Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

František Machala; Vladimír Slezák

Join-closed and meet-closed subsets in complete lattices

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 43 (2004), No. 1, 113--117

Persistent URL: http://dml.cz/dmlcz/132940

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **43** (2004) 113–117

Join-Closed and Meet-Closed Subsets in Complete Lattices *

František MACHALA [1], Vladimír SLEZÁK [2]

Department of Algebra and Geometry,
Faculty of Science,
Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: [1]*F.Machala@seznam.cz*
[2]*slezakv@seznam.cz*

(Received January 21, 2004)

Abstract

To every subset A of a complete lattice L we assign subsets $J(A)$, $M(A)$ and define join-closed and meet-closed sets in L. Some properties of such sets are proved. Join- and meet-closed sets in power-set lattices are characterized. The connections about join-independent (meet-independent) and join-closed (meet-closed) subsets are also presented in this paper.

Key words: Complete lattices, join-closed and meet-closed sets.

2000 Mathematics Subject Classification: 06B23, 08A02, 08A05

Let $(L, \leq)$ be a complete lattice in which $\bigvee A, \bigwedge A$ denote the supremum and the infimum of any subset $A \subseteq L$, respectively. The least and the greatest elements in $(L, \leq)$ are denoted by 0, 1, respectively. If $A \subseteq L$, $A \neq \emptyset$, then we put $A_x := A \setminus \{x\}$ for $x \in A$ and

$$J(A) = \Big\{ \bigvee A_x \mid x \in A \Big\}, \quad M(A) = \Big\{ \bigwedge A_x \mid x \in A \Big\}.$$

Instead of $M(J(A))$, $J(M(A))$ we write just $MJ(A), JM(A)$. If we put $P_x = (J(A))_{\bigvee A_x} = \{\bigvee A_a \mid a \in A_x\}$, then $MJ(A) = \{\bigwedge P_x \mid x \in A\}$. Dually, $R_x = (M(A))_{\bigwedge A_x} = \{\bigwedge A_a \mid a \in A_x\}$ and $JM(A) = \{\bigvee R_x \mid x \in A\}$. It is easy to see that $x \leq \bigwedge P_x$ and $\bigvee R_x \leq x$ for all $x \in A$, thus $\bigvee R_x \leq \bigwedge P_x$.

*Supported by the Council of Czech Government J14/98:153100011.

Proposition 1 *If $A \subseteq L$, $|A| > 2$, then $\bigvee M(A) \leq \bigwedge J(A)$.*

Proof Consider $x \in A$ and $z \in A_x$. By assumption, there exists an element $y \in A_x$ distinct from z. From $x, z \in A_y$ we get $\bigwedge A_z \leq \bigvee R_y$ and $\bigwedge P_y \leq \bigvee A_x$, thus $\bigwedge A_z \leq \bigvee A_x$. We also have $\bigwedge A_x \leq \bigvee A_x$ and hence $\bigwedge A_z \leq \bigvee A_x$ for all $z \in A$. We have obtained the relation $\bigvee M(A) \leq \bigvee A_x$ holding for all $x \in A$. Thus $\bigvee M(A) \leq \bigwedge J(A)$. □

Definition 1 A set $A \subseteq L$ is said to be *meet-closed* iff $MJ(A) = A$. Similarly, $A \subseteq L$ is *join-closed* iff $JM(A) = A$. In brief, we call them M-closed and J-closed, respectively.

Remark 1 A set $A = \{x\}$ is M-closed (J-closed) if and only if $x = 1$ ($x = 0$). If $A = \{x, y\}$, then $J(A) = A = M(A)$ and A is both M-closed and J-closed.

Proposition 2 *A subset $A \subseteq L$ is M-closed if and only if $x = \bigwedge P_x$ for all $x \in A$.*

Proof 1. If $x = \bigwedge P_x$ for all $x \in A$, then $MJ(A) = \{x \mid x \in A\} = A$.

2. Assume that $MJ(A) = A$ and consider $x \in A$. It follows from $\bigwedge P_x \in A$ that $\bigwedge P_x = y$ for a certain $y \in A$ and since $x \leq \bigwedge P_x$ we have $x \leq y$. Let us suppose that $x \neq y$. Then $\bigvee A_y \in P_x$ which yields $y \leq \bigvee A_y$. From $y \leq \bigwedge P_y$ we obtain $y \leq \bigwedge J(A)$. Consequently (with respect to $P_x \subseteq J(A)$), $\bigwedge J(A) \leq \bigwedge P_x = y$ and $y = \bigwedge J(A)$. There exists $z \in A$ such that $x = \bigwedge P_z$. Then $y \leq \bigwedge P_z$, i. e. $y \leq x$ which contradicts the assumption $x < y$. Thus $x = \bigwedge P_x$. □

Remark 2 The notions of M-closed and J-closed sets are dual, hence each assertion about M-closed and J-closed sets admits its corresponding dual one. Therefore, a set $A \subseteq L$ is J-closed iff $x = \bigvee R_x$ for all $x \in A$. In what follows the dual results will not be stated explicitly.

Proposition 3 *If $A \subseteq L$, then the set $M(A)$ is M-closed.*

Proof If we put $Q_x = (JM(A))_{\bigvee R_x} = \{\bigvee R_y \mid y \in A_x\}$, then $MJM(A) = \{\bigwedge Q_x \mid x \in A\}$. Consider $x \in A$. Then $\bigwedge Q_x \leq \bigvee R_y \leq y$ for all $y \in A_x$ which implies $\bigwedge Q_x \leq \bigwedge A_x$. Furthermore, $\bigwedge A_x \in R_y$, thus $\bigwedge A_x \leq \bigvee R_y$ and $\bigwedge A_x \leq \bigwedge Q_x$. We have obtained $\bigwedge Q_x = \bigwedge A_x$ and $MJM(A) = \{\bigwedge A_x \mid x \in A\} = M(A)$. □

Proposition 4 *If a set $A \subseteq L$, $|A| > 1$, is M-closed, then $\bigwedge J(A) = \bigwedge A$.*

Proof Let us consider $x \in A$. Then there exists $y \in A_x$ such that $\bigwedge A \leq y \leq \bigvee A_x$. Thus $\bigwedge A \leq \bigwedge J(A)$. We also have $P_x \subseteq J(A)$ and $x = \bigwedge P_x$ which yields $\bigwedge J(A) \leq x$ and $\bigwedge J(A) \leq \bigwedge A$. □

Remark 3 A set $A \subseteq L$ is M-closed if and only if $A \cup \{\bigwedge A\}$ is M-closed.

Proposition 5 *Every subset of an M-closed set containing at least two elements is M-closed.*

Proof Let X be a subset of an M-closed set $A \subseteq L$. If $|X| = 2$, then X is M-closed by Remark 1. Let $|X| > 2$. Consider $x \in X$ and denote $Q_x = \{\bigvee X_l \mid l \in X_x\}$, $y = \bigwedge Q_x$. Since $x \leq \bigvee X_l$ for all $l \in X_x$ we have $x \leq y$. Obviously, $X_l \subseteq A_l$ for all $l \in X_x$, which yields $y \leq \bigvee X_l \leq \bigvee A_l$. If $m \in A \setminus X$, then $X_l \subseteq X \subseteq A_m$ and $y \leq \bigvee X_l \leq \bigvee A_m$ for any $l \in X_x$. If $a \in A_x$, then either $a \in X_x$ or $a \in A \setminus X$. Thus $y \leq \bigvee A_a$ and $y \leq \bigwedge P_x = x$. It means that $x = \bigwedge Q_x$ and the set X is M-closed. □

Proposition 6 *Let $A \subseteq L$, $|A| > 1$, be an M-closed set, X_i, $i \in J$, be non-empty subsets of A such that $\bigcap_{i \in J} X_i = \emptyset$ and $\mathcal{X} = \{\bigvee X_i \mid i \in J\}$. Then $\bigwedge \mathcal{X} = \bigwedge A$.*

Proof It is easy to see that $\bigwedge A \leq \bigwedge \mathcal{X}$. For each $i \in J$ and $x \in A \setminus X_i$ we have $X_i \subseteq A_x$ and hence $\bigvee X_i \leq \bigvee A_x$. It follows from $\bigcap_{i \in J} X_i = \emptyset$ that $\bigcup_{i \in J}(A \setminus X_i) = A$ and $\bigwedge \mathcal{X} \leq \bigvee A_y$ for all $y \in A$. Thus $\bigwedge \mathcal{X} \leq \bigwedge J(A)$ and, according to Proposition 4, $\bigwedge \mathcal{X} \leq \bigwedge A$. □

Corollary 1 *Let $A \subseteq L$, $|A| > 1$, be an M-closed set. Then $\bigwedge X = \bigwedge A$ for any $X \subseteq A$, $|X| \geq 2$.*

Definition 2 A subset $A \subseteq L$ is said to be *join-independent (meet-independent)* if and only if $x \not\leq \bigvee A_x$ ($\bigwedge A_x \not\leq x$) for all $x \in A$.

Remark 4 The concept of independence have been studied in various types of lattices motivated by applications in algebra and geometry (refer to [1, 2, 3, 4, 8]). Definition 2 is given in [5] and some other related results are presented in [6, 7].

Remark 5 Join- and meet-independence are dual notions, hence each of the following results holds also dually.

Remark 6 If a set $A \subseteq L$ is join-independent, then $J(A)$ is meet-independent. (See [5, 6].)

Proposition 7 *If a set $A \subseteq L$, $|A| > 2$, is meet-independent, then it is not M-closed.*

Proof Let A be a meet-independent set. Suppose that it is also M-closed. Then $x = \bigwedge P_x$ for all $x \in A$. It follows from $P_x \subseteq J(A)$ that $\bigwedge J(A) \leq \bigwedge P_x$. Since $\bigvee M(A) \leq \bigwedge J(A)$ (Proposition 1) we have $\bigwedge A_x \leq \bigvee M(A) \leq \bigwedge J(A) \leq x$ which contradicts the meet-independence of A. □

Let A be a set. In what follows we denote the power set of A by $\mathcal{P}(A)$. Then $(\mathcal{P}(A), \subseteq)$ is a complete lattice with lattice operations $\cup, \cap$.

Proposition 8 *Let A be a set and $X = \{X_i \mid i \in J\} \subseteq \mathcal{P}(A)$ where $|J| > 1$. The set X is M-closed in $(\mathcal{P}(A), \subseteq)$ if and only if $X_k \cap X_l = \bigcap X$ for every two distinct elements k, l of J.*

Proof It is evident that $J(X) = \{\bigcup X_{X_i} \mid i \in J\} = \{\bigcup_{j \in J \setminus \{i\}} X_j \mid i \in J\}$, $P_{X_i} = \{\bigcup X_{X_j} \mid j \in J \setminus \{i\}\} = \{\bigcup_{m \in J \setminus \{j\}} X_m \mid j \in J \setminus \{i\}\}$ and $MJ(X) = \{\bigcap P_{X_i} \mid i \in J\}$.

1. Assume that $X = MJ(X)$. If $|J| = 2$, then $X = \{X_1, X_2\}$ and $\bigcap X = X_1 \cap X_2$. For $|J| > 2$ we have $X_i = \bigcap P_{X_i}$ for all $i \in J$ by Proposition 2. Consider any two distinct elements $k, l \in J$. Then $\bigcap X \subseteq X_k \cap X_l$. Let $x \in X_k \cap X_l$. If $i \in J$ is distinct from k, l, then for each $j \in J \setminus \{i\}$ either $X_k \subseteq \bigcup X_{X_j}$ or $X_l \subseteq \bigcup X_{X_j}$ and hence $x \in \bigcap P_{X_i}$ and $x \in X_i$. Since it holds for all $i \in J$ distinct from k, l we have $x \in \bigcap X$ which yields $\bigcap X = X_k \cap X_l$.

2. Assume that $\bigcap X = X_k \cap X_l$ for any $k, l \in J$, $k \neq l$. In case of $|J| = 2$ this equality always holds and X is M-closed by Remark 1. Let $|J| > 2$. Consider $i \in J$ and denote $X^j = \{X_m \mid m \in J \setminus \{i, j\}\}$ for all $j \in J \setminus \{i\}$. Then $P_{X_i} = \{X_i \cup (\bigcup X^j) \mid j \in J \setminus \{i\}\}$. Let $x \in \bigcap \{\bigcup X^j \mid j \in J \setminus \{i\}\}$, i. e. $x \in X_k$ for a certain $k \in J \setminus \{i\}$. However, x belongs to another set X_l, $l \in J \setminus \{i\}, l \neq k$. Indeed, otherwise we get $x \notin \bigcup X^k$ which is a contradiction. Thus $x \in X_k \cap X_l$ and, by assumption, also $x \in X_i$. It follows from $X_i \subseteq \bigcap P_{X_i}$ that $X_i = \bigcap P_{X_i}$ and the set X is M-closed by Proposition 2. □

Let $A \subseteq L$ be join-independent set. Consider a mapping $\psi : \mathcal{P}(A) \to L$ given by $\psi(X) = \bigvee X$ for all non-empty subsets $X \in \mathcal{P}(A)$ and $\psi(\emptyset) = \bigwedge A$. According to [5], $(\psi(\mathcal{P}(A)), \leq)$ is a complete lattice isomorphic to $(\mathcal{P}(A), \subseteq)$ which is also a complete join subsemilattice of $(L, \leq)$.

Proposition 9 *Let a set $A \subseteq L$ be join-independent and consider subsets $X = \{X_i \mid i \in J\} \subseteq \mathcal{P}(A)$, $\mathcal{X} = \{\psi(X_i) \mid i \in J\} \subseteq L$. The following statements are equivalent:*

(i) *X is join-independent in $(\mathcal{P}(A), \subseteq)$.*

(ii) $X_i \not\subseteq \bigcup_{j \in J \setminus \{i\}} X_j$ *for all $i \in J$.*

(iii) *$\mathcal{X}$ is join-independent in $(L, \leq)$.*

Proof It is obvious.

Proposition 10 *Let a join-independent set $A \subseteq L$, $|A| > 2$, be M-closed in $(L, \leq)$. The following statements are equivalent:*

(i) *The set $L_1 = \psi(\mathcal{P}(A))$ is a sublattice in $(L, \leq)$.*

(ii) *The image of any M-closed set in $(\mathcal{P}(A), \subseteq)$ of cardinality 3 under the mapping ψ is M-closed in $(L, \leq)$.*

(iii) *The image of any join-independent M-closed set in $(\mathcal{P}(A), \subseteq)$ of cardinality 3 under the mapping ψ is M-closed in $(L, \leq)$.*

Proof $(i) \Rightarrow (ii)$ Let $X = \{X_1, X_2, X_3\} \subseteq \mathcal{P}(A)$ be an M-closed set. According to Proposition 2, for each $i \in \{1,2,3\}$ we have $\bigcap P_{X_i} = (X_i \cup X_j) \cap (X_i \cup X_k) = X_i$ where $j,k \in \{1,2,3\}$ and i,j,k are pairwise distinct. If $\psi(X) = \{\psi(X_1), \psi(X_2), \psi(X_3)\}$, then in $(L, \leq)$ there we have

$$\begin{aligned}\bigwedge P_{\psi(X_i)} &= (\psi(X_i) \vee \psi(X_j)) \wedge (\psi(X_i) \vee \psi(X_k)) = \psi(X_i \cup X_j) \wedge \psi(X_i \cup X_k) \\ &= \psi((X_i \cup X_j) \cap (X_i \cup X_k)) = \psi(X_i).\end{aligned}$$

Thus, by Proposition 2, the set $\psi(X)$ is M-closed in $(L, \leq)$.

$(ii) \Rightarrow (iii)$ Obvious.

$(iii) \Rightarrow (i)$ Since $\psi(\mathcal{P}(A))$ is a join subsemilattice in $(L, \leq)$ it suffices to prove that the infimum of any two elements of L_1 in $(L, \leq)$ belongs to L_1. Consider $\psi(X_1), \psi(X_2)$ for $X_1, X_2 \in \mathcal{P}(A)$. Let us put $Y = X_1 \cap X_2$. If for instance $Y = X_1$, then $X_1 \subseteq X_2$ and $\psi(X_1) = \psi(X_1) \wedge \psi(X_2)$. Further let us suppose that $Y \neq X_1, X_2$ which also means that $X_1, X_2 \neq \emptyset$. If $Y = \emptyset$, then $\psi(X_1) \wedge \psi(X_2) = \bigwedge A$ by Proposition 6. Assume that $Y \neq \emptyset$ and denote $X_1' = X_1 \setminus Y$, $X_2' = X_2 \setminus Y$, $X = \{Y, X_1', X_2'\}$. The set X is join-independent in $(\mathcal{P}(A), \subseteq)$ by Proposition 9. It follows from $Y \cap X_1' = Y \cap X_2' = X_1' \cap X_2' = \bigcap X = \emptyset$ that (by Proposition 8) X is M-closed in $(\mathcal{P}(A), \subseteq)$. According our assumption, the set $\psi(X) = \{\psi(Y), \psi(X_1'), \psi(X_2')\}$ is M-closed in $(L, \leq)$. Thus $\psi(X_1) \wedge \psi(X_2) = \psi(Y \cup X_1') \wedge \psi(Y \cup X_2') = (\psi(Y) \vee \psi(X_1')) \wedge (\psi(Y) \vee \psi(X_2')) = \bigwedge P_{\psi(Y)} = \psi(Y)$. □

References

[1] Crawley, P., Dilworth, R. P.: Algebraic Theory of Lattices. *Englewood Cliffs*, 1973.

[2] Czédli, G., Huhn, A. P., Schmidt, E. T.: *Weakly independent sets in lattices.* Algebra Univers. **20** (1985), 194–196.

[3] Dlab, V.: *Lattice formulation of general algebraic dependence.* Czech. Math. Journal **20**, 95 (1970), 603–615.

[4] Grätzer, G.: General Lattice Theory. *Birkhäuser Verlag*, 1998.

[5] Machala, F.: *Join-independent and meet-independent sets in complete lattices.* Order **18** (2001), 269–274.

[6] Machala, F., Slezák, V.: *Lattice-inadmissible incidence structures.* Discuss. Math. (submitted).

[7] Slezák, V.: *On the special context of independent sets.* Discuss. Math., Gen. Algebra and Appl. **21** (2001), 115–122.

[8] Szász, G.: Introduction to Lattice Theory. *Akadémiai Kiadó*, Budapest, 1963.