Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 43 (2004), No. 1, 7--20

Persistent URL: `http://dml.cz/dmlcz/132943`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **43** (2004) 7–20

Further Ultimate Boundedness of Solutions of some System of Third Order Nonlinear Ordinary Differential Equations

A. U. AFUWAPE[1], M. O. OMEIKE[2]

[1] *Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria*
e-mail: aafuwape@oauife.edu.ng

[2] *Department of Mathematical Sciences, University of Agriculture*
Abeokuta, Nigeria
e-mail: moomeike@yahoo.com

(Received September 3, 2003)

Abstract

In this paper, we shall give sufficient conditions for the ultimate boundedness of solutions for some system of third order non-linear ordinary differential equations of the form

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

where $X, F(\ddot{X})$, $G(\dot{X})$, $H(X)$, $P(t, X, \dot{X}, \ddot{X})$ are real n-vectors with F, G, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous in their respective arguments. We do not necessarily require that $F(\ddot{X}), G(\dot{X})$ and $H(X)$ are differentiable. Using the basic tools of a complete Lyapunov Function, earlier results are generalized.

Key words: Ultimate boundedness, complete Lyapunov functions, nonlinear third order system.

2000 Mathematics Subject Classification: 34D40, 34D20, 34C25

1 Introduction

In a sequence of results, Afuwape [1, 2, 3], Ezeilo [5], Ezeilo and Tejumola [8, 9], Meng [10] and Tiryaki [12] studied particular cases of the third-order nonlinear system of differential equations of the form

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.1}$$

where $X, F(\ddot{X}), G(\dot{X}), H(X), P(t, X, \dot{X}, \ddot{X})$ are real n-vectors with $F, G, H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous in the respective arguments.

Boundedness and Periodicity results were discussed by imposing differentiability conditions in [5, 8, 9, 12] on the nonlinear functions in the particular cases of (1.1), while not necessarily differentiable conditions were imposed in [1, 3, 10] for the study of ultimate boundedness of particular cases of (1.1). Furthermore, the Lyapunov second method was used with the aid of a suitable differentiable Lyapunov function.

For $n = 1$ and $f(\ddot{x}) = a\ddot{x}, g(\dot{x}) = b\dot{x}$ this reduces to

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1.2}$$

which was studied by Ezeilo [6,7]. In [7], Ezeilo studied the ultimate boundedness and convergence of solutions of (1.2) by assuming

$$\frac{h(\xi + \eta) - h(\eta)}{\xi} \in I_0 \tag{1.3}$$

for some designated $\xi, \eta (\neq 0)$ with $I_0 \equiv [\delta, kab]$ where $\delta > 0$ is an arbitrary constant and $0 < k < 1$. I_0 is a subset of the generalized Routh–Hurwitz interval $(0, ab)$.

When $\eta = 0$, $\xi \neq 0$ in (1.3) we have

$$H_0 = H_0(\xi) \equiv \frac{\{h(\xi) - h(0)\}}{\xi} \tag{1.4}$$

and

$$H_0 = \frac{h(\xi)}{\xi} \quad \text{if } h(0) = 0. \tag{1.5}$$

On the other hand if $F(\ddot{X}) = A\ddot{X}, G(\dot{X}) = B\dot{X}$ in (1.1) we have

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.6}$$

where A, B are real symmetric $n \times n$ matrices.

Afuwape [1] and Meng [10] studied (1.6) for the ultimate boundedness and periodicity of solutions for which H is of class $C(\mathbb{R}^n)$ by satisfying

$$H(X) = H(Y) + A(X, Y)(X - Y) \tag{1.7}$$

where $A(X,Y)$ is a real $n \times n$ operator for any X, Y in $\mathbb{R}^n$, and having real eigenvalues $\lambda_i(A(X,Y))$ $(i = 1, 2, \dots, n)$.

It was assumed that these eigeuvalues satisfy

$$0 < \delta_h \leq \lambda_i(A(X,X)) \leq \Delta_h \tag{1.8}$$

with δ_h, Δ_h as fixed constants.

Moreover, the matrices A, B have real positive eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$ respectively with $\delta_a = \min \lambda_i(A), \delta_b = \min \lambda_i(B), \Delta_a = \max \lambda_i(A)$, $\Delta_b = \max \lambda_i(B), i = 1, 2, \dots, n$ and that for some constant $k(< 1)$ the "generalized" Routh–Hurwitz condition,

$$\Delta_h \leq k\delta_a\delta_g \tag{1.9}$$

was satisfied. Furthermore, when $F(\ddot{X}) = A\ddot{X}$ in (1.1) we have

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.10}$$

where A is a real symmetric $n \times n$ matrix.

In [3], Afuwape studied (1.10) for the ultimate boundedness of solutions for which G, H are of class $C(\mathbb{R}^n)$ by satisfying

$$G(Y_1) = G(Y_2) + B_g(Y_1, Y_2)(Y_1 - Y_2) \tag{1.11a}$$

$$H(X_1) = H(X_2) + C_h(X_1, X_2)(X_1 - X_2) \tag{1.11b}$$

where $B_g(Y_1, Y_2)$, $C_h(X_1, X_2)$ are $n \times n$ real continuous operators, having real eigenvalues $\lambda_i(B_g(Y_1, Y_2))$, $\lambda_i(C_h(X_1, X_2))$, $(i = 1, 2, \dots, n)$ respectively and which satisfy

$$0 < \delta_g \leq \lambda_i(B_g(Y_1, Y_2)) \leq \Delta_g \tag{1.12a}$$

$$0 < \delta_h \leq \lambda_i(C_h(X_1, X_2)) \leq \Delta_h \tag{1.12b}$$

with $\delta_g, \delta_h, \Delta_g, \Delta_h$ as fixed constants.

Also, the matrix A has real positive eigenvalues $\lambda_i(A)$ with $\delta_a = \min \lambda_i(A)$, $\Delta_a = \max \lambda_i(A)$, $i = 1, 2, \dots, n$ and that for some constant $k(< 1)$ the "generalized" Routh Hurwitz condition (1.9) was satisfied.

In this paper, we shall extend earlier results of [1, 3, 5, 8, 9, 10, 12] to systems of the form (1.1) and for which generalized Routh–Hurwitz condition (1.9) is satisfied. A new differentiable Lyapunov function which is a modification of the one used in [10] is used to prove ultimate boundedness of solutions of (1.1). In addition to (1.11a) and (1.11b) we assume that F is of class $C(\mathbb{R}^n)$ and satisfies

$$F(Z_1) = F(Z_2) + A_f(Z_1, Z_2)(Z_1 - Z_2) \tag{1.11c}$$

where $A_f(Z_1, Z_2)$ is $n \times n$ real continuous operator having real eigenvalues $\lambda_i(A_f(Z_1, Z_2))$ $(i = 1, 2, \dots, n)$. These real eigenvalues satisfy

$$0 < \delta_f \leq \lambda_i(A_f(Z_1, Z_2)) \leq \Delta_f \tag{1.12c}$$

with δ_f, Δ_f as fixed constants.

Furthermore, these eigenvalues satisfy, for some constant $k(k < 1$, defined later) the "generalized" Routh–Hurtwitz condition (1.9).

Finally, we shall assume that $P(t, X, Y, Z)$ satisfies

$$\|P(t,X,Y,Z)\| \leq p_1(t) + p_2(t)\left\{\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2\right\}^{\rho/2} + p_3(t)\left\{\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2\right\}^{1/2} \tag{1.13}$$

for any X, Y, Z in $\mathbb{R}^n$, where $p_1(t), p_2(t), p_3(t)$ are continuous functions in t and $0 \leq \rho \leq 1$.

Remark 1 The estimate (1.13) reduces to [8, 1.3 (3)] if $p_3(t) = \delta_0$. When specialized to the case $n = 1$, the estimate (1.13) reduces to estimate (4.96) of [11, p. 339] if $p_3(t) = q$.

2 Notations

We shall use the notations as given in [1]. Throughout this paper, δ's and Δ's with or without suffices will denote positive constants whose magnitudes depend on vector functions F, G, H and P. The δ's and Δ's with numerical or alphabetical suffices shall retain fixed magnitudes, while those without suffices are not necessarily the same at each occurrences.

Finally, we shall denote the scalar product $\langle X, Y\rangle$ of any vectors X, Y in $\mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^n x_i y_i$. In particular, $\langle X, X\rangle = \|X\|^2$.

3 Statement of the results

Our first main result in this paper is the following:

Theorem 1 *Suppose $F(0) = G(0) = H(0) = 0$, and that*

(i) there exist $n \times n$ real continuous operators

$$A_f(Z_1, Z_2), \quad B_g(Y_1, Y_2), \quad C_h(X_1, X_2)$$

for any vectors $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ in $\mathbb{R}^n$, such that the functions F, G, H are of class $C(\mathbb{R}^n)$, satisfy (1.11a,b,c), with the eigenvalues, $\lambda_i(A_f(Z_1, Z_2))$, $\lambda_i(B_g(Y_1, Y_2))$, $\lambda_i(C_h(X_1, X_2))$ $(i = 1, 2, \ldots, n)$ satisfying (1.12a,b,c);

(ii) the operators A_f, B_g and C_h are associative and commute pairwise, and

(iii) the vector function P satisfies inequality (1.13) for all X, Y, Z in $\mathbb{R}^n$, where $p_1(t)$, $p_2(t)$ and $p_3(t)$ are continuous functions of t, with $0 \leq \rho < 1$.

Then, there exist constants $\rho_3, \Delta_1, \Delta_2, \Delta_3$ such that if $|p_3(t)| \leq \rho_3$, for all t in $\mathbb{R}$, with ρ_3 chosen small enough, then every solution $X(t)$ of (1.1) with $X(t_0) =$

X_0, $\dot{X}(t_0) = Y_0$, $\ddot{X}(t_0) = Z_0$, and for any constant r, whatever in the range $\frac{1}{2} \le r \le 1$, satisfies

$$\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)^2\|\}^r \le \Delta_1 \exp\{-\Delta_2(t - t_0)\}$$
$$+ \Delta_3 \int_{t_0}^{t} \left\{p_1^{2r}(\tau) + p_2^{2r/(1-\rho)}(\tau)\right\} \exp\{-\Delta_2(t - \tau)\}\, d\tau; \tag{3.1}$$

for all $t \ge t_0 \ge 0$, where $\Delta_1 \equiv \Delta_1(X_0, Y_0, Z_0)$.

Remark 2 (1) When specialized to the case $n = 1$ with P dependent only on t the above estimate (3.1) reduces to the estimate (4.86) of [11, Theorem (4.24) p. 335].

(2) In fact this result generalizes Theorem 1 of [3] if $\rho_3 = \delta_0$: A number of quite important results can be deduced from the above. For example, we have

Corollary 1 *If $P \equiv 0$ and all the conditions of Theorem 1 hold, then every solution $X(t)$ of (1.1) satisfies*

$$\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)^2\|\} \longrightarrow 0 \tag{3.2}$$

as $t \to \infty$, provided that ρ_3 is small enough.

Indeed by setting $\rho_1(t) = 0 = \rho_2(t)$ in (1.13), we have that

$$\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)^2\|\}^r \le \Delta_1 \exp\{-\Delta_2(t - t_0)\}, \quad t \ge t_0$$

from which (3.2) follows on letting $t \to \infty$.

Remark 3 When specialized to the case $n = 1$ with $p_1(t) = p_2(t) = 0$ i.e. satisfying condition (C'') of [11, Theorem 4.25] then the above estimate (3.2) reduces to the estimate (4.97) of [11, Theorem 4.25].

Further, if $P \ne 0$, but such that

$$\int_{t}^{t+\mu} \left\{p_1^{\nu}(\tau) + p_2^{\nu/(1-\rho)}(\tau)\right\} d\tau \longrightarrow 0 \tag{3.3}$$

as $t \to \infty$, then we have

Corollary 2 *Suppose that there are some fixed constants ν $(1 \le \nu \le 2)$, and $\mu > 0$, such that (3.3) is true, and all the conditions of Theorem 1 hold. Then, every solution $X(t)$ of (1.1) satisfies (3.2) as $t \to \infty$.*

Remark 4 This result is a direct generalization of [6, Theorem 2] when specialized to the case $n = 1$. Its proof can be obtained from (3.1) by using an obvious modification of the arguments in [6, §3.2].

The next result is on the ultimate boundedness of solutions of (1.1).

Theorem 2 *Suppose that $F(0) = G(0) = H(0) = 0$ and that all the conditions of Theorem 1 hold. Suppose further that $|p_3(t)| \leq \rho_3$ for all t in $\mathbb{R}$ with ρ_3 sufficiently small and that the functions $p_1(t), p_2(t)$ satisfy*

$$|p_1(t)| \leq \delta_0 \quad \textit{and} \quad |p_2(t)| \leq \delta_1$$

for all t in $\mathbb{R}$.

Then, there exists a constant Δ_4 such that every solution $X(t)$ of (1.1) ultimately satisfies.

$$\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\} \leq \Delta_4 \tag{3.4}$$

Remark 5 (1) If $|p_1(t)| \leq \delta_0$, $|p_2(t)| \leq \delta_1$ and $|p_3(t)| \leq \rho_3$, with ρ_3 sufficiently small, then Theorem 2 reduces to Corollary 3 of [8] for which equation (1.6) was considered.

(2) If $\rho = 0$ in (1.13) we have the estimates (3.6) of [1, Theorem 1] which improves on estimates (3.4) of [1, Theorem 1] and (1.8) of [10, Theorem 1]. Thus, Theorem 2 reduces to Theorem 1 of [1,10] for which (1.6) was considered. Moreover, the estimate (1.13) is a generalization of all the bounds on $P(t, X, Y, Z)$ mentioned earlier.

4 Some preliminary results

We shall state, for completeness, some standard results needed in the proofs of our results.

Lemma 1 (1,§4) *Let Q, D be real symmetric commuting $n \times n$ matrices. Then,*

(i) for any X in $\mathbb{R}^n$,

$$\delta_d\|X\|^2 \leq \langle DX, X\rangle \leq \Delta_d\|X\|^2 \tag{4.1}$$

where δ_d, Δ_d are respectively, the least and greatest eigenvalues, of matrix D;

(ii) the eigenvalues $\lambda_i(QD)$, $(i = 1, 2, \ldots, n)$ of the product matrix QD are all real and satisfy

$$\min_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \leq \lambda_i(QD) \leq \max_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \tag{4.2}$$

(iii) the eigenvalues $\lambda_i(Q + D)$, $(i = 1, 2, \ldots, n)$ of the sum of Q and D are all real and satisfy

$$\begin{aligned}\left\{\min_{1\leq j\leq n} \lambda_j(Q) + \min_{1\leq k\leq n} \lambda_k(D)\right\} &\leq \lambda_i(Q + D) \\ &\leq \left\{\max_{1\leq k\leq n} \lambda_j(Q) + \max_{1\leq k\leq n} \lambda_k(D)\right\}\end{aligned} \tag{4.3}$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are respectively the eigenvalues of Q and D.

5 The function V

Our main tool in the proof of the results is the continuous function $V = V(X,Y,Z)$ defined for any X, Y, Z in $\mathbb{R}^n$ by

$$2V = \beta(1-\beta)\delta_g^2\|X\|^2 + \beta\delta_g\|Y\|^2 + \alpha\delta_g\delta_f^{-1}\|Y\|^2 + \alpha\delta_f^{-1}\|Z\|^2 + \|Z + \delta_f Y + (1-\beta)\delta_g X\|^2. \quad (5.1)$$

where $0 < \beta < 1$ and $\alpha > 0$

The following result is immediate from (5.1):

Lemma 2 *Assume that all the hypothesis on vectors $F(Z), G(Y)$ and $H(X)$ in Theorem 1 are satisfied. Then, there exist positive constants δ_2 and δ_3 such that*

$$\delta_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq 2V \leq \delta_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \quad (5.2)$$

Proof The proof follows if we use Lemma 1 repeatedly and then choose

$$\delta_2 = \min\left\{\beta(1-\beta)\delta_g^2; \delta_g(\beta + \alpha\delta_f^{-1}); \alpha\delta_f^{-1}\right\}$$

and

$$\delta_3 = \max\left\{\delta_g(1-\beta)(1+\delta_g+\delta_f); \delta_g(\beta+\alpha\delta_f^{-1}) + \delta_f[1+\delta_g(1-\beta)+\delta_f]; 1 + \alpha\delta_f^{-1} + \delta_f + \delta_g(1-\beta)\right\}$$

□

6 Proof of Theorem 1

Let us replace system of differential equations of form (1.1) in the equivalent system form

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = -F(Z) - G(Y) - H(X) + P(t,X,Y,Z) \quad (6.1)$$

for which a typical solution will be $(X(t), Y(t), Z(t))$.

To prove Theorem 1, it suffices to show that the function V (defined in (5.1)) satisfies for any solution $(X(t), Y(t), Z(t))$ of (6.1) and for any r in the range $\frac{1}{2} \leq r \leq 1$.

$$\dot{V} \leq -\delta_4\psi^2 + \delta_5\left\{p_1^{2r}(t) + p_2^{\frac{2r}{(1-\rho)}}(t)\right\}\psi^{2(1-r)} \quad (6.2)$$

for some constants δ_4, δ_5 where $\psi^2 = \{\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2\}$. We note that from Lemma 2, (6.2) becomes

$$\dot{V} \leq -\delta_6 V + \delta_7\left\{p_1^{2r}(t) + p_2^{\frac{2r}{(1-\rho)}}(t)\right\}V^{(1-r)} \quad (6.3)$$

with $\delta_6 = \delta_2\delta_4$ and $\delta_7 = \delta_3\delta_5$. If we choose $U = V^r$, this reduces to

$$\dot{U} \leq -r\delta_6 U + r\delta_7\left\{p_1^{2r}(t) + p_2^{\frac{2r}{(1-\rho)}}(t)\right\}. \quad (6.3)$$

which can be solved for U to obtain

$$U(t) \leq U(t_0)\exp\{-r\delta_6(t-t_0)\} + \Delta_5 \int_{t_0}^{t} \left\{p_1^{2r}(\tau) + p^{\frac{2r}{(1-\rho)}}(\tau)\right\} \exp\{-r\delta_6(t-\tau)\}\, d\tau \tag{6.4}$$

for all $t \geq t_0$.

Rewriting this with $V^r = U$ and applying Lemma 2, we shall obtain (3.1) with

$$\Delta_1 = \delta\{\|X(t_0)\|^2 + \|Y(t_0)\|^2 + \|Z(t_0)\|^2\}^r;$$
$$\Delta_2 = r\delta_6 \quad \text{and} \quad \Delta_3 = \delta\Delta_5$$

Thus the proof of Theorem 1 is complete as soon as inequality (6.2) is proved.

7 The derivative of V and the proof of (6.2)

Let $(X(t), Y(t), Z(t))$ be any solution of (6.1). The total derivative of V, with respect to t along the solution path after simplification is

$$\dot{V} = -W_1 - W_2 - W_3 - W_4 - W_5 - W_6 - W_7 + W_8 \tag{7.1}$$

where

$$\begin{aligned}
W_1 &= \{\gamma_1\delta_g(1-\beta)\langle X, H(X)\rangle + \eta_1\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle \\
&\quad + \xi_1\alpha\delta_f^{-1}\langle Z, F(Z)\rangle + \langle Z, F(Z) - \delta_f Z\rangle\Big\} \\
W_2 &= \Big\{\gamma_2\delta_g(1-\beta)\langle X, H(X)\rangle + \xi_2\alpha\delta_f^{-1}\langle Z, F(Z)\rangle + (1+\alpha\delta_f^{-1})\langle Z, H(X)\rangle\Big\} \\
W_3 &= \{\gamma_3\delta_g(1-\beta)\langle X, H(X)\rangle + \eta_2\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle + \delta_f\langle Y, H(X)\rangle\} \\
W_4 &= \Big\{\gamma_4\delta_g(1-\beta)\langle X, H(X)\rangle + \xi_3\alpha\delta_f^{-1}\langle Z, F(Z)\rangle \\
&\quad + \delta_g(1-\beta)\langle X, F(Z) - \delta_f Z\rangle\} \\
W_5 &= \{\gamma_5\delta_g(1-\beta)\langle X, H(X)\rangle + \eta_3\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle \\
&\quad + \delta_g(1-\beta)\langle X, G(Y) - \delta_g Y\rangle\} \\
W_6 &= \Big\{\xi_4\alpha\delta_f^{-1}\langle Z, F(Z)\rangle + \eta_4\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle \\
&\quad + (1+\alpha\delta_f^{-1})\langle Z, G(Y) - \delta_g Y\rangle\Big\} \\
W_7 &= \Big\{\xi_5\alpha\delta_f^{-1}\langle Z, F(Z)\rangle + \eta_5\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle + \delta_f\langle Y, F(Z) - \delta_f Z\rangle\} \\
W_8 &= \Big\{\langle(1-\beta)\delta_g X + \delta_f Y + (1+\alpha\delta_f^{-1})Z,\ P(t, X, Y, Z)\rangle\Big\}
\end{aligned}$$

with $\xi_i, \eta_i, \gamma_i;\ (i = 1, 2, 3, 4, 5)$ are strictly positive constants such that

$$\sum_{i=1}^{5} \xi_i = 1; \quad \sum_{i=1}^{5} \eta_i = 1 \quad \text{and} \quad \sum_{i=1}^{5} \gamma_i = 1.$$

To arrive at (6.2), we first prove the following:

Lemma 3 *Subject to a conveniently chosen value of k in (1.9), we have for all X, Y, Z in $\mathbb{R}^n$*

$$W_j \geq 0, \quad (j = 2, 3, 4, 5, 6, 7).$$

Proof For strictly positive constants k_1, k_2, conveniently chosen later, we have

$$\begin{aligned} &\langle (1 + \alpha\delta_f^{-1})Z, H(X)\rangle = \\ &= \|k_1(1 + \alpha\delta_f^{-1})^{1/2} Z + 2^{-1}k_1^{-1}(1 + \alpha\delta_f^{-1})^{1/2} H(X)\|^2 \\ &- \langle k_1^2(1 + \alpha\delta_f^{-1})Z, Z\rangle - \langle 4^{-1}k_1^{-2}(1 + \alpha\delta_f^{-1})H(X),\ H(X)\rangle \end{aligned} \qquad (7.2a)$$

and

$$\begin{aligned} \langle \delta_f Y, H(X)\rangle &= \|k_2\delta_f^{1/2} Y + 2^{-1}k_2^{-1}\delta^{1/2} H(X)\|^2 \\ &- \langle k_2^2\delta_f Y, Y\rangle - \langle 4^{-1}k_2^{-2}\delta_f H(X), H(X)\rangle. \end{aligned} \qquad (7.2b)$$

Now, using (1.11) and the assumptions that $F(0) = G(0) = H(0) = 0$, we have

$$\begin{aligned} W_2 &= \|k_1(1 + \alpha\delta_f^{-1})^{1/2} Z + 2^{-1}k_1^{-1}(1 + \alpha\delta_f^{-1})^{1/2} H(X)\|^2 \\ &+ \langle Z, \xi_2\alpha\delta_f^{-1}F(Z) - k_1^2(1 + \alpha\delta_f^{-1})Z\rangle \\ &+ \langle H(X), \gamma_2\delta_g(1 - \beta)X - 4^{-1}k_1^{-2}(1 + \alpha\delta_f^{-1})H(X)\rangle \end{aligned} \qquad (7.3a)$$

and

$$\begin{aligned} W_3 &= \|k_2\delta_f^{1/2} Y + 2^{-1}k_2^{-1}\delta^{1/2} H(X)\|^2 \\ &+ \langle Y, \eta_2\delta_f[G(Y) - \delta_g(1 - \beta)Y] - k_2^2\delta_f Y\rangle \\ &+ \langle H(X), \gamma_3\delta_g(1 - \beta)X - 4^{-1}k_2^{-2}\delta_f H(X)\rangle. \end{aligned} \qquad (7.3b)$$

Furthermore, by using Lemma 1 repeatedly, we obtain for all X, Z in $\mathbb{R}^n$,

$$W_2 \geq 0 \qquad (7.4a)$$

if $k_1^2 \leq \frac{\xi_2\alpha\delta_f}{\alpha+\delta_f}$ with

$$\Delta_h \leq \frac{4\gamma_2\xi_2\alpha(1 - \beta)\delta_f^2\delta_g}{(\alpha + \delta_f)^2} \qquad (7.5a)$$

and for all X, Y in $\mathbb{R}^n$,

$$W_3 \geq 0. \qquad (7.4b)$$

If $k_2^2 \leq \eta_2\beta\delta_g$ with

$$\Delta_h \leq 4\gamma_3\eta_2\beta(1 - \beta)\delta_g^2/\delta_f. \qquad (7.5b)$$

Combining all the inequalities in (7.3) and (7.4), we have for all X, Y, Z in $\mathbb{R}^n$, $W_2 \geq 0$ and $W_3 \geq 0$, if $\Delta_h \leq k\delta_f\delta_g$ with

$$k = \min\left\{\frac{4\gamma_2\xi_2\alpha(1 - \beta)\delta_f}{(\alpha + \delta_f)^2}; \frac{4\eta_2\gamma_3\beta(1 - \beta)\delta_g}{\delta_f^2}\right\} < 1. \qquad (7.6)$$

To complete the proof of Lemma 3, we need to show that for all X, Y, Z in $\mathbb{R}^n$

$$W_i \geq 0 \quad (i = 4, 5, 6, 7).$$

By hypothesis (1.11) the assumptions that $F(0) = G(0) = H(0) = 0$, and for strictly positive constants k_3, k_4, k_5, k_6 conveniently chosen later, we have

$$\begin{aligned}
\langle \delta_g(1-\beta)X, F(Z) - \delta_f Z\rangle &= \langle \delta_g(1-\beta)X, [A_f(Z,O) - \delta_f I]Z\rangle \\
&= \|2^{-1}k_3^{-1}\delta_g^{1/2}(1-\beta)^{1/2}[A_f(Z,O) - \delta_f I]^{1/2}X \\
&\quad + k_3\delta_g^{1/2}(1-\beta)^{1/2}[A_f(Z,O) - \delta_f I]^{1/2}Z\|^2 \\
&\quad - \langle 4^{-1}k_3^{-2}\delta_g(1-\beta)[A_f(Z,O) - \delta_f I]X, X\rangle \\
&\quad - \langle k_3^2\delta_g(1-\beta)[A_f(Z,O) - \delta_f I]Z, Z\rangle \qquad (7.7a)
\end{aligned}$$

$$\begin{aligned}
\delta_g(1-\beta)\langle X, G(Y) - \delta_g Y\rangle &= \langle \delta_g(1-\beta)X, [B_g(Y,O) - \delta_g I]Y\rangle \\
&= \|2^{-1}k_4^{-1}\delta_g^{1/2}(1-\beta)^{1/2}[B_g(Y,O) - \delta_g I]^{1/2}X \\
&\quad + k_4\delta_g^{1/2}(1-\beta)^{1/2}[B_g(Y,O) - \delta_g I]^{1/2}Y\|^2 \\
&\quad - \langle 4^{-1}k_4^{-2}\delta_g(1-\beta)[B_g(Y,O) - \delta_g I]X, X\rangle \\
&\quad - \langle k_4^2\delta_g(1-\beta)[B_g(Y,O) - \delta_g I]Y, Y\rangle \qquad (7.7b)
\end{aligned}$$

$$\begin{aligned}
(1+\alpha\delta_f^{-1})\langle Z, G(Y) - \delta_g Y\rangle &= \langle (1+\alpha\delta_f^{-1})Z, [B_g(Y,O) - \delta_g I]Y\rangle \\
&= \|2^{-1}k_5^{-1}(1+\alpha\delta_f^{-1})^{1/2}[B_g(Y,O) - \delta_g I]^{1/2}Z \\
&\quad + k_5(1+\alpha\delta_f^{-1})^{1/2}[B_g(Y,O) - \delta_g I]^{1/2}Y\|^2 \\
&\quad - \langle 4^{-1}k_5^{-2}(1+\alpha\delta_f^{-1})[B_g(Y,O) - \delta_g I]Z, Z\rangle \\
&\quad - \langle k_5^2(1+\alpha\delta_f^{-1})[B_g(Y,O) - \delta_g I]Y, Y\rangle \qquad (7.7c)
\end{aligned}$$

$$\begin{aligned}
&\delta_f\langle Y, F(Z) - \delta_f Z\rangle = \langle \delta_f Y, [A_f(Z,O) - \delta_f I]Z\rangle \\
&= \|2^{-1}k_6^{-1}\delta_f^{1/2}[A_f(Z,O) - \delta_f I]^{1/2}Y + k_6\delta_f^{1/2}[A_f(Z,O) - \delta_f I]^{1/2}Z\|^2 \\
&\quad - \langle 4^{-1}k_6^{-2}\delta_f[A_f(Z,O) - \delta_f I]Y, Y\rangle \\
&\quad - \langle k_6^2\delta_f[A_f(Z,O) - \delta_f I]Z, Z\rangle. \qquad (7.7d)
\end{aligned}$$

Thus,

$$\begin{aligned}
W_4 &= \|2^{-1}k_3^{-1}\delta_g^{1/2}(1-\beta)^{1/2}[A_f(Z,O) - \delta_f I]^{1/2}X \\
&\quad + k_3\delta_g^{1/2}(1-\beta)^{1/2}[A_f(Z,O) - \delta_f I]^{1/2}Z\|^2 \\
&\quad + \langle X, \{\gamma_4\delta_g(1-\beta)C_h(X,O) - 4^{-1}k_3^{-2}\delta_g(1-\beta)[A_f(Z,O) - \delta_f I]\}X\rangle \\
&\quad + \langle Z, \{\xi_3\alpha\delta_g^{-1}A_f(Z,O) - k_3^2\delta_g(1-\beta)[A_f(Z,O) - \delta_f I]\}Z\rangle \qquad (7.8a)
\end{aligned}$$

$$
\begin{aligned}
W_5 = \|2^{-1}k_4^{-1}\delta_g^{1/2}(1-\beta)^{1/2}[B_g(Y,O)-\delta_g I]^{1/2}X \\
+ k_4\delta_g^{1/2}(1-\beta)^{1/2}[B_g(Y,O)-\delta_g I]^{1/2}Y\|^2 \\
+ \langle X, \{\gamma_5\delta_g(1-\beta)C_h(X,0) - 4^{-1}k_4^{-2}\delta_g(1-\beta)[B_g(Y,O)-\delta_g I]\}X\rangle \\
+ \langle Y, \{\eta_3\delta_f[B_g(Y,O)-\delta_g(1-\beta)I] - k_4^2\delta_g(1-\beta)[B_g(Y,O)-\delta_g I]\}Y\rangle
\end{aligned} \tag{7.8b}
$$

$$
\begin{aligned}
W_6 = \|2^{-1}k_5^{-1}(1+\alpha\delta_f^{-1})^{1/2}[B_g(Y,O)-\delta_g I]^{1/2}Z \\
+ k_5(1+\alpha\delta_f^{-1})^{1/2}[B_g(Y,O)-\delta_g I]^{1/2}Y\|^2 \\
+ \langle Z, \{\xi_4\alpha\delta_g^{-1}A_f(Z,O) - 4^{-1}k_5^{-2}(1+\alpha\delta_f^{-1})[B_g(Y,O)-\delta_g I]\}Z\rangle \\
+ \langle Y, \{\eta_4\delta_f[B_g(Y,O)-\delta_g(1-\beta)I] \\
- k_5^2(1+\alpha\delta_f^{-1})[B_g(Y,O)-\delta_g I]\}Y\rangle
\end{aligned} \tag{7.8c}
$$

and

$$
\begin{aligned}
W_7 = \|2^{-1}k_6^{-1}\delta_f^{1/2}[A_f(Z,O)-\delta_f I]^{1/2}Y + k_6\delta_f^{1/2}[A_f(Z,0)-\delta_f I]^{1/2}Z\|^2 \\
+ \langle Y, \{\eta_5\delta_f[B_g(Y,O)-\delta_g(1-\beta)I] - 4^{-1}k_6^{-2}\delta_f[A_f(Z,O)-\delta_f I]\}Y\rangle \\
+ \langle Z, \{\xi_5\alpha\delta_f^{-1}A_f(Z,O) - k_6^2\delta_f[A_f(Z,O)-\delta_f I]\}Z\rangle.
\end{aligned} \tag{7.8d}
$$

Thus, for all X, Z in $\mathbb{R}^n$

$$W_4 \geq 0 \tag{7.9a}$$

if

$$\frac{\Delta_f - \delta_f}{4\gamma_4\delta_h} \leq k_3^2 \leq \frac{\xi_3\alpha}{(1-\beta)(\delta_g-\delta_f)}. \tag{7.10a}$$

For all X, Y in $\mathbb{R}^n$

$$W_5 \geq 0 \tag{7.9b}$$

if

$$\frac{\Delta_g - \delta_g}{4\gamma_5\delta_h} \leq k_4^2 \leq \frac{\eta_3\beta\delta_f}{(1-\beta)(\Delta_g-\delta_g)}\,. \tag{7.10b}$$

For all Y, Z in $\mathbb{R}^n$

$$W_6 \geq 0 \tag{7.9c}$$

if

$$\frac{\delta_g(\alpha+\delta_f)(\Delta_g-\delta_g)}{4\xi_4\alpha\delta_f^2} \leq k_5^2 \leq \frac{\beta\eta_4\delta_g\delta_f^2}{(\alpha+\delta_f)(\Delta_g-\delta_g)}\,. \tag{7.10c}$$

Also, for all Y, Z in $\mathbb{R}^n$

$$W_7 \geq 0 \tag{7.9d}$$

if

$$\frac{\Delta_f - \delta_f}{4\eta_5\beta\delta_g} \leq k_6^2 \leq \frac{\alpha\xi_5}{\delta_f(\Delta_f-\delta_f)}\,. \tag{7.10d}$$

This completes the proof of Lemma 3. □

We are now left with the estimates for W_1 and W_8.

From (7.1), we clearly have

$$W_1 \geq \gamma_1\delta_g\delta_h(1-\beta)\|X\|^2 + \eta_1\delta_f\delta_g\beta\|Y\|^2 + \xi_1\alpha\|Z\|^2$$
$$\geq \delta_8(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \qquad (7.11)$$

where $\delta_8 = \min\{\gamma_1\delta_g\delta_h; \eta_1\delta_f\delta_g\beta; \xi_1\alpha\}$. For the remaining part of the proof of (6.2); let us for convenience denote $(\|X\|^2 + \|Y\|^2 + \|Z\|^2)$ by ψ^2.

Since $P(t, X, Y, Z)$ satisfies (1.5), Schwarz's inequality gives for W_8.

$$|W_8| \leq \left\{(1-\beta)\delta_g\|X\| + \delta_f\|Y\| + (1+\alpha\delta_1^{-1})\|Z\|\right\}\|P(t, X, Y, Z)\|$$
$$\leq 3^{1/2}\delta_9\left\{p_3(t)\psi^2 + p_2(t)\psi^{(1+\rho)} + p_1(t)\psi\right\}; \qquad (7.12)$$

where $\delta_9 = \max\left\{(1-\beta)\delta_g; \delta_f; (1+\alpha\delta_f^{-1})\right\}$.

Combining inequalities (7.3), (7.11) and (7.13) with the assumption that $|p_3(t)| \leq \rho_3$ for all t in $\mathbb{R}$, we obtain from (7.1) that

$$\dot{V} \leq -(\delta_8 - 3^{1/2}\delta_9\rho_3)\psi^2 + 3^{1/2}\delta_9\left\{p_2(t)\psi^{(1+\rho)} + p_1(t)\psi\right\}. \qquad (7.14)$$

This we can rewrite as

$$\dot{V} \leq -\delta_{10}\psi^2 + \psi_1 + \psi_2 \qquad (7.15)$$

where

$$3\delta_{10} = \delta_8 - 3^{1/2}\delta_9\rho_3, \qquad \psi_1 = \{\delta_{11}p_1(t) - \delta_{10}\psi\}\psi;$$

and

$$\psi_2 = \left\{\delta_{11}p_2(t)\psi^{(1+\rho)} - \delta_{10}\psi^2\right\}.$$

If we choose ρ_3 small enough such that $\delta_{10} > 0$ (following [6, p. 306]), with the necessary modification we obtain

$$\psi_1 \leq \delta_{12}\psi^{2(1-r)}p_1^{2r}(t) \qquad (7.16a)$$

and

$$\psi_2 \leq \delta_{13}\psi^{2(1-r)}p_2^{2r/(1-\rho)}(t) \qquad (7.16b)$$

for any constant r in the range $\frac{1}{2} \leq r \leq 1$.

Thus, (7.15) reduces to

$$\dot{V} \leq -\delta_{10}\psi^2 + \delta_{14}\left\{p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t)\right\}\psi^{2(1-r)} \qquad (7.17)$$

with

$$\delta_{14} = \max\{\delta_{12}; \delta_{13}\}$$

This is (6.2) with $\delta_4 = \delta_{10}$ and $\delta_5 = \delta_{14}$.

8 Proof of Theorem 2

As pointed out in [1], to prove Theorem 2, if suffices to prove that the function V satisfies

(i) $V(X,Y,Z) \to \infty$ as $(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \to \infty$; and

(ii) $\dot{V} \leq -1$

along paths of any solution $(X(t), Y(t), Z(t))$ of (6.1) for which $(\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2)$ is large enough. We only need to concern ourselves with property (ii), since by Lemma 2, inequality (5.3), property (i) has been taken care of.

If all the conditions of Theorem 1 are satisfied, then, for any solution $(X(t), Y(t), Z(t))$ of (6.1), $\dot{V}$ satisfies inequality (7.17). That is

$$\dot{V} \leq -\delta_{10}\psi^2 + \delta_{14}\left\{p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t)\right\}\psi^{2(1-r)}$$

for any r in the range $\frac{1}{2} \leq r \leq 1$.

Now, if $p_1(t)$ and $p_2(t)$ are bounded for all t in $\mathbb{R}$, then there exists some constant $\delta_{15} > 0$ such that

$$\dot{V} \leq -\delta_{10}\psi^2 + \delta_{15}\psi^{2(1-r)} \leq -1$$

if

$$\psi \geq \delta_{16} > (\delta_{10}^{-1}\delta_{15})^{1/2r}.$$

Thus property (ii) is proved for V, and this completes the proof of Theorem 2.

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