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On Special Almost Geodesic Mappings of Type π_1 of Spaces with Affine Connection *

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Abstract

N. S. Sinyukov [5] introduced the concept of an *almost geodesic mapping* of a space A_n with an affine connection without torsion onto $\overline{A}_n$ and found three types: π_1, π_2 and π_3. The authors of [1] proved completness of that classification for $n > 5$.

By definition, special types of mappings π_1 are characterized by equations

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = a_{ij}\delta^h_k,$$

where $P^h_{ij} \equiv \overline{\Gamma}^h_{ij} - \Gamma^h_{ij}$ is the deformation tensor of affine connections of the spaces A_n and $\overline{A}_n$.

In this paper geometric objects which preserve these mappings are found and also closed classes of such spaces are described.

Key words: Almost geodesic mappings, affine connection space.

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1 Introduction

In this paper the theory of almost geodesic mappings of type π_1^* of spaces with affine connection without torsion is studied. These mappings are a special case of almost geodesic mappings of type π_1 introduced by N. S. Sinyukov [5].

General properties of mappings π_1^* are shown and invariant objects with respect to these mappings are found. Mappings π_1^* of spaces of constant curvature and affine spaces are studied.

First we recall basic formulas and properties of the theory of almost geodesic mappings of spaces A_n with affine connection which are described in [5], [6].

A curve ℓ defined in a space with affine connection A_n is called *almost geodesic* if there exists a two-dimensional parallel distribution along ℓ, to which the tangent vector of this curve belongs at every point.

A diffeomorphism $f\colon A_n \to \overline{A}_n$ is an *almost geodesic mapping* if, as a result of f, every geodesic of the space A_n passes into an almost geodesic curve of the space $\overline{A}_n$.

A mapping f from A_n onto $\overline{A}_n$ is almost geodesic if and only if, in a common coordinate system $x \equiv (x^1, x^2, \dots, x^n)$ with respect to the mapping f, the connection deformation tensor $P_{ij}^h(x) \equiv \overline{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x)$ satisfies the relations [5]

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma \equiv a\, P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b\,\lambda^h,$$

where $A^h_{ijk} \equiv P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}$, Γ^h_{ij} $(\overline{\Gamma}^h_{ij})$ are components of the affine connection of spaces A_n $(\overline{A}_n)$, λ^h is any vector, a and b are some functions of variables x^h and λ^h. Hereafter "," denotes the covariant derivative with respect to the connection of the space A_n.

Three types of almost geodesic mappings, π_1, π_2 and π_3, were found in [5]. We proved [1] that for $n > 5$ other types do not exist. Almost geodesic mappings of type π_1 are characterized by the following conditions

$$A^h_{(ijk)} = \delta^h_{(i}\, a_{jk)} + b_{(i}\, P^h_{jk)},$$

where a_{ij} is a symmetric tensor, b_i is a covector, δ^h_i is the Kronecker symbol, (ijk) is the symmetrization of indices.

Many papers are dedicated to study of mappings π_2 and π_3 (see [5], [6], [4]) in contrast to mappings π_1. The reason is that basic equations of these mappings are difficult to study. Therefore in this paper we deal with a special case of mappings π_1. This special case does not imply that geodesic mappings are either π_2 or π_3 mappings.

2 Almost geodesic mappings π_1^*

Let a diffeomorphism from A_n onto $\overline{A}_n$ satisfy

$$P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k} = a_{ij}\,\delta^h_k, \tag{1}$$

where a_{ij} is a symmetric tensor.

Diffeomorphisms of this kind are a special case of almost geodesic mappings of type π_1. We denote them by π_1^*.

Let us derive the integrability condition arising from (1). We differentiate (1) covariantly by x^m and then alternate with respect to the indices k and m. Next in the integrability condition of (1) we contract with respect to the indices h and m. After editing we have

$$(n-1)\, a_{ij,k} = P^{\alpha}_{ij} R_{\alpha k} - P^{\beta}_{\alpha(i} R^{\alpha}_{j)\beta k} - (n-1)\, P^{\alpha}_{ij} a_{\alpha k}, \tag{2}$$

where R^h_{ijk} is the Riemannian tensor in A_n, $R_{ij} \equiv R^{\alpha}_{ij\alpha}$ is the Ricci tensor, $(i\,j)$ denotes the symmetrization of indices.

Evidently, equations (1) and (2) represent a system of differential equations of Cauchy type in the space A_n which is solvable with respect to unknown functions $P^h_{ij}(x)$ and $a_{ij}(x)$, which, naturally, satisfy the algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \qquad a_{ij}(x) = a_{ji}(x). \tag{3}$$

We have

Theorem 1 *The space A_n with affine connection admits an almost geodesic mapping π_1^* onto $\overline{A}_n$ if and only if there exists a solution P^h_{ij} and a_{ij} of system of Cauchy type (1) and (2) satisfying (3).*

Integrability conditions of this system have the form

$$-P^{\alpha}_{ij} R^h_{\alpha km} + P^h_{\alpha(i} R^{\alpha}_{j)km} =$$
$$\frac{1}{n-1}\left[(P^{\alpha}_{ij} R_{\alpha m} - P^{\beta}_{\alpha(i} R^{\alpha}_{j)m\beta})\delta^h_k - (P^{\alpha}_{ij} R_{\alpha k} - P^{\beta}_{\alpha(i} R^{\alpha}_{j)k\beta})\delta^h_m \right],$$

$$(n-1) a_{\alpha(i} R^{\alpha}_{j)km} = P^{\alpha}_{ij} R^h_{\alpha[k,m]} + P^{\beta}_{\alpha(i} R^{\alpha}_{j)mk,\beta}$$
$$+ R_{[mk]} a_{ij} + P^{\beta}_{\gamma[m} R^h_{|i|k]\beta} P^{\gamma}_{\alpha j} + P^{\alpha}_{ij} P^{\beta}_{\alpha\gamma} R^{\gamma}_{[km]\beta} - P^{\alpha}_{ij} P^{\beta}_{\gamma[k} R^{\gamma}_{|\alpha|m]\beta},$$

where $[i\,j]$ denotes the alternation of indices.

3 Invariant object of mappings π_1^*

If P^h_{ij} is the deformation tensor ([5]) then Riemannian tensors R^h_{ijk} and $\overline{R}^h_{ijk}$ of spaces A_n and $\overline{A}_n$ satisfy the following condition

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{i[k,j]} + P^{\alpha}_{i[k} P^h_{j]\alpha}. \tag{4}$$

Using formulas (1) and (4) we obtain

$$\overline{\overset{*}{W}}{}^h_{ijk} = \overset{*}{W}{}^h_{ijk}, \tag{5}$$

where

$$\overset{*}{W}{}^h_{ijk} \equiv R^h_{ijk} - \frac{1}{n-1} R_{i[j}\delta^h_{k]}, \qquad \overline{\overset{*}{W}}{}^h_{ijk} \equiv \overline{R}^h_{ijk} - \frac{1}{n-1} \overline{R}_{i[j}\delta^h_{k]}. \tag{6}$$

Clearly, $\overset{*}{W}{}^h_{ijk}$ and $\overset{*}{\overline{W}}{}^h_{ijk}$ is a tensor of type $\binom{1}{3}$ in the space A_n and $\overline{A}_n$, respectively.

Condition (5) shows that this tensor is invariant with respect to almost geodesic mappings π_1^*.

We contract condition (5) in indices h and i to obtain the equality

$$W_{ij} = \overline{W}_{ij}, \tag{7}$$

where

$$W_{ij} \equiv R_{[ij]}, \qquad \overline{W}_{ij} \equiv \overline{R}_{[ij]}, \tag{8}$$

Subtract (7) from (5) to write

$$W^h_{ijk} = \overline{W}^h_{ijk}, \tag{9}$$

where W^h_{ijk} and $\overline{W}^h_{ijk}$ are Weyl projective curvature tensors of spaces A_n and $\overline{A}_n$, respectivelly. We get

Theorem 2 *The Weyl projective curvature tensor W^h_{ijk} and tensors $\overset{*}{W}{}^h_{ijk}$ and W_{ij}, which are defined by (6) and (8), are invariant with respect to almost geodesic mappings π_1^*.*

4 Mappings π_1^* of affine and projective-euclidean spaces

From Theorem 2 it follows

Theorem 3 *If a projective-euclidean space or equiaffine space admits an almost geodesic mapping π_1^* onto $\overline{A}_n$ then $\overline{A}_n$ is also a projective-euclidean space or an equiaffine space.*

The proof of Theorem 3, evidently, follows from the condition $W^h_{ijk} = 0$ in the projective-euclidean space and from the condition $W_{ij} = 0$ in the equiaffine space.

It means that projective-euclidean spaces and equiaffine spaces make up closed classes with respect to mappings π_1^*.

Clearly, the Riemannian tensor is preserved by mappings π_1^* if and only if the tensor a_{ij} vanishes. In this case basic equations have the form

$$P^h_{ij,k} = -P^\alpha_{ij} P^h_{\alpha k}. \tag{10}$$

Equations (10) are completely integrable in the affine space. Evidently, these equations have a solution for any initial conditions $P^h_{ij}(x_o)$.

If the initial conditions are such that $P^h_{ij}(x_o) \not\equiv \delta^h_{(i}\psi_{j)}(x_o)$ then every solution generates a mapping π_1^* which is not a geodesic mapping of the affine space A_n onto the affine space $\overline{A}_n$. Therefore we can write

Theorem 4 *Mappings π_1^* of an affine space A_n onto itself exist. All lines map into planar curves (not necessary lines).*

Moreover, integrability conditions (1) and (2) in affine space are always true. We obtain

Theorem 5 *Riemannian spaces V_n with non constant curvature admit non geodesic mappings π_1^* which are necessarilly mappings of type π_3 and preserve the quadratic complex of geodesics.*

Proof Let a Riemannian space V_n with non constant curvature K admit a non geodesic mapping π_1^*. Integrability conditions (1) then have the form

$$K(P^h_{k(i}g_{j)l} - P^h_{l(i}g_{j)k}) + \delta^h_l B_{ijk} - \delta^h_k B_{ijl} = 0, \tag{11}$$

where $B_{ijk} \equiv a_{ij,k} + P^\alpha_{ij}(a_{\alpha k} + K\, g_{\alpha k})$, g_{ij} is the metric tensor of the space V_n.

From the last formula it follows

$$P^h_{ij} = P^h g_{ij} \tag{12}$$

where P^h is a vector. Then the mapping is F-planar [4]. Clearly, on the basis of results in [1], such mappings are almost geodesic mappings of type π_3. It is proved in the paper [1] that mappings $\pi_1 \cap \pi_3$ preserve the quadratic complex of geodesics [3].

After substituting (12) in (1) we have

$$P^h_{,k} + P^h P_k = \alpha \delta^h_k,$$

where α is a function, P_k is a covector.

These conditions characterize concircular vector fields P^h, which always exist in spaces with constant curvature. □

5 Examples of almost geodesic mappings π_1^*

We present an example of an almost geodesic mapping of type π_1^* of an affine space A_n onto an affine space $\overline{A}_n$.

Let $x^1, x^2, \ldots, x^n$ and $\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n$ be affine coordinate in A_n and $\overline{A}_n$, respectively.

The mapping

$$\overline{x}^h = \frac{1}{2} C^h_\alpha (x^\alpha - C^\alpha)^2 + x^h_o, \tag{13}$$

where C^h_i, C^h, x^h_o are some constants, $x^h \neq C^h$, and the determinant $\det|C^h_i| \not\equiv 0$, defines an almost geodesic mapping π_1^* of the space A_n onto $\overline{A}_n$.

We can prove directly that the deformation tensor P^h_{ij} in the coordinate system $x^1, x^2, \ldots, x^n$ has the form

$$P^i_{ii} = \frac{1}{x^i - C^i}, \qquad i = \overline{1,n},$$

and the other components are equal to zero.

Evidently, the tensor P^h_{ij} corresponds to equations (10). This mapping is not of type π_2 or π_3.

Lines in the space A_n which are defined by equations $x^h = a^h + b^h\,t$ where t is the parameter, map into parabolas (or lines) of the space $\overline{A}_n$, which are defined by equations

$$\overline{x}^h = D^h + E^h\,t + F^h\,t^2$$

where

$$D^h = \frac{1}{2}C^h_\alpha(a^\alpha - C^\alpha)^2, \quad E^h = C^h_\alpha(a^\alpha - C^\alpha)b^\alpha, \quad F^h = \frac{1}{2}C^h_\alpha(b^\alpha)^2$$

in this mapping.

The image is a line if vectors E^h and F^h are collinear.

Finally we remark that formula (13) generates a system of almost geodesic mappings of type π_1 of planar spaces if the coefficients C^h_i, C^h and x^h_o are continuous.

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