\alpha(u_1v_1)=7r+2\ ,\qquad \alpha(u_{2r}v_{2r})=6r+3\ .$$

It is easy to see that α is magic with $\sigma = 9r + 5$. It remains to show that there is no super-magic α. If α were super-magic the values of α would be $a, a+1, a+2, \ldots, a+3m-1$. We would have

$$2m\sigma(\alpha) = 2\sum_{i=0}^{3m-1}(a+i)$$

or

$$\sigma(\alpha) = \tfrac{3}{2}(2a + 3m - 1)\,.$$

This contradicts the fact that $\sigma(\alpha)$ is an integer.

b) If m is odd we can, using similar considerations as in Theorem 2 — case b), also get a contradiction. This completes the proof.

Concluding this paper, let us investigate prime-magic graphs. B.M. Stewart [13] shows that the graph M_6 (i.e. the complete bipartite graph of type $\langle 3, 3\rangle$) is prime-magic with $\sigma = 139$. W. Sierpiński [11] gives an example of a magic square with three rows formed only from prime numbers and satisfying $\sigma = 1077$. What minimal value of σ can be assigned to the graph $\langle 3,3\rangle$?

Theorem 4a. *The smallest natural number σ divisible by three which can be assigned to the prime-magic graph $\langle 3,3\rangle$ is $\sigma = 99$.*

Proof. Let us express α in terms of matrices, $A = (a_{ij})$. If $\sigma(\alpha)$ is odd and divisible by three then either all the nine prime numbers a_{ij} are $\equiv 1$ or all the nine of them are $\equiv -1 \pmod 3$. Thus either

$$3\sigma \geqq 7 + 13 + 19 + 31 + 37 + 43 + 61 + 67 + 73$$

or

$$3\sigma \geqq 5 + 11 + 23 + 29 + 41 + 47 + 53 + 59\,;$$

this results in $\sigma \geqq 99$. The matrix

$$\begin{bmatrix} 71, & 5, & 23 \\ 11, & 41, & 47 \\ 17, & 53, & 29 \end{bmatrix}$$

shows that $\sigma = 99$ is the minimum value, indeed.

Theorem 4b. *The smallest natural number σ not divisible by three which can be assigned to the prime-magic graph $\langle 3,3\rangle$ is $\sigma = 53$.*

Proof. If $\sigma \equiv 1 \pmod 3$ then either six elements of A are $\equiv 1$ and the other are $\equiv -1 \pmod 3$ or we have $a_{ij} = 3$ for some i, j. In the first case we get

$$3\sigma \geqq (7 + 13 + 19 + 31 + 37 + 43) + (5 + 11 + 17)$$

or $\sigma \geqq 61$. In the second case there are, besides $a_{ij} = 3$, four other elements of A being $\equiv 1$ and four of them being $\equiv -1 \pmod 3$. Thus

$$3\sigma \geqq 3 + (7 + 13 + 19 + 31) + (5 + 11 + 17 + 23)$$

or $\sigma \geqq 43$. Let us discuss the cases $\sigma = 43$ and $\sigma = 49$. For $\sigma = 43$ only primes 5, 11, 17 and 23 are available to decompose the number $43 - a_{ij} = 40$ into a sum in two different ways. However, this is impossible and we have a contradiction. For $\sigma = 49$ the primes 5, 11, 17, 23, 29 and 41 are available, but the number $49 - a_{ij} = 46$ cannot be decomposed in two different ways.

If $\sigma \equiv -1 \pmod 3$ then the consideration is similar as before: we can show that there is no σ satisfying $\sigma < 53$. For $\sigma = 53$ we have

$$\begin{bmatrix} 3, & 13, & 37 \\ 19, & 23, & 11 \\ 31, & 17, & 5 \end{bmatrix}$$

and the proof is thus completed.

Added in proof. Meanwhile M. Doob has shown that both M_{2m}, m even, and P_{2m}, m odd, are magic (Proc. 5th S-E Conf. Combinatorics, Graph Theory, and Computing, pp. 361—374).

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О МАГИЧЕСКИХ ГРАФАХ

Иржи Седлачек

Резюме

На конференции в Смоленице в 1963 г. автор ввел понятие магического графа, который тесно связан с хорошо известными магическими квадратами. В работах [13, 15] предложена классификация этих графов в различные типы (псевдомагические, магические, сверхмагические, магические по отношению к простым числам и т.п.). В последующем этими графами занимались и другие авторы — см. например [2, 7, 12]. В настоящей статье мы описываем дальнейшие классы магических и сверхмагических графов.