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ON ISOMETRIES OF LATTICES

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Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

Metrics defined on lattices with values in an abelian partially ordered group are dealt with in this note. Šmarda [7] investigated metrics defined on lattice ordered groups with values in an abelian lattice ordered group. Swamy [6] studied a metric $d(x, y)$ on an abelian lattice ordered group G with values in G, where $d(x, y)=|x-y|$ for each $x, y \in G$.

1. Preliminaries

Let L be a lattice and let H be an abelian partially ordered group. Let v be a mapping of L into H such that

$$v(x)-v(x\wedge y)=v(x\vee y)-v(y)$$

is valid for each pair of elements $x, y \in L$. Then v is said to be a H-valuation on L. If $v(x)<v(y)$ whenever $x<y$, then v is called a positive H-valuation.

Let d be a mapping of $L\times L$ into H which satisfies the following conditions for each $x, y, z \in L$:

(i) $d(x, y)=d(y, x)$;

(ii) $d(x, y)\geqq 0$, and $d(x, y)=0$ iff $x=y$;

(iii) $d(x, y)+d(y, z)\geqq d(x, z)$;

(iv) $d(x\wedge z, y\wedge z)+d(x\vee z, y\vee z)\leqq d(x, y)$.

Then d is called an H-metric on L.

1.1. Lemma. *Let v: $L\to H$ be a positive H-valuation. Then the lattice L is modular.*

The proof is the same as in [1], p. 232, Thm. 2.

The relation between H-metrics and positive H-valuations is given by the following propositions 1.2 and 1.3.

1.2. Proposition. *Let v be a positive H-valuation on L. Put*

$$d(x, y)=v(x\vee y)-v(x\wedge y)$$

for each $x, y \in L$. Then d is an H-metric on L.

The proof can be performed by using the same steps as in [1], p. 230, Thm. 1 (cf. also [1], p. 234, Exercise 4).

1.3. Proposition. *Let d be an H-metric on L. Let $x_0 \in L$. For each $x \in L$ we set*

$$v(x) = d(x_0, x_0 \vee x) - d(x_0 \vee x, x).$$

Then v is a positive H-valuation on L.

Proof. At first we verify that

$$(\alpha) \qquad x \leqq z \leqq y \Rightarrow d(x, z) + d(z, y) = d(x, y)$$

is valid. In fact, from (iv) and from $x \leqq z \leqq y$ we obtain

$$d(x, z) + d(z, y) \leqq d(x, y),$$

and hence in view of (iii), $d(x, z) + d(z, y) = d(x, y)$ holds.

Now we prove that for each pair $x, y \in L$ the relation

$$(\beta) \qquad d(x, x \wedge y) = d(x \vee y, y)$$

is valid. Namely, from (iv) we infer (by putting $v_1 = x \vee y$ and $u_1 = x \wedge y$ that

$$d(v_1 \wedge y, x \wedge y) + d(v_1 \vee y, x \vee y) \leqq d(v_1, x)$$
$$d(u_1 \wedge x, y \wedge x) + d(u_1 \vee x, y \vee x) \leqq d(u_1, y),$$

hence $d(y, u_1) \leqq d(v_1, x)$ and $d(x, v_1) \leqq d(u_1, y)$, implying that ($\beta$) holds.

From (α) it follows that for each $t \in L$ with $t \geqq x_0 \vee x$ the relation

$$(\gamma) \qquad v(x) = d(x_0, t) - d(t, x)$$

is valid. Moreover, (α) and (γ) imply that whenever $x < y$, then

$$v(y) - v(x) = d(x, y) > 0$$

holds. Therefore in view of (β),

$$v(x \vee y) - v(y) = d(x \vee y, y) = d(x, x \wedge y) = v(x) - v(y).$$

We have verified that v is a positive H-valuation on L.

1.4. Lemma. *Let d be an H-metric on L. Let $x, y \in L$, $u_1 = x \wedge y$, $v_1 = x \vee y$. Then $d(x, y) = d(x, u_1) + d(u_1, y) = d(x, v_1) + d(v_1, y)$ and $d(x, y) = d(u_1, v_1)$.*

Proof. From (iv) we obtain (by putting $x = z$)

$$d(x, y) \geqq d(x, u_1) + d(x, v_1).$$

Hence in view of (β) and (iii),

$$d(x, y) \geqq d(x, u_1) + d(u_1, y) \geqq d(x, y).$$

The relation $d(x, y)=d(x, v_1)+d(v_1, y)$ can be verified analogously. From (α) and (β) we infer $d(u_1, v_1)=d(u_1, x)+d(x, v_1)=d(u_1, x)+d(u_1, y)=d(x, y)$.

Let L' be a lattice and let d' be a H-metric on L'. Suppose that $h: L \to L'$ is bijection such that

$$d(x, y)=d'(h(x), h(y))$$

is valid for each $x, y \in L$. Then h is said to be an H-isometry.

If the abelian partially ordered group H is fixed, then we use the term valuation instead of H-valuation, and in a similar sense we apply the notions of metric and isometry.

The results of this section will be applied below without quotation.

2. Basic properties of isometries

Let L, L', d, d' and H be as in § 1 and let h be an isomery of L onto L'. For each $x \in L$ we put $h(x)=x'$. (If $z' \in L'$, then z is the element of L with $h(z)=z'$.)

2.1. Lemma. *Let $x, y, a \in L$, $x \leqq a \leqq y$, $x' \leqq y'$. Then $x' \leqq a' \leqq y'$.*

Proof. We have $d(x, a)+d(a, y)=d(x, y)$, hence

$$d'(x', a')+d'(a', y')=d'(x', y').$$

Put $x' \wedge a'=u'$, $y' \vee a'=v'$. Then

$$\begin{aligned} d'(x', y') &\leqq d'(u', v')=d'(u', a')+d'(a', v') \leqq \\ &\leqq d'(x', u')+d'(u', a')+d'(y', v')+d'(a', v')= \\ &=d'(x', a')+d'(a', y')=d'(x', y'), \end{aligned}$$

whence $d'(x', u')=0$ and $d'(y', v')=0$. Therefore $u'=x'$ and $v'=y'$; thus $x' \leqq a' \leqq y'$.

Analogously we can prove

2.1′. Lemma. *Let $x, y, a \in L$, $x \leqq a \leqq y$, $x' \geqq y'$. Then $x' \geqq a' \geqq y'$.*

2.2. Lemma. *Let $x, y \in L$, $x \leqq y$, $x' \wedge y'=u'$. Then $x \leqq u \leqq y$.*

Proof. Put $x \wedge u=u_1$, $y \vee v=v_1$. We have $d'(x', u')+d'(u', y')=d'(x', y')$, hence $d(x, u)+d(u, y)=d(x, y)$. Thus

$$\begin{aligned} d(x, y) &\leqq d(u_1, v_1)=d(u_1, u)+d(u, v_1) \leqq d(x, u_1)+d(u_1, u)+ \\ &+d(y, v_1)+d(v_1, u)=d(x, u)+d(u, y)=d(x, y), \end{aligned}$$

whence $d(x, u_1)=0$ and $d(y, v_1)=0$, implying $x=u_1$ and $y=v_1$. Therefore $x \leqq u \leqq y$.

Similarly we can prove

2.2.′. Lemma. *Let $x, y \in L$, $x \leqq y$, $x' \vee y' = v'$. Then $x \leqq v \leqq y$.*

2.3. Lemma. *Let x, y, u, v be as in 2.2 and 2.2′. Then $u \wedge v = x$ and $u \vee v = y$.*

Proof. Put $u \wedge v = x_1$, $u \vee v = y_1$. In view of 2.2 and 2.2′ we have $x_1, y_1 \in [x, y]$. Hence

$$\begin{gathered} d(x, y) = d(x, x_1) + d(x_1, y_1) + d(y_1, y) = \\ = d(x, x_1) + d(u, v) + d(y_1, y) = d(x, x_1) + d'(u', v') + d(y_1, y) = \\ = d(x, x_1) + d(x, y) + d(y_1, y). \end{gathered}$$

Thus $d(x, x_1) = 0$ and $d(y, y_1) = 0$; hence $x = x_1$ and $y = y_1$.

2.4. Lemma. *Let $a, b \in L$, $a \wedge b = u$, $a \vee b = v$, $a' \leqq u'$, $u' \leqq b'$. Then $u' \wedge v' = a'$ and $u' \vee v' = b'$.*

Proof. We have $a' \leqq b'$. From 2.3 we obtain (if h is replaced by h^{-1}) that $u' \wedge v' = a'$ and $u' \vee v' = b'$ is valid.

2.5. Lemma. *Let $a, b \in L$, $a \wedge b = u$, $a \vee b = v$, $u' \leqq a'$, $u' \leqq b'$. Then $a' \wedge b' = u'$ and $a' \vee b' = v'$.*

Proof. a) Denote $a' \wedge b' = u_1'$. In view of 2.1 (if we replace h by h^{-1}) the relation $u_1 \in [u, a] \cap [u, b]$ must be valid, hence $u = u_1$.

b) Denote $a_1' = a' \vee v'$, $b_1' = b' \vee v'$. According to 2.2′ we have $a_1 \in [a, v]$ and $b_1 \in [b, v]$. Put $u_0 = a_1 \wedge b_1$. In view of 2.1 (with h replaced by h^{-1}) the relations $u' \leqq u_0' \leqq a_1'$, $u_0' \leqq b_1'$ are valid; hence from a) we obtain $u_0' = a_1' \wedge b_1'$. Further we have $v' \leqq a_1'$, $v' \leqq b_1'$, thus $v' \leqq u_0'$. Because of $u_0 \leqq a_1 \leqq v$, from 2.1′ (and with h replaced by h^{-1}) we infer that $v' \leqq a_1' \leqq u_0'$ must be valid. Thus $u_0 = a_1$. Similarly we obtain $b_1 = u_0$. In view of $u_0 \in [a, v] \cap [b, v]$ we get $u_0 = v$, hence $a' \leqq v'$ and $b' \leqq v'$. Thus according to the dual of a), $v' = a' \vee b'$ is valid.

The assertions dual to 2.4 and 2.5 can be proved analogously.

2.6. Lemma. *Let $x, y, z \in L$, $x_1 = x \vee z$, $y_1 = y \vee z$, $x \leqq y$, $x' \leqq y'$. Then $x_1' \leqq y_1'$.*

Proof. Put $x_1 \wedge y = p$, $p' \vee x_1' = v'$. In view of 2.2′ we have $p \leqq v \leqq x_1$. Next, lemma 2.1 (with h replaced by h^{-1}) yields $p' \leqq y'$. Denote $v_1 = y \vee v$. According to 2.5, $v_1' = y' \vee v'$.

From the relations $v \leqq v_1$, $v' \leqq v_1'$, $v \leqq x_1$, $v' \geqq x_1'$ we infer (using 2.1 and the dual of 2.1) that $v_1 \wedge x_1 = v$ is valid. Thus (since $v_1 \vee x_1 = y_1$) in view of 2.4 $x_1' \leqq y_1'$ holds.

3. Direct product decompositions corresponding to isometries

Again, let L, L', d, d', H and h be as above.

Let $x, y \in L$, $u = x \wedge y$, $v = x \vee y$. If $u' \leqq v'$ (or $u' \geqq u'$, respectively), then we put xR_1y (xR_2y).

From 2.6 (and using duality) we obtain:

3.1. Lemma. *Let $x, y, z \in L$, $i \in \{1, 2\}$, xR_iy. Then $x \vee zR_iy \vee z$ and $x \wedge zR_iy \wedge z$.*

Both R_1 and R_2 are obviously reflexive and symmetric.

3.2. Lemma. *The relation R_1 is transitive.*

Proof. Let x, y, $z \in L$, xR_1y, yR_1z. Put $x \wedge y = u_1$, $x \vee y = v_1$, $y \wedge z = u_2$, $y \vee z = v_2$, $u_1 \wedge u_2 = u_3$, $v_1 \vee v_2 = v_3$.

In view of 2.1, we have u_1R_1y and u_2R_1y, hence $u_1R_1(u_1 \vee u_2)$ and $u_2R_1(u_1 \vee u_2)$. From this and from the dual of 2.5 we infer that $u_3R_1u_1$ and $u_3R_1u_2$ holds. We have u_1R_1x and u_2R_1z, thus u_3R_1x and u_3R_1z. Since $u_3 \leqq x \wedge z$, using 2.1 again we obtain that $(x \wedge z)' \leqq x'$ is valid. By a dual argument, $x' \leqq (x \vee z)'$ holds. Hence xR_1z.

Similarly we can verify that R_2 is transitive. Hence according to 3.1, both R_1 and R_2 are congruence relations on L.

3.3. Lemma. *Let x, $y \in L$, xR_1y and xR_2y. Then $x = y$.*

Proof. We have $(x \wedge y)' \leqq (x \vee y)'$ and, at the same time, $(x \wedge y)' \geqq (x \vee y)'$. Thus $x \wedge y = x \vee y$ and hence $x = y$.

3.4. Lemma. *$R_1 \vee R_2$ is the greatest congruence relation on L.*

Proof. Let a, $b \in L$; we have to verify that $aR_1 \vee R_2b$ holds. Put $x = a \wedge b$, $y = a \vee b$, $u' = x' \wedge y'$. In view of 2.2, the relations xR_2u, uR_1y are valid, hence $xR_1 \vee R_2y$ holds. Thus $aR_1 \vee R_2b$.

3.5. Lemma. *The congruence relations R_1 and R_2 are permutable.*

Proof. Let $x, y \in L$, $x \leqq y$, $u' = x' \wedge y'$, $v' = x' \vee y'$. In the proof of 3.4 we have already verified that xR_2u and uR_1y is valid. In view of 2.2′, the relations xR_1v and vR_1y hold. Now it suffices to apply Ex. 10, p. 163, [1].

For $x \in L$ and $i \in \{1, 2\}$ we denote $\{z \in L: zR_ix\} = x(R_i)$. Let φ_i: $L \to L/R_i$ be the mapping defined by $\varphi_i(x) = x(R_i)$ for each $x \in L$. From 3.3, 3.4, 3.5 and [1], Thm. 5, Chap. VII we obtain:

3.6. Theorem. *The mapping φ': $L \to L/R_1 \times L/R_2$ defined by $\varphi(x) = (\varphi_1(x), \varphi_2(x))$ for each $x \in L$ is an isomorphism of the lattice L onto the lattice $L/R_1 \times L/R_2$.*

Now we introduce analogous notions and denotations for the lattice L'.

Let x', $y' \in L'$, $u' = x' \wedge y'$, $v' = x' \vee y'$. If $u \leqq v$ (or $u \geqq v$), then we put $x'R_1'y'$ ($x'R_2'y'$). Analogously as we did for R_1 and R_2 we can verify that R_1' and R_2' are congruence relations on the lattice L'; if φ_1', φ_2' and φ' are defined similarly as φ_1, φ_2 and φ, then we have

3.6′. Theorem. *The mapping φ: $L' \to L'/R_1' \times L'/R_2'$ defined by $\varphi'(x') = (\varphi_1'(x'), \varphi_2'(x'))$ for each $x' \in L'$ is an isomorphism of L' onto $L'/R_1 \times L'/R_2$.*

3.7. Lemma. *Let x, $y \in L$. Then xR_1y iff $x'R_1'y'$.*

Proof. Let xR_1y, $u = x \wedge y$, $v = x \vee y$. Hence $u' \leqq v'$. From 2.1 and from the assertion that we obtain from 2.1 if we replace h by h^{-1} it follows that the mapping h maps isomorphically the interval $[u, v]$ of L onto the interval $[u', v']$ of L'; in particular, the relations $u' = x' \wedge y'$ and $v' = x' \vee y'$ are valid. Thus $x'R_1'y'$. The inverse implication can be verified similarly.

Analogously we can prove (by applying 2.1′):

3.7′. Lemma. *Let $x, y \in L$. Then xR_2y iff $x'R_2'y'$.*

For each lattice K, we denote by $K^{\sim}$ the lattice dual to K.

Let us consider the mappings $\psi_1: L'/R_1' \to L/R_1$ and $\psi_2: L'/R_2' \to L/R_2$ defined by $\psi_1(x'(R_1')) = x(R_1)$ and $\psi_2(x'(R_2')) = x(R_2)$. From 3.7 and 3.7′ it follows that ψ_1 and ψ_2 are correctly defined; moreover, from the definitions of R_i and R_i' $(i = 1, 2)$ we infer that ψ_1 is a dual isomorphism and ψ_2 is an isomorphism. Therefore from 3.6′ we obtain

3.8. Theorem. *Let ψ: $L' \to L/R_1 \times L/R_2$ be the mapping defined by $\psi(x') = (x(R_1), x(R_2))$ for each $x' \in L'$. Then* (i) *ψ is an isomorphism of the lattice L onto the lattice $(L/R_1)^{\sim} \times (L/R_2)$;* (ii) *$\varphi = h \circ \psi$.*

Theorems 3.8 and 3.6 yield:

3.9. Corollary. *Let h be an isometry of a lattice L onto a lattice L'. Then there exist lattices A, B and direct product representations f_1: $L \to A \times B$, f_2: $L' \to A^{\sim} \times B$, such that $f_1 = h \circ f_2$.*

Let R be the additive group of all reals with the natural linear order. R-isometries of lattices were investigated in [2]. The method of proving theorem 1 in [2] essentially differs from the direct method applied above (in [2] a result of M. Kolibiar [5] on the betweenness relation in lattices was used).

4. Isometries of *l*-groups

Let G and G' be abelian lattice ordered groups and let H be an abelian partially ordered group. The corresponding lattices (i.e., the structures that we obtain from G and G' if the group operations are not taken under consideration) will be denoted by $L(G)$ and $L(G')$. Let h, d and d' be as in § 1 with the difference that L and L' are replaced by $L(G)$ and $L(G')$. Then Corollary 3.9 is valid (with L and L' replaced by $L(G)$ and $L(G')$).

Let us introduce the following denotation. Let L, P, Q be lattices and let us have a direct product representation f: $L \to P \times Q$. Let $z_0 \in L$, $f(z_0) = (p_0, q_0)$. We denote by $P_L[z_0]$ the set of all elements $z \in L$ such that $f(z) = (p, q)$ with $q = q_0$. The set $Q_L[z_0]$ is defined analogously. Both $P_L[z_0]$ and $Q_L[z_0]$ are sublattices of L. For each $z \in L$ we put $f[z_0](z) = (p_1, q_1)$, where (under the above denotations) we have $p_1 \in P_L[z_0]$, $q_1 \in Q_L[z_0]$, $f(p_1) = (p, q_0)$ and $f(q_1) = (p_0, q)$. Then $f[z_0]$: $L \to P_L[z_0] \times Q_L[z_0]$ is a direct product representation of the lattice L.

If G is an l-group and $L = L(G)$, then we write $P_G[z_0]$ instead of $P_{L(G)}[z_0]$.

4.1. Proposition. *Assume that we have a direct product representation f: $L(G)\to P\times Q$. Then $P_G[0]$ and $Q_G[0]$ are l-subgroups of G and the mapping $f[0]$: $G\to P_G[0]\times Q_G[0]$ is a direct product representation of the l-group G.*

This was proved in [4] (Thm. 3) (under different denotations).

From 4.1 and 3.9 we infer:

4.2. Theorem. *Let h be an H-isometry of an l-group G onto an l-group G'.* (i) *There are lattices A, B and direct product representations f_1: $L(G)\to A\times B$, f_2: $L(G')\to A^{\sim}\times B$ such that $f_1=h\circ f_2$.* (ii) *$A_G[0]$ and $B_G[0]$ are l-subgroups of G, and $A_{G'}[0]$, $B_{G'}[0]$ are l-subgroups of G'; for the l-groups G and G' we have direct product representations*

$$f_1[0]: G\to A_G[0]\times B_G[0],$$
$$f_2[0]: G'\to A^{\sim}_{G'}[0]\times B_{G'}[0].$$

If we put $H=G$, then the 'simplest' positive H-valuation v_0 on G is the identity. The H-metric d_0 corresponding to v_0 (cf. Lemma 1.2) is defined by

$$d_0(x, y)=x\vee y-x\wedge y=|x-y|$$

for each x, $y\in G$. In the particular case when also $G'=G$ we have the following result:

4.3. Proposition (Cf. [3]). *Let h: $G\to G$ be a bijection such that $h(0)=0$ and $d_0(h(x), h(y))=d_0(x, y)$ for each pair of elements x, $y\in G$. Then there exists a direct product representation f: $G\to A\times B$ such that, whenever $x\in G$ and $f(x)=(a, b)$, then $f(h(x))=(-a, b)$.*

Let us now return to the general case dealt with in Theorem 4.2 above. The lattices $L(A_G[0])$ and $L(A_{G'}[0])$ are isomorphic, and so are the lattices $L(B_G[0])$ and $L(B_{G'}[0])$. Motivated by Proposition 4.3 we can now consider the question whether $A_G[0]$ and $A_{G'}[0]$ must be also isomorphic as groups, and similarly for $B_G[0]$ and $B_{G'}[0]$. The following example shows that the answer is negative.

Let R_0 and N_0 be the additive group of all rational numbers or of all integers, respectively; both R_0 and N_0 are considered with the natural linear order. Let G be the lexicographic product $R_0\circ N_0$ (the elements of G are pairs (x, y) with $x\in R_0$, $y\in N_0$, the group operation is performed coordinatewise, and $(x_1, y_1)\leqq(x_2, y_2)$ if either $y_1<y_2$, or $y_1=y_2$ and $x_1\leqq x_2$). Put $G'=H=R_0$. There exists an isomorphism h of $L(G)$ onto $L(G')$. For z_1, $z_2\in G$ and x_1, $x_2\in G'$ we put

$$d(z_1, z_2)=|h(z_1)-h(z_2)|,\quad d'(x_1, x_2)=|x_1-x_2|.$$

Then d is a H-metric on G, d' is a H-metric on G' and h is an H-isometry of G onto G'. Both G and G' are directly indecomposable and they fail to be isomorphic.

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ОБ ИЗОМЕТРИЯХ РЕШЕТОК

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Резюме

В статье исследуются метрики на решетках со значениями в частично упорядоченной абелевой группе. Найдены соотношения между изометриями и прямыми разложениями.