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ATOMS OF CYCLIC CONNECTIVITY IN CUBIC GRAPHS

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ABSTRACT. Let G be a connected cubic graph with cyclic connectivity k. An induced subgraph P of G is called a cyclic part of G if there exists a cycle-separating k-edge cut S such that P is a component of $G - S$. A cyclic part minimal under inclusion is called an atom. We establish various properties of atoms and cyclic parts in cubic graphs. Among other things we show that a cyclic part which is not a k-cycle has at least $2k - 3$ vertices, or $k = 6$, and it is isomorphic to the graph Θ_8 consisting of three internally disjoint paths of length 3 joining two vertices. Using this bound we prove that, if two cyclic parts intersect, then either each of them contains a smaller cyclic part, or one of them or their complements is isomorphic to a k-cycle, or, if $k = 6$, to Θ_8. These two results have a number of corollaries which include results of Aldred, Holton and Jackson [1], Fouquet and Thuillier [3], and McCuaig [7]. For instance, we show that the cyclic connectivity of a connected cubic vertex-transitive graph is equal to its girth, and the same is true for cubic edge-transitive graphs (see [10]).

1. Introduction

When examining the connectivity properties of a graph it is sometimes necessary to avoid trivial cuts, i.e., those producing a component which consists of a single vertex. If an edge cut gives rise to a component which is a tree with more than one vertex, then a little modification at one of its pendant vertices yields a smaller cut. Thus a minimum non-trivial edge cut-set separates the graph into components each of which contains a cycle. This simple consideration suggests the definition of the edge cyclic connectivity of a graph to be the minimum number of edges separating two cycles. Vertex cyclic connectivity can be introduced analogously.

The definition of cyclic connectivity first appeared in Tutte's paper [11]. Since then, this concept has occurred in various contexts including hamiltonian problems, edge colourings of graphs, maximum genus, the existence of cycles through

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prescribed edges, and others. For cubic graphs, cyclic connectivity is especially important, being a natural extension of the usual vertex and edge-connectivity. Indeed, if the cyclic connectivity of a cubic graph does not exceed 3, then it coincides with the ordinary connectivity. But while the usual connectivity is bounded from above by 3, cyclic connectivity is unbounded.

In spite of that, not much is known about cyclic connectivity in general, even in the case of cubic graphs. So far, the most extensive investigation of cyclic connectivity of cubic graphs is due to Fouquet and Thuillier [3], and McCuaig [7], who established several important results of general character. (Some of them have been independently obtained in [9] using a different method.)

Our approach to the investigation of cyclic connectivity is based on the concept of an atom, the smallest induced subgraph separated by a minimum cut-set. Although the term atom may receive different meanings depending on type of connectivity under consideration, various properties of the different kinds of atoms are much alike. For example, it is obvious that the image of an atom under a graph automorphism is again an atom. Therefore the study of atoms is important for, and in fact has been motivated by, the investigation of connectivity properties of transitive graphs.

For the ordinary edge-connectivity it can be easily shown that distinct atoms are vertex-disjoint (see Mader [6] or Lovász [5; 12.14]). The same is true for atoms of vertex connectivity but the proof is less straightforward (see Mader [6] and Watkins [13]). The present paper resolves a similar problem for the cyclic connectivity of cubic graphs and provides the necessary background for a detailed treatment of properties of cyclic connectivity in transitive cubic graphs (see [10]).

While atoms of vertex or edge-connectivity have been mainly employed in connection with graph automorphisms, the results of this paper indicate that atoms of cyclic connectivity may also be important for understanding cyclic connectivity of general cubic graphs. Various properties of cubic graphs, including those not presented here, can be obtained and formulated in terms of atoms. This is true in particular for the results of Fouquet and Thuillier [3] and McCuaig [7], whose approach to cyclic connectivity is based on independent edge cuts. Atoms and, more generally, cyclic parts provide in a certain sense a complementary view of the problem. Moreover, they seem to have the advantage that an atom or a cyclic part allows to grasp both the corresponding edge cut and the subgraphs obtained by deleting the cut. This leads to simpler formulations and proofs of several results in this area and to establishing new ones.

In this paper, we develop a theory of atoms and cyclic parts in cubic graphs. Our investigation culminates in proving two important results. The first of them, Theorem 7, yields a lower bound on the number of vertices of a non-trivial cyclic part, i.e., one which is not a single shortest cycle. A slightly stronger bound is obtained for a non-trivial atom. In [3; Proposition 4] Fouquet and Thuillier established a result of a similar type for cyclic connectivity greater than 4.

By using Theorem 7 we prove our second main result, Theorem 9. This theorem deals with the mutual position of cyclic parts in a cubic graph. Roughly speaking, it says that if two cyclic parts of a cubic graph intersect, then either each of them contains a smaller cyclic part, or one of them or their complements is a trivial atom. Thus it either localizes a trivial atom, or enables us to reduce a large cyclic part to a smaller one. This reduction process eventually ends with an atom. Therefore we call our Theorem 9 "Reduction Theorem".

Theorem 9, as well as its proof, has been inspired by McCuaig's lemma on crossing independent edge cuts [7; Lemma 2.1]. Although this is indeed a technical lemma rather than a theorem, it plays an important role in the theory presented in [7]. By taking our approach based on atoms, omitting unnecessary details from the formulation and pushing the proof a step further, we obtain a result providing a deep insight into the structure of atoms in cubic graphs. This result has various important consequences. In particular, it serves as a strong tool in our investigation of atoms in transitive cubic graphs [10] and is in the background of the main results of [7].

The present paper is divided into four sections. The next section contains basic definitions and elementary results on cyclic connectivity of cubic graphs. The third section contains two main results and their proofs: the lower bound on the number of vertices of a non-trivial cyclic part of a cubic graph, and the Reduction Theorem. The final section is devoted to corollaries of the main results, especially of the Reduction Theorem.

Besides new results, this paper contains also some older results proved by various authors. We include them partly for the sake of completeness, partly because our present proofs are new or shorter.

2. Basic properties of cyclic connectivity

In this paper, we deal with connected finite graphs, allowing both multiple edges and loops. However, we mostly confine to cubic graphs and their subdivisions.

Let $G = (V, E)$ be a connected graph. We say that a set B of edges of G is *cycle-separating* if $G - B$ is disconnected, and at least two of its components contain cycles. Note, however, that there exist graphs possessing no cycle-separating edge cuts. These graphs have been characterized by Lovász in [4]. We shall formulate a partial result for cubic graphs in Proposition 1, together with a short independent proof (see also [7], [9] and [8]).

For X being a set of vertices or a subgraph of a graph G, let δX denote the set of all edges in G with exactly one end in X. If H is an induced subgraph of G, let H' denote its *complement*, the subgraph of G induced by the set $V - V(H)$. Finally, let Θ_2 denote the graph consisting of two vertices and three parallel edges.

PROPOSITION 1. *A connected cubic graph G has no cycle-separating edge cut if and only if it is isomorphic to one of K_4, $K_{3,3}$ or Θ_2. In fact, if C is a shortest cycle in G, then δC is a cycle-separating edge cut unless G is K_4 or $K_{3,3}$ or Θ_2.*

Proof. It is obvious that K_4, $K_{3,3}$ and Θ_2 have no cycle-separating edge cuts. For the converse, assume that C is a shortest cycle in G and that δC is not a cycle-separating cut. Then C' is a forest. Thus C' is either empty, or it contains an isolated vertex or two vertices of valency one. Let g denote the length of C. If C' is empty, then obviously $g = 2$ and $G \cong \Theta_2$. Let C' contain an isolated vertex u. Then u is adjacent to three vertices on C, and at least two of its neighbours, say x and y, have distance $\leq g/3$ on C. If P is the corresponding segment of C, then $uxPyu$ is a cycle, say, of length h. Now, $g \leq h \leq g/3 + 2$ whence $g = 3$. Since $|\delta C| = 3$, it is readily seen that $G \cong K_4$. Similarly, if G has two vertices of valency one, we deduce that $g \leq g/2 + 2$ and, consequently, that $g = 4$. Thus $|\delta C| = 4$, $C' \cong K_2$, and G is isomorphic to $K_{3,3}$. □

We say that a connected graph $G = (V, E)$ is *cyclically k-edge-connected* if no set of fewer than k edges is cycle-separating in G. Let $\beta(G) = |E| - |V| + 1$ be the Betti number (=cycle rank) of G. Clearly, deleting any set of $k \geq \beta(G)$ edges yields either a disconnected graph or a graph without cycles. Thus, if G contains a cycle-separating set, then it contains one with no more than $\beta(G) - 1$ edges. The *edge cyclic connectivity* of G, $\zeta(G)$, is the largest integer $k \leq \beta(G)$ for which G is cyclically k-edge-connected. For instance, $K_{3,3}$ is cyclically k-edge-connected for each positive integer k, but its edge cyclic connectivity is 4. In fact, $\zeta(G) = \beta(G)$ if and only if G has no cycle-separating edge cut. By Proposition 1, there are exactly three such cubic graphs.

Let $g(G)$ denote the *girth* of a graph G, i.e., the length of a shortest cycle in G. Using Proposition 1 and examining the three exceptional cases we obtain the following

PROPOSITION 2. *For every connected cubic graph G, $\zeta(G) \leq g(G)$.*

Let G be a connected cubic graph and suppose it contains a cut-set J consisting of independent edges. Then the minimum valency of $G - J$ is 2, and, consequently, each component of $G - J$ contains a cycle. Thus each independent edge cut is cycle-separating. On the other hand, observe that, if B is a minimum cycle-separating edge cut in a connected cubic graph G, then it consists of independent edges. Indeed, if B contains two adjacent edges e and f, let h be the third edge adjacent to both e and f. Then the set $\big(B - \{e, f\}\big) \cup \{h\}$ is again cycle-separating, contradicting the minimality property of B. Therefore, in the study of edge cyclic connectivity of cubic graphs, we may restrict ourselves to independent edge cuts. For the same reason, in the case of cubic graphs, we could replace in the definition of $\zeta(G)$ cycle-separating edge cuts by independent edge cuts.

Yet another version of cyclic connectivity has been defined by Tutte [12; p. 70] and termed vertex cyclic connectivity. A set X of vertices of a connected graph G is called a *cycle-separating vertex cut* if G can be expressed as a union of two edge disjoint subgraphs H and K, both containing cycles, such that $H \cap K = X$. We say that G is *cyclically k-vertex-connected* if it contains no cycle-separating vertex cut with fewer than k vertices.

Although the cyclic connectivity of a cubic graph can be introduced in various ways, the following result shows that the different approaches are interchangeable.

PROPOSITION 3. ([7]) *Let G be a connected cubic graph not isomorphic to K_4, $K_{3,3}$ or Θ_2. Then the following conditions are equivalent:*

(i) *G is cyclically k-edge-connected.*
(ii) *G is cyclically k-vertex-connected.*
(iii) *Each independent edge cut has at least k edges.*

Proof. Let m_E and m_V be the cardinality of a minimum cycle-separating edge cut and vertex cut in G, respectively, and let m_I be the minimum number of edges in an independent edge cut. We shall show that $m_E \geq m_I \geq m_V \geq m_E$, thereby proving the theorem.

As we have already observed, every minimum cycle-separating edge cut consists of independent edges. Hence $m_E \geq m_I$.

Now, let B be a minimum independent edge cut of G. Since each component $G - B$ has minimum valency 2, it contains a cycle. Let H be a component of $G - B$, and let U be the set of vertices of H incident with an edge in B. Define G_1 to be the subgraph of G induced by $V(H) \cup U$ and G_2 the subgraph of G induced by $\big(V(G) - V(H)\big) \cup U$. Then $G_1 \cup G_2 = G$, $G_1 \cap G_2 = U$, and both G_1 and G_2 contain cycles. Thus U is a cycle-separating vertex cut. The minimality of B ensures that $|U| \leq |B|$, which implies that $m_I = |B| \geq |U| \geq m_V$.

Finally, let W be a minimum cycle-separating vertex cut of G. Then there exist subgraphs H_1 and H_2 in G, both containing cycles, such that $H_1 \cup H_2 = G$ and $H_1 \cap H_2 = W$. Let P be the set of edges of G that are pendant in H_1 or H_2. Obviously, $|P| \leq |W|$. It is clear that $H_1 - P$ and $H_2 - P$ are components of $G - P$, and that both contain cycles, for a pendant edge does not lie in a cycle. Thus P is a cycle-separating set of edges and $m_V \geq m_E$.

This completes the proof. □

The previous theorem implies that the edge cyclic connectivity of a cubic graph is equal to its vertex cyclic connectivity. Therefore, in the rest of the paper, we shall use the term *cyclic connectivity* for edge cyclic connectivity and *cycle-separating cut* for cycle-separating edge cut. Moreover, it can easily be seen that the cyclic connectivity of a graph is invariant under subdivisions and adjoining new 1-valent vertices. This fact guarantees that most of our subsequent results can be modified for graphs with maximum valency 3.

In the following lines, we introduce the central notion of this paper. Let B be a minimum cycle-separating cut in a cubic graph G. Clearly, $B = \delta P$ for some induced subgraph P of G. We shall refer to P as a *cyclic part* of G. Thus a cyclic part is an induced subgraph P of G such that δP is a cycle-separating cut with $|\delta P| = \zeta(G)$. A cyclic part minimal under inclusion will be called an *atom*.

It is fairly obvious that each atom must be connected. The next observation, partly due to McCuaig [7], shows that a little bit more can be said.

PROPOSITION 4. *Let G be a connected cubic graph. Then each cyclic part of G is connected and each atom is a block. Moreover, if $\zeta(G) \geq 3$, then each cyclic part is a block.*

Proof. Let P be a cyclic part of G. If P is disconnected, then it has a component Q containing a cycle. Note that $\delta Q \subseteq \delta P$ for P is an induced subgraph of G. Moreover, $|\delta Q| < |\delta P|$ since $Q \neq P$. Thus δQ is a cycle-separating cut with $|\delta Q| < \zeta(G)$, which is a contradiction.

Now, let P be a cyclic part of G containing a cut-vertex. Since the maximum valency of P is at most 3, there is a cut-edge e in P. Let P_1 and P_2 be the components of $P - e$. Clearly,

$$|\delta P_1| + |\delta P_2| = |\delta P| + 2 = \zeta(G) + 2 .$$

On the other hand, δP is an independent set of edges. It follows that the minimum valency of both P_1 and P_2 is 2, implying that each of them contains a cycle. Hence $|\delta P_1| \geq \zeta(G)$ and $|\delta P_2| \geq \zeta(G)$, with sharp inequalities occurring whenever P is an atom. As a result,

$$2\zeta(G) \leq |\delta P_1| + |\delta P_2| \leq \zeta(G) + 2 ,$$

and, if P is an atom,

$$2\zeta(G) + 2 \leq |\delta P_1| + |\delta P_2| \leq \zeta(G) + 2 .$$

By solving these inequalities with respect to $\zeta(G)$, we obtain $\zeta(G) \leq 2$ and $\zeta(G) \leq 0$, respectively. The second possibility cannot happen, which means that an atom cannot have a cut-vertex. A cyclic part may have a cut vertex only when $\zeta(G) \leq 2$. The result follows. □

A cyclic part (in particular, an atom) is said to be *trivial* if it consists of a single cycle (necessarily a shortest one). Otherwise it is *non-trivial*. Note that two atoms of the same graph may differ in the number of vertices. Thus we are led to consider atoms with minimum number of vertices, which we call *proper atoms*. This concept enables us to give a simple answer to the natural question suggested by Proposition 2: For which graphs the cyclic connectivity is strictly smaller than the girth?

PROPOSITION 5. ([9]) *Let G be a connected cubic graph of girth g. Then $\zeta(G) < g$ if and only if G has a non-trivial proper atom.*

Proof. If G has a trivial atom, say C, then $\zeta(G) = |\delta C| \geq g$. By Proposition 2, $\zeta(G) = g$. If G has no atom, then Proposition 1 shows that G is either K_4 or $K_{3,3}$ or Θ_2, and we have $\zeta(G) = g$ again.

For the converse, assume that $\zeta(G) = g$, but G has a non-trivial proper atom, say A. Clearly, G is different from K_4, $K_{3,3}$ and Θ_2. By Proposition 1, if C is a cycle of length g, then δC is cycle-separating. Thus $|A| \leq g$. On the other hand, A contains a cycle D with $|D| < |A|$ since it is non-trivial. Therefore $g \geq |A| > |D| \geq g$, which is absurd. □

The previous proposition has an interesting consequence:

COROLLARY 6. ([9]) *In a cubic graph, either all proper atoms are trivial or all proper atoms are non-trivial or there are no atoms.*

Proof. Assume there exists a cubic graph which contains a trivial (proper) atom as well as a non-trivial proper atom. Then, by Proposition 5, its cyclic connectivity is at the same time equal to the girth and smaller than the girth, a contradiction. □

Observe that there do exist cubic graphs having both trivial and non-trivial atoms. Of course this can only happen in a graph with cyclic connectivity equal to girth. The simplest such example can be constructed by first subdividing each edge of Θ_2 with one additional vertex and subsequently by joining the 2-valent vertices of the resulting graph (denoted by Θ_5) to the vertices of a triangle to form a cubic graph. Thus the adjective "proper" cannot be omitted from Proposition 5. On the other hand, by comparing Proposition 5 and Corollary 6, we get that if G is a cubic graph with $\zeta(G) < g$, then all its atoms are non-trivial.

3. Main results

In this section, we prove our two main contributions to the study of atoms of cyclic connectivity in cubic graphs, Theorem 7 and Theorem 9. In both of them, certain subdivisions of the graph Θ_2 occur as exceptional cases. It will be therefore convenient to fix the following notation. Let Θ_5 be the graph of order 5 obtained from Θ_2 by replacing each edge with a path of length 2, and let Θ_8 be the graph of order 8 obtained similarly using paths of length 3.

Finally, for an arbitrary graph K, let $c(K)$ be the number of components of K, $e(K)$ be the number of edges of K, and let $v(K)$ and $v_i(K)$ be the number of vertices and the number of i-valent vertices of K, respectively.

Now, we can state our first main result.

THEOREM 7. *Let G be a connected cubic graph with cyclic connectivity k, and let P be a non-trivial cyclic part of G. Then either $v(P) \geq 2k-3$, or $k=6$, P is isomorphic to Θ_8 and $v(P)=2k-4$. Moreover, if P is an atom, then $v(P) \geq 2k$, or $k=3$, P is isomorphic to Θ_5 and $v(P)=2k-1$.*

Proof. Since P is a non-trivial cyclic part of G, it is homeomorphic to a cubic graph H. If there are two disjoint cycles in H, then, by Proposition 2, $v(P) \geq 2k$ and the theorem holds. Therefore we may assume that any two cycles in H have a vertex in common. Thus H is isomorphic to one of Θ_2, K_4, or $K_{3,3}$.

If H is isomorphic to K_4 or $K_{3,3}$, then P contains a subgraph K which is a subdivision of K_4. We shall estimate $2e(K)$ in two ways. First, we have

$$2e(K)=2v_2(K)+3v_3(K)=2v(K)+v_3(K)=2v(K)+4. \tag{1}$$

On the other hand, if C_1, C_2, C_3 and C_4 are the cycles of K corresponding to the triangles of K_4, then every i-valent vertex of K is contained in exactly i cycles from C_1, C_2, C_3 and C_4. Since $v(C_i) \geq k$, it follows that

$$2e(K)=\sum v(C_i) \geq 4\zeta(G)=4k. \tag{2}$$

By comparing (1) with (2), we obtain

$$4k \leq 2e(K)=2v(K)+4 \leq 2v(P)+4,$$

whence $v(P) \geq 2k-2$. Thus, in this case, the theorem holds.

It remains to consider the case that $H \cong \Theta_2$. Suppose $v(P) \leq 2k-4$. Then,

$$2k-4 \geq v(P)=v_2(P)+2=k+2,$$

whence $k \geq 6$. To establish the reverse inequality, we compute $2e(H)$ in two ways. On one hand,

$$2e(H)=2v_2(P)+3v_3(P)=2|\delta P|+6=2k+6. \tag{3}$$

On the other hand, let C_1, C_2 and C_3 be the cycles in P. Again, $v(C_i) \geq k$, and analogously as before we have

$$2e(H) = \sum v(C_i) \geq 3k\,. \tag{4}$$

From (3) and (4), it follows that

$$3k \leq 2e(H) = 2k + 6\,,$$

that is, $k \leq 6$. Thus $k = 6$. To summarize, P is homeomorphic to Θ_2, $v(P) = 8$ and, by Proposition 2, it has girth at least 6. But this is only possible when P is isomorphic to Θ_8. This proves the first part of the theorem.

From now on, suppose that P is an atom. In order to finish the proof we point out two important properties of non-trivial atoms.

PROPOSITION 8. *Let P be a non-trivial atom of a cubic graph G. Then the following hold:*

(i) *No two 2-valent vertices of P are adjacent.*
(ii) *If P is not isomorphic to Θ_5, then no three 2-valent vertices of P have a common neighbour.*

Proof. Let H be the cubic graph homeomorphic with P. Then to each edge e of H there corresponds a path Q_e in P whose end-vertices belong to $v_3(P)$ and internal vertices to $v_2(P)$. First, we show that the length of every path Q_e is at most 2, thereby proving (i).

Let J be the set of all internal vertices of Q_e. As every vertex of J is incident with exactly one edge in δP, we have

$$|\delta(P - J)| = |\delta P| - |J| + 2 = \zeta(G) - |J| + 2\,. \tag{5}$$

On the other hand, since $P - J$ is a subdivision of $H - e$, it contains a cycle. However, P is an atom, so

$$|\delta(P - J)| \geq \zeta(G) + 1\,. \tag{6}$$

By comparing (6) with (7) we obtain

$$\zeta(G) + 1 \leq \zeta(G) \; - |J| + 2\,,$$

whence

$$|J| \leq 1\,. \tag{7}$$

Thus, the path Q_e in P corresponding to an edge e of H has length at most 2. This proves (i). (If the cyclic part P were not an atom, then instead of (6) we would get $|\delta(P - J)| \geq \zeta(G)$ implying that $|J| \leq 2$. This shows that in a non-trivial cyclic part each edge of H corresponds to a path of length at most 3.)

Let S be the set of all edges in H for which the length of Q_e is 2. Obviously,

$$|S| = v_2(P) = |\delta P| = \zeta(G). \tag{8}$$

Denote by H_S the subgraph of H formed by the edges in S and the vertices incident with them. Now, we show that if H_S contains a vertex of valency 3, then $P = \Theta_5$.

Suppose v is a 3-valent vertex of H_S. Then v is adjacent in P with three vertices $w, x, y \in V_2(P)$. Obviously,

$$\bigl|\delta\bigl(P - \{v, x, y, w\}\bigr)\bigr| = |\delta P| = \zeta(G).$$

Since P is an atom, $P - \{v, x, y, w\}$ is acyclic, and so is $H - v$. However, $H - v$ is in fact a single vertex. Indeed, in the opposite case, $H - v$ would contain at least two vertices of valency not exceeding 1, implying that $3 = |\delta v| = |\delta(H - v)| \geq 4$, a contradiction. Thus $H - v$ is an isolated vertex and H is isomorphic to Θ_2. Note that $v_2(P) = 3$ because (7) implies $v_2(P) \leq 3$ and (8) implies $v_2(P) \geq 3$. In summary, if H_S contains a 3-valent vertex, then P is isomorphic to Θ_5. This proves (ii). □

Proof of Theorem 7 continued. Let P be a non-trivial atom of G. The second part of our theorem obviously holds if $P \cong \Theta_5$. Now, assume that P is not isomorphic to Θ_5 and let H be the cubic graph homeomorphic with P. Then, using Proposition 8 and the notation of its proof, every vertex of H_S has valency at most 2. Hence, $v(H_S) \geq e(H_S)$. Combining this observation with (8) we obtain

$$v_3(P) = v_3(H) = v(H) \geq v(H_S) \geq e(H_S) = |S| = \zeta(G),$$

and, consequently,

$$v(P) = v_2(P) + v_3(P) \geq |S| + \zeta(G) = 2\zeta(G) = 2k,$$

which completes the proof of Theorem 7. □

In a certain sense, Theorem 7 could be strengthened. The reason is that non-isomorphic cyclic parts P with $v(P) < 2\zeta(G)$ can easily be listed. This follows from the fact that, in this case, P is either a k-cycle, or is isomorphic to a graph $H \in \{\Theta_2, K_4, K_{3,3}\}$, and each edge of H corresponds in P to a path of length at most 3. For $k \geq 5$ such a list can be found in Fouquet and Thuillier [3; Fig. 3 and Fig. 4].

Here is our second main result, the Reduction Theorem.

THEOREM 9. *Let G be a connected cubic graph with cyclic connectivity k. If A and B are distinct cyclic parts of G with non-empty intersection, then one of the following cases occurs:*

(i) *At least one of $A \cap B$ and $A \cap B' = A - B$ is a cyclic part of G contained in A.*

(ii) *Among the cyclic parts A, A', B and B' at least one is a k-cycle.*
(iii) $k = 6$ *and at least one of A, A', B and B' is isomorphic to* Θ_8.

P r o o f . Set $G_0 = A \cap B$, $G_1 = A \cap B'$, $G_2 = A' \cap B'$ and $G_3 = A' \cap B$, where the indices are taken modulo 4. Note that G_0 is non-empty. If G_1 or G_3 is empty, then either $A \subseteq B$ or $B \subseteq A$, respectively, and hence $A \cap B$ is a cyclic part. If G_2 is empty, then $A - B = B'$ is a cyclic part. In both cases, part (i) of the theorem holds. Thus we can assume that G_i is non-empty for all $i = 0, 1, 2$ and 3.

Choosing any index i, $0 \le i \le 3$, one can find among A, A', B and B' a cyclic part D_i which contains both G_i and G_{i+1}. There is, in fact, exactly one cyclic part D_i as required and, moreover, $V(D_i) = V(G_i) \cup V(G_{i+1})$. A crucial step of our proof is to establish the following lemma, which is, basically, a part of Lemma 2.1 in [7].

Lemma. *Assume that both G_i and G_{i+1} are acyclic. If $|\delta G_i| \le k$ and $|\delta G_{i+1}| \le k$, then D_i is either a k-cycle or it is isomorphic to Θ_8.*

P r o o f . We show that $v(D_i) \le 2k - 4$. Since δA and δB are independent sets of edges, it is immediate that every G_j $(0 \le j \le 3)$ has minimum valency greater than or equal to 1. Thus

$$|\delta G_j| = 2v_1(G_j) + v_2(G_j)\,. \tag{9}$$

Note that for every forest F with maximum valency ≤ 3 one has

$$2v_0(F) + v_1(F) - v_3(F) = 2c(F)\,, \tag{10}$$

which can easily be proved by induction. For $j = i$ and $j = i + 1$, (10) implies that $v_3(G_j) = v_1(G_j) - 2c(G_j)$, whence

$$v(G_j) = v_1(G_j) + v_2(G_j) + v_3(G_j) = 2v_1(G_j) + v_2(G_j) - 2c(G_j)\,.$$

By combining this with (9), for $j = i$ and $j = i + 1$ we obtain

$$v(G_j) = |\delta G_j| - 2c(G_j)\,. \tag{11}$$

But $|\delta G_i| \le k$ and $|\delta G_{i+1}| \le k$, so from (11) it follows that

$$\begin{aligned} v(D_i) = v(G_i) + v(G_{i+1}) &= |\delta G_i| - 2c(G_i) + |\delta G_{i+1}| - 2c(G_{i+1}) \\ &\le k - 2 + k - 2 \le 2k - 4\,. \end{aligned}$$

Thus we have proved that

$$v(D_i) \le 2k - 4\,. \tag{12}$$

If D_i is not a k-cycle, then it is a non-trivial cyclic part of G. But now, Theorem 7 and inequality (12) imply that it is isomorphic to Θ_8. □

Proof of Theorem 9 continued. It is well known ([5; 6.48 (a)]) that for any two subsets X and Y of $V(G)$ it holds that

$$|\delta(X \cup Y)| \leq |\delta X| + |\delta Y| - |\delta(X \cap Y)|.$$

Setting $X = A$ and $Y = B$ and using the fact that $\delta(X \cup Y) = \delta(X' \cap Y') = \delta(G_2)$ we have

$$|\delta G_2| \leq 2k - |\delta G_0|.$$

Using an appropriate substitution for X and Y we obtain that

$$|\delta G_{i+2}| \leq 2k - |\delta G_i| \tag{13}$$

for any $i \in \{0, 1, 2, 3\}$ taken modulo 4.

Now, suppose that neither G_0 nor G_1 is a cyclic part of G. We distinguish four cases covering all the possible relations between $|\delta G_i|$ and k, where $i = 0, 1$.

Case 1. $|\delta G_0| > k$ and $|\delta G_1| > k$. Then by virtue of (13) we have $|\delta G_2| < k$ and $|\delta G_3| < k$, which implies that both G_2 and G_3 are acyclic. Employing our Lemma, we can deduce that $A' = D_2$ is either a k-cycle, or $k = 6$ and A' is isomorphic to Θ_8. This yields (ii) or (iii).

Case 2. $|\delta G_0| > k$ and $|\delta G_1| \leq k$. From (13), it follows that $|\delta G_2| < k$. Thus G_2 is acyclic while G_1 is acyclic by the assumption of this case. Our Lemma now yields that $B' = D_1$ is either a k-cycle, or is isomorphic to Θ_8. Again, (ii) or (iii) is satisfied.

Case 3. $|\delta G_0| \leq k$ and $|\delta G_1| > k$. In this case, (ii) or (iii) is satisfied with $D_3 = B$.

Case 4. $|\delta G_0| \leq k$ and $|\delta G_1| \leq k$. Now, (ii) or (iii) is satisfied with $D_0 = A$.

The proof of Theorem 9 is complete. □

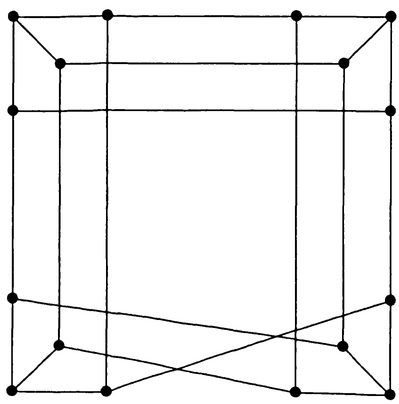

Figure 1.

We finally remark that case (iii) of Theorem 9 really occurs and is not contained in cases (i) and (ii). To see this, consider the example in Fig. 1. If u_0, u_1, u_2 and u_3 are the "corner" vertices of the graph in Fig. 1, set G_i to be the subgraph induced by u_i and its neighbours. With the notation of the above proof, it is easy to see that the cyclic parts D_0, D_1, D_2 and D_3 are all isomorphic to Θ_8.

4. Corollaries

We start this section with two corollaries of Theorem 7. The essential part of this section, however, will be devoted to consequences of Theorem 9.

The following result improves Corollary 1 of [3].

COROLLARY 10. *Let G be a connected cubic graph with $\zeta(G) < g(G)$, and let P be a cyclic part of G. Then either $v(P) \geq 2\zeta(G)$, or $\zeta(G) = 3$ and P is isomorphic to Θ_5 and $v(P) = 5 = 2\zeta(G) - 1$.*

Proof. Since $\zeta(G) < g(G)$, there exists a non-trivial atom $A \subseteq P$, and the result follows from Theorem 7. □

Our next theorem imposes a restriction on the possible choice of atoms for a cubic graph with given cyclic connectivity.

THEOREM 11. *Let A be a non-trivial atom of a connected cubic graph G. Then $\zeta(A) \geq \zeta(G)/2 + 1$, or A is isomorphic to Θ_5 and $\zeta(A) = 2 = \zeta(G)/2 + 1/2$. In each case, $\zeta(A) > \zeta(G)/2$.*

Proof. Let us first examine the case when A is homeomorphic to none of Θ_2, K_4 and $K_{3,3}$. Then there exists a cycle-separating cut B in A such that $|B| = \zeta(A)$. If A_1 and A_2 are the components of $A - B$, then

$$|\delta A_1| + |\delta A_2| = |\delta A| + 2|\delta A_1 \cap \delta A_2| = \zeta(G) + 2|B| = \zeta(G) + 2\zeta(A). \qquad (14)$$

On the other hand, A is an atom, so $|\delta A_i| \geq \zeta(G) + 1$ for $i = 1, 2$. By using (14) we now obtain

$$2\zeta(G) + 2 \leq |\delta A_1| + |\delta A_2| = \zeta(G) + 2\zeta(A),$$

whence

$$\zeta(A) \geq \zeta(G)/2 + 1,$$

as required.

It remains to handle the case when A is homeomorphic to one of the three exceptional graphs. Denoting this graph by H we have

$$\begin{aligned}\zeta(A) = \beta(A) = \beta(H) = v(H)/2 + 1 &= v_3(A)/2 + 1 \\ &= \big(v(A) - v_2(A)\big)/2 + 1 \\ &= \big(v(A) - \zeta(G)\big)/2 + 1 .\end{aligned}$$

But now, if A is not isomorphic to Θ_5, Theorem 7 implies that

$$\zeta(A) = \big(v(A) - \zeta(G)\big)/2 + 1 \geq \zeta(G)/2 + 1,$$

and the theorem follows. □

Sometimes the bound given in Theorem 11 can be improved. For instance, from Theorem 11 it follows that no cyclically 7-connected cubic graph contains an atom homeomorphic to Θ_2, K_4 or $K_{3,3}$. Proposition 8 implies, however, that in a non-trivial atom $A \not\cong \Theta_5$ the set of edges each of which is incident with a 2-valent vertex forms a subgraph with maximum valency 2. But then it is easy to see that the cyclic connectivity 7 can be replaced by 6, and this is best possible.

We now pass to corollaries of Theorem 9. Here, we concentrate only on results of general character, without imposing further restrictions on graphs. Besides establishing new results, we reprove two theorems of McCuaig [7]. Our Reduction Theorem enables us not only to shorten the original proofs but also sheds new light on them. For instance, one of McCuaig's theorems is included as an implication between two of three equivalent statements. Finally, we present a sample of results on vertex-transitive cubic graphs. Further results can be found in [10].

The following concept and two notations are due to McCuaig [7]. A cyclic part A of a connected cubic graph G is an *end* of G (a $\zeta(G)$-*end* in the original terminology) if no edge of A belongs to a minimum cycle-separating cut. Obviously, each end is an atom. Moreover, it is easy to see that any two distinct ends are disjoint (see also [7; Lemma 6.2]).

In a different context, McCuaig considers two special subgraphs, which we denote by $M(G)$ and $N(G)$ (originally $N_{\zeta(G)}(G)$ and $N'_{\zeta(G)}(G)$). For a connected cubic graph G let $M(G)$ be the subgraph of G formed by the set of edges belonging to some minimum cycle-separating cut. Edges in $M(G)$ are sometimes called non-removable, whereas those not belonging to $M(G)$ are called removable ([3], [2]).

Further, let $N(G)$ be the subgraph of $M(G)$ formed by the set consisting of edges that are contained in no cycle of length $\zeta(G)$.

The following theorem relates the three objects to each other and contains Theorem 2.3 of [7] as the implication (ii) $\Longrightarrow$ (iii).

THEOREM 12. *Let G be a connected cubic graph of girth g. The following conditions are equivalent.*

(i) *Each atom of G is an end.*

(ii) *No minimum cycle-separating cut of G contains some edge of a cycle of length $\zeta(G)$, i.e., $M(G) = N(G)$.*

(iii) *No minimum cycle-separating cut contains two edges in the same component of $M(G)$.*

Proof.

(ii) $\Longrightarrow$ (i):
Assuming (ii), let A be an atom which is not an end. Then there exists a cyclic part $B \neq A$ such that some edge e of δB is in A. Since A is an atom, Theorem 9 implies that $\zeta(G) = g$, and there exists a cyclic part P among A, A', B and B' which is a g-cycle or Θ_8. In both cases, each edge of P lies on a g-cycle. On the other hand, one of the cuts δA and δB separates P while the other intersects P. Thus one of those cuts contains an edge of a g-cycle, contradicting (ii).

(i) $\Longrightarrow$ (iii):
Suppose that (i) holds and that G has a minimum cycle-separating cut S comprising two edges e and f in the same component of $M(G)$. Choose e and f so that their distance in $M(G)$ is minimum and let Q be the shortest path of $M(G)$ joining them. The choice of e and f implies that Q lies entirely in one of the cyclic parts separated by S. Call it A. Since e and f are independent, Q is not a single vertex, and therefore A contains an edge h of $M(G)$. By the definition of $M(G)$, there is a cyclic part B in G such that h is in δB. Obviously, $A \neq B$ and A intersects B, so we can employ Theorem 9. From the choice of e and f it follows that the case (i) of Theorem 9 cannot occur. Thus $\zeta(G) = g$ and one of A, A', B and B' is a g-cycle or Θ_8. In both cases, there is a g-cycle C containing an edge of δA or an edge of δB. Thus C is not an end. Since C is an atom, we have arrived at a contradiction.

(iii) $\Longrightarrow$ (ii):
Assume that some $\zeta(G)$-cycle C contains an edge $e = uv$ of $M(G)$. Since G contains a $\zeta(G)$-cycle, we have $\zeta(G) \geq g$. By Proposition 2, $\zeta(G) \leq g$ and so $\zeta(G) = g$. Thus C is a g-cycle. Let f and h be the edges of δC incident with u and v, respectively. Since $\delta C \subseteq M(G)$, feh is a path in $M(G)$ implying that f and h belong to the same component of $M(G)$.

This completes the proof. □

The following result is Theorem 2.2 of McCuaig [7].

THEOREM 13. ([7]) *Let G be a connected cubic graph. Then $N(G)$ is a forest.*

Proof. Suppose that there is a cycle C contained in $N(G)$. Then, for any edge e_1 in C, there exists another edge e_2 in C such that both e_1 and e_2 belong to a common minimum cycle-separating cut S. Choose e_1 and e_2 to have the minimum distance in C and let P be the portion of C realizing this distance. Since e_1 and e_2 are independent, P contains an edge, say f_1. Again, there exists another edge f_2 in C (which by the choice of e_1 and e_2 is contained in $C - (P \cup e_1 \cup e_2)$ such that f_1 and f_2 belong to some minimum cycle-separating cut T. Let A be the component of $G - S$ containing f_1 and let B be the component of $G - T$ containing e_1. Clearly, A and B are distinct cyclic parts of G with $A \cap B \neq \emptyset$. Moreover, the edges f_1, f_2, e_1 and e_2 lie in A, A', B and B', respectively. The choice of e_1 and e_2 implies that neither $A \cap B$ nor $A - B$ is a cyclic part. Thus from Theorem 9 it follows that one of A, A', B and B' is either a cycle of length $\zeta(G)$ or Θ_8. In either case, one of the edges f_1, f_2, e_1 and e_2 lies on a cycle of length $\zeta(G)$, which is a contradiction. □

Cubic graphs with cyclic connectivity smaller than girth differ from other cubic graphs in many remarkable properties. The following theorem summarizes some of them. Part (a) is a generalization of the main result of [9] which states that distinct non-trivial proper atoms are disjoint. Part (b) coincides with Theorem 6.2 of [7], and part (c) combines Theorem 2.3 of [7] with Proposition 7 of [3]. Note that from the proof it readily follows that the assumption $\zeta(G) < g(G)$ can be replaced by any of the three equivalent statements of Theorem 12.

Let us call an atom A of a cubic graph G *detached* if there exists no atom $B \neq A$ having a non-empty intersection with A. Note that each end is a detached atom (cf. [7; Theorem 6.1]). Fig. 2 shows that the converse is false.

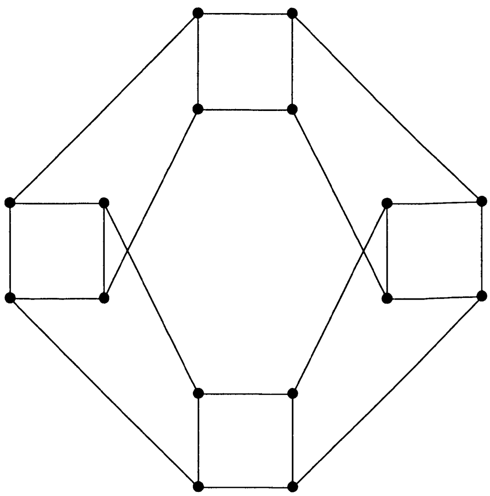

Figure 2.

THEOREM 14. *If G is a connected cubic graph with $\zeta(G) < g(G)$, then the following hold:*

(a) *Every atom in G is an end and hence is detached.*
(b) *G has at least two ends.*
(c) *$M(G)$ is a forest with at least $\zeta(G)$ components and each component of $M(G)$ is an induced subgraph of G.*

Proof. Let g be the girth of G. Since $\zeta(G) < g$, G contains no $\zeta(G)$-cycles, and hence $M(G) = N(G)$. Thus, the three equivalent statements (i) – (iii) of Theorem 12 hold. In particular, (i) is true, implying (a). (Observe, however, that (a) can also be directly deduced from Theorem 9.)

By our assumption, G is not isomorphic to Θ_2, K_4 or $K_{3,3}$. Therefore G contains an atom. This atom and any atom contained in its complement are the two required ends of G.

Finally, Theorem 13 combined with (ii) of Theorem 12 implies that $M(G)$ is a forest. By (iii) of the same theorem, the number of components cannot be smaller than $\zeta(G)$. The proof that each component of $M(G)$ is an induced subgraph of G is similar to that of Theorem 13.

This completes the proof. □

The following corollary coincides with Theorem 1 of Aldred, Holton and Jackson [1].

COROLLARY 15. *Let G be a connected cubic graph with cyclic connectivity k. If $M(G)$ intersects an atom of G, then $g(G) = k$.*

Proof. By the assumption, G contains an atom that is not an end. Theorem 14 (a) then implies that k cannot be smaller than $g(G)$. Thus the statement follows. □

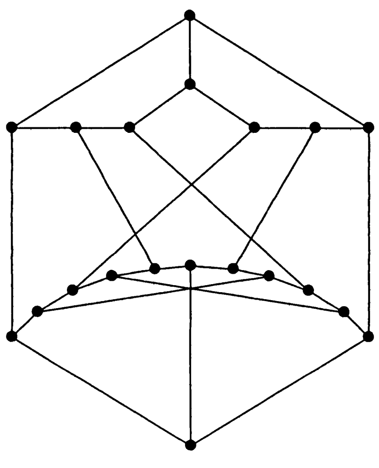

Figure 3.

From Proposition 5 and Corollary 6, we know that, if $\zeta(G) < g(G)$, then each atom in G is non-trivial, and, from Theorem 14 (a), that it is detached. Thus one could expect that every non-trivial atom in a cubic graph must be detached. The graph in Fig. 3 shows that this is not the case. It is easy to see that its cyclic connectivity is 5, it contains exactly two 5-cycles, and their complements are distinct non-trivial atoms with non-empty intersection. The following theorem shows that this example is typical.

THEOREM 16. *Let G be a connected cubic graph with cyclic connectivity k and girth g. If G contains two distinct non-trivial atoms with non-empty intersection, then $k = g$ and there exists a non-trivial atom A in G whose complement A' is isomorphic either to a k-cycle, or, if $k = 6$, to Θ_8.*

Proof. If $k = \zeta(G) < g$, then, by Theorem 14, any two distinct atoms of G are disjoint. Therefore $k = g$. Now let A and B be two non-trivial atoms with $A \cap B \neq \emptyset$. Clearly, none of A and B is a k-cycle, and, if $k = 6$, none of them is isomorphic to Θ_8. Using Theorem 9 for A and B and realizing that A is an atom we see that (ii) or (iii) of that theorem is true. Hence, one of A' or B' is either a k-cycle or Θ_8. This completes the proof. □

In our final result, we apply Theorem 14 (a) to determining the cyclic connectivity of any cubic vertex-transitive or edge-transitive graph. For further results in this direction see [10].

THEOREM 17. *Let G be a cubic vertex-transitive or edge-transitive graph of girth g. Then G has no non-trivial proper atom. In particular, $\zeta(G) = g$.*

Proof. We prove the theorem only for vertex-transitive graphs. The case of edge-transitive graphs is handled in [10, Theorems 8 and 14].

Assume that G contains a non-trivial proper atom A. Thus, by Proposition 5, we have $\zeta(G) < g$. If x and y are two vertices of A, then there is an automorphism ϕ sending x to y. Obviously, $\phi(A)$ is an atom, too, and $A \cap \phi(A) \neq \emptyset$. However, by virtue of Theorem 14 (a), A is detached. Therefore $\phi(A) = A$, and hence A is vertex-transitive. Since A contains a 2-valent vertex, all the vertices of A are 2-valent, by vertex-transitivity. Thus A is a cycle, which contradicts the non-triviality of A. □

The previous result suggests the question whether the adjective "proper" in the statement of Theorem 17 can be omitted. We conjecture that the answer is "yes". To support this conjecture we prove in [10] that a vertex-transitive or edge-transitive cubic graph with sufficiently many vertices as compared to its girth (at least $g \cdot 2^{(g/2)-1} + 3$) has no non-trivial atom. The proof is based on Theorem 16.

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