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FUBINI THEOREMS FOR BORNOLOGICAL MEASURES

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ABSTRACT. A theorem about the existence of the tensor product of bornological measures is proved and two Fubini theorems are stated for a bilinear integral in convex bornological spaces.

It is well known that the tensor product of two vector measures need not always exist, even in the case of measures valued in the same Hilbert space and being the bilinear mapping (used in its definition) the corresponding inner product (see for instance [4] and [11]). Huneycutt has proved in [13] the existence of the tensor product of two Banach valued measures of bounded variation, offering an integral representation of this product measure, proving also a Fubini theorem in that context. Several authors have given sufficient conditions for the existence of the tensor product measure, including the case of measures valued in locally convex spaces ([8], [9], [10], [12], [13], [21], [22] and others). In [20] a bilinear integral is defined in the context of the locally convex spaces which contains the countable case of the Bartle integral [3] and which allows to state the existence of the tensor product of two measures valued in locally convex spaces under certain conditions, extending the results mentioned before about this question. Later some Fubini theorems have been established in [12] for this integral.

As it is pointed out in [5], vector measures and integrable functions (with respect to scalar measures) with values in a large class of locally convex spaces are actually measures or functions valued in a normed space E_B. This bornological character is even more remarkable in the Radon-Nikodym type theorems for locally convex spaces, where the proofs usually involve an embedding in an appropriate Banach space E_B and an application of the result for Banach spaces.

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Also, the results stated in [2] about the dual of the L^p spaces for functions valued in locally convex spaces (and scalar measures) make clear the usefulness of the study made in [1] about the L^p spaces for functions valued in convex bornological spaces (and scalar measures).

Similar facts and the pronounced bornological character of the bilinear integration theory developed in [20] show the fitness of making a development of a similar bilinear integration theory in the context of the convex bornological spaces, which presents between other applications the possibility of obtaining results about the representation of bounded linear operators and the derivation of bornological measures not only with respect to scalar measures (as it is made in [6]) but also with respect to bornological measures.

Clearly the integration defined in [20] can be obtained from the integral introduced here considering in the locally convex topological vector spaces the corresponding von Neumann bornologies. Also in the particular case of scalar measures, the integrable functions used here coincide with the bornological Bochner integrable functions defined in [5]. In this paper a theorem about the existence and the integral representation of the tensor product of two bornological measures is proved, and two Fubini theorems are stated for functions valued in convex bornological spaces and bornological measures. These results contain in several cases (for instance if the considered locally convex topological vector spaces satisfy the strict Mackey condition (in the sense of Definition 3 of [14])) the corresponding results of [12], [19] and [20].

For questions about bornological spaces we remit ourselves to [15], [16], [17] and [18], and for the properties of bornological measures to [5] and [6].

Preliminaries

Let Σ be a σ-algebra of subsets of a set Ω and E a separated convex bornological space that we will ever suppose to be regular (i.e. its bornological dual separates the points of E, see for instance [18]). A mapping $m: \Sigma \to E$ is said to be a ***bornological measure*** (see [5]) if

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

holds for all sequences $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint members of Σ.

It is clear that if E is a locally convex topological vector space which satisfy the strict Mackey condition (see [14]), then every E-valued topological (countable additive) measure is also a bornological measure when we consider the space E endowed with its von Neumann bornology.

The bornological measures present some special peculiarities in comparison with topological measures, thus the property about the boundedness of the range of a measure with values in a locally convex topological vector space and the Orlicz-Pettis theorem for these kind of measures do not admit a direct translation for bornological measures. For instance, if the space $L^1([0,1], \mu)$ is endowed with the compact bornology, where μ denotes the Lebesgue measure, and Σ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$, then the set mapping $m\colon \Sigma \to L^1([0,1], \mu)$ with $m(A) = \chi_A$, is a bornological measure but $m(\Sigma)$ is not relatively compact (see [5]). More questions about bornological measures, relations with the topological measures and examples can be found in [5] and [6].

Tensor product of bornological measures

Let $(X, \mathfrak{B}_1)$, $(Y, \mathfrak{B}_2)$ and $(Z, \mathfrak{B}_3)$ be three separated convex bornological spaces, Z being complete, and consider a bounded bilinear mapping b from $X \times Y$ into Z (we will write xy for $b(x,y)$) and an Y-valued bornological measure β defined on a σ-algebra Σ of subsets of a set Ω. Following Grothendieck's notation, for every $B \in \mathfrak{B}_1$ X_B denotes (and similarly in the other spaces) the subspace of X generated by B endowed with the topology defined by the Minkowski functional q_B of B (in X_B).

The measure β is said to be *of bounded b-semivariation* if

$$I_b(S_{B_1}) = \left\{ \int_\Omega s \, \mathrm{d}\beta : \ s \in S_{B_1} \right\} \in \mathfrak{B}_3$$

for every absolutely convex bounded set $B_1 \in \mathfrak{B}_1$, where S_{B_1} denotes the family of all B_1-valued simple functions defined on Ω (the simple functions and their integrals are defined as usual). From now on the measure β will be assumed to be of bounded semivariation.

A set $N \in \Sigma$ is a *(β, b)-null set* if $\|\beta\|_{B_1,B_3} = 0$ for every pair of absolutely convex sets $B_1 \in \mathfrak{B}_1$ and $B_3 \in \mathfrak{B}_3$ such that $I_b(S_{B_1}) \subseteq B_3$, and

$$\|\beta\|_{B_1,B_3}(A) = \sup\left\{ q_{B_3}\left(\int_\Omega s \, \mathrm{d}\beta \right) : \ s \in S_{B_1}, \ s\chi_{\Omega\setminus A} \equiv 0 \right\}$$

for every $A \in \Sigma$. We say that the measure β satisfies the *$(^*, b)$-condition* if for every pair of absolutely convex sets $B_1 \in \mathfrak{B}_1$ and $B_3 \in \mathfrak{B}_3$ with $I_b(S_{B_1}) \subseteq B_3$, there exists a measure $\nu_{B_1,B_3}\colon \Sigma \to \mathbb{R}^+$ such that $\|\beta\|_{B_1,B_3} \ll \nu_{B_1,B_3}$ (i.e.

$\lim_{\nu_{B_1,B_3}(A)\to 0} \|\beta\|_{B_1,B_3}(A) = 0$). If there exists a measure $\nu\colon \Sigma \to \mathbb{R}^+$ such that $\|\beta\|_{B_1,B_3} \ll \nu$ for every pair of absolutely convex sets $B_1 \in \mathfrak{B}_1$ and $B_3 \in \mathfrak{B}_3$ verifying $I_b(S_{B_1}) \subseteq B_3$, then it is said that the measure β satisfies the $(^{**}, b)$*-condition* and ν is called a *control measure* of β (then we write $\|\beta\|_b \ll \nu$).

DEFINITION 1. *A function $f\colon \Omega \to X$ is said to be (β, b)-measurable if there exists a sequence (f_n) of X-valued simple functions which is a.e. Mackey convergent to f (i.e. there exists a (β, b)-null set $N \in \Sigma$ and an absolutely convex bounded set $B_1 \in \mathfrak{B}_1$ such that $(f_n(t))$ converges to $f(t)$ in X_{B_1} for all $t \in \Omega \setminus N$).*

A (β, b)-measurable function $f\colon \Omega \to X$ is (β, b)*-integrable* if there exists a sequence (f_n) of X-valued simple functions, a (β, b)-null set $N \in \Sigma$ and an absolutely convex bounded set $B_1 \in \mathfrak{B}_1$ such that $(f_n(t))$ converges to $f(t)$ in X_{B_1} for all $t \in \Omega \setminus N$ and

$$\lim_{\|\beta\|_{B_1,B_3}(A)\to 0} q_{B_3}\left(\int_A f_n \, d\beta\right) = 0,$$

this limit being uniform in $n \in \mathbb{N}$, for every completing absolutely convex bounded set $B_3 \in \mathfrak{B}_3$ (note that (Z_{B_3}, q_{B_3}) is a Banach space since B_3 is completing) such that $I_b(S_{B_1}) \subseteq B_3$. Such a sequence is called a B_1*-approximating sequence of the function* f. If the measure β satisfies the $(^*, b)$-condition, then

$$\lim_n \int_A f_n \, d\beta = \int_A f \, d\beta$$

for every $A \in \Sigma$. When $N = \emptyset$ the function f is said to be *strong* (β, b)*-integrable.*

It can be proved in standard way that every essentially bounded (β, b)-measurable function is (β, b)-integrable, from which there follows easily the following:

BOUNDED CONVERGENCE THEOREM. *If (f_n) is a sequence of (β, b)-integrable functions which is a.e. Mackey convergent to a function $f\colon \Omega \to X$, $\bigcup_{n\in\mathbb{N}} f_n(\Omega) \in \mathfrak{B}_1$ and there exists a B_1-approximating sequence of f_n, for every $n \in \mathbb{N}$, B_1 being an absolutely convex bounded subset of X which contains $\bigcup_{n\in\mathbb{N}} f_n(\Omega)$, then the function f is also (β, b)-integrable, there* *exists a B_1-approximating sequence of f and*

$$\lim_n \int_A f_n \, d\beta = \int_A f \, d\beta$$

holds for every $A \in \Sigma$.

If Δ is a σ-algebra of subsets of a set Ω' and $\alpha\colon \Delta \to X$ is a bornological measure, then in standard way, if there exists one and only one bornological measure $\alpha \otimes \beta\colon \Delta \otimes \Sigma \to Z$ (as usual $\Delta \otimes \Sigma$ denotes the σ-algebra generated by $\Delta \times \Sigma$) such that the equality

$$\alpha \otimes \beta(A \times C) = b(\alpha(A), \beta(C))$$

holds for every pair $(A, C) \in \Delta \times \Sigma$, then the measure $\alpha \otimes \beta$ is called the *tensor product* (*bornological*) *measure of α and β.*

THEOREM 2. *Let Δ be a σ-algebra of subsets of a set Ω' and $\alpha\colon \Delta \to X$ a bornological measure. If the measure β satisfies the $(^{**}, b)$-condition and there exists an absolutely convex set $B_1 \in \mathfrak{B}_1$ such that $\alpha\colon \Delta \to X_{B_1}$ is a countably additive vector measure*[1], *then there exists the tensor product measure $\alpha \otimes \beta\colon \Delta \otimes \Sigma \to Z$, the mapping $\alpha(U^\bullet)\colon \Omega \to X$, defined by $\alpha(U^\bullet)(t) = \alpha(U^t)$ for every $t \in \Omega$, is (β, b)-integrable and*

$$\alpha \otimes \beta(U) = \int_\Omega \alpha(U^t) \, d\beta \tag{2.1}$$

holds for every $U \in \Delta \otimes \Sigma$ (U^t being as usual the t-section of U, for every $t \in \Omega$).

P r o o f. Let $B \in \mathfrak{B}_1$ be an absolutely convex set such that $\alpha(\Delta) \cup B_1 \subseteq B$ (remark that $\alpha(\Delta)$ is a bounded set of (X_{B_1}, q_{B_1})) and let us denote by $\mathfrak{C}$ the family of all measurable sets $U \in \Delta \otimes \Sigma$ such that the function $\alpha(U^\bullet)$ is (β, b)-integrable and it has a B-approximating sequence. Then clearly $\Delta \times \Sigma \subseteq \mathfrak{C}$ and $\Omega' \times \Omega \setminus U \in \mathfrak{C}$ for every $U \in \mathfrak{C}$. Moreover, if $(U_n) \subseteq \mathfrak{C}$ is an increasing sequence the same happens with (U_n^t) for every $t \in \Omega$, and therefore the equality

$$\alpha\left[\left(\bigcup_{n \in \mathbb{N}} U_n\right)^t\right] = \lim_n \alpha(U_n^t)$$

[1] It follows from [5, Corollary 8] that this condition holds in particular if the space X is infra-Schwartz (see [15]) or if it has a basis formed by completing absolutely convex bounded sets B_i such that X_{B_i} does not contain any copy of l^∞.

holds for every $t \in \Omega$, and it follows from the bounded convergence theorem that the function $\alpha\left[\left(\bigcup_{n\in\mathbb{N}} U_n\right)^{\bullet}\right]$ is (β, b)-integrable and it has a B-approximating sequence, so $\bigcup_{n\in\mathbb{N}} U_n \in \mathfrak{C}$ and $A \otimes \Sigma \subseteq \mathfrak{C}$. If $\lambda\colon \Delta \otimes \Sigma \to Z$ is defined by

$$\lambda(U) = \int_\Omega \alpha(U^t)\ \mathrm{d}\beta\,,$$

then λ is a bornological measure, since for every sequence (U_n) of pairwise disjoint sets of $A \otimes \Sigma$ it is deduced from the bounded convergence theorem that

$$\lambda\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \lim_n \int_\Omega \alpha\left[\left(\bigcup_{k=1}^{n} U_k\right)^t\right]\ \mathrm{d}\beta = \lim_n \sum_{k=1}^{n} \int_\Omega \alpha(U_k^t)\ \mathrm{d}\beta = \sum_{n=1}^{\infty} \lambda(U_n)\,,$$

from where the result follows easily.

Fubini theorems

Let $(X_i, \mathfrak{B}_i)$ be a separated convex bornological space for $i = 1, \ldots 6$, which will be supposed complete if $i = 4, 5, 6$, and consider four bounded bilinear mappings $b_1\colon X_1 \times X_2 \to X_4$, $b_2\colon X_3 \times X_1 \to X_5$, $b_3\colon X_5 \times X_2 \to X_6$ and $b_4\colon X_3 \times X_4 \to X_6$ such that

$$b_3\big[b_2(x_3, x_1), x_2\big] = b_4\big[x_3, b_1(x_1, x_2)\big]$$

for every $x_i \in X_i$ $(i = 1, 2, 3)$.

$\alpha\colon \Delta \to X_1$ and $\beta\colon \Sigma \to X_2$ will denote two bornological measures (Δ and Σ are as before two σ-algebras of subset of Ω' and Ω respectively) verifying the following conditions:

- i) There exists an absolutely convex set $B_1 \in \mathfrak{B}_1$ such that $\alpha\colon \Delta \to (X_1)_{B_1}$ is a countably additive vector measure. Let us denote by $B_1' \in \mathfrak{B}_1$ an absolutely convex set such that $\alpha(\Delta) \cup B_1 \subseteq B_1'$.
- ii) The measure β is of bounded b_3-semivariation and it satisfies the $(^{**}, b_i)$-condition for $i = 1, 3$.

Under the last two conditions, Theorem 2 states the existence of the product measure $\alpha \otimes \beta$ whose integral representation is given by (2.1).

PROPOSITION 3. *Let $x_3 \in X_3$ and $U \in \Delta \times \Sigma$, then the function $b_2\big[x_3, \alpha(U^{\bullet})\big]$ is (β, b_3)-integrable and*

$$\int_\Omega b_2\big[x_3, \alpha(u^t)\big]\ \mathrm{d}\beta = b_4\big[x_3, \alpha \otimes \beta(U)\big]\,. \tag{3.1}$$

Proof. Let $B_5 = b_2(x_3, B_1')$ and $\mathfrak{C}$ the family of all sets $U \in \Delta \otimes \Sigma$ such that $b_2\left[x_3, \alpha(U^{\bullet})\right]$ is a (β, b_3)-integrable function having a B_5-approximating sequence and (3.1) holds. If $U = A_1 \times A_2 \in \Delta \times \Sigma$, then $\alpha(U^{\bullet}) = \alpha(A_1)\chi_{A_2}$, $b_2\left[x_3, \alpha(U)\right] = b_3\left[x_3, \alpha(A_1)\right]$ and

$$b_4\left[x_3, \alpha \otimes \beta(A_1 \times A_2)\right] = b_4\left[x_3, b_1\left(\alpha(A_1), \beta(A_2)\right)\right] = b_3\left[b_2\left(x_3, \alpha(A_1)\right), \beta(A_2)\right]$$
$$= \int\limits_{\Omega} b_2\left[x_3, \alpha(U^{\bullet})\right] \, \mathrm{d}\beta .$$

Moreover, if $U \in \mathfrak{C}$, then it is easily proved that $\Omega' \times \Omega \setminus U \in \mathfrak{C}$ and for every increasing sequence $(U_n) \subseteq \mathfrak{C}$ we have that $\left(b_2\left[x_3, \alpha(U_n^t)\right]\right)$ converges to $b_2\left[x_3, \alpha\left(\left(\bigcup_{n\in\mathbb{N}} U_n\right)^t\right)\right]$ (in $(X_5)_{B_5}$) for every $t \in \Omega$ and

$$\bigcup_{t\in\Omega} b_2\left[x_3, \alpha(U_n^t)\right] \in b_2\left[x_3, \alpha(\Delta)\right] \subseteq B_5 ,$$

from where it is deduced by the bounded convergence theorem that the function $\alpha\left[\left(\bigcup_{n\in\mathbb{N}} U_n\right)^{\bullet}\right]$ is (β, b_3)-integrable, it has a B_5-approximating sequence and

$$b_4\left[x_3, \alpha \otimes \beta\left(\bigcup_{n\in\mathbb{N}} U_n\right)\right] = \lim_n b_4\left[x_3, \alpha \otimes \beta(U_n)\right] = \lim_n \int\limits_{\Omega} b_2\left[x_3, \alpha(U_n^t)\right] \, \mathrm{d}\beta$$
$$= \int\limits_{\Omega} b_2\left[x_3, \alpha\left(\left(\bigcup_{n\in\mathbb{N}} U_n\right)^t\right)\right] \, \mathrm{d}\beta .$$

Now the result follows immediately.

PROPOSITION 4. *If* $f = \sum_{k=1}^{n} x_k \chi_{U_k} : \Omega' \times \Omega \to X_3$ *is a simple function, then the function* $f_t = \sum_{k=1}^{n} x_k \chi_{U_k} t \colon \Omega' \to X_3$ *is* (α, b_2)*-integrable for every* $t \in \Omega$, *the function*

$$F(t) = \int\limits_{\Omega'} f_t \, \mathrm{d}\alpha = \sum_{k=1}^{n} x_k \alpha(U_k^t)$$

is (β, b_3)*-integrable and*

$$\int\limits_{\Omega' \times \Omega} f \, \mathrm{d}\alpha \otimes \beta = \int\limits_{\Omega} \left(\int\limits_{\Omega'} f_t \, \mathrm{d}\alpha \right) \mathrm{d}\beta . \tag{4.1}$$

P r o o f . It is an immediate consequence of Proposition 3.

PROPOSITION 5. *Let $(U_n)_{n\in\mathbb{N}} \subseteq \Delta \otimes \Sigma$ be a pairwise disjoint sequence and $B_3 \in \mathfrak{B}_3$, $B_6 \in \mathfrak{B}_6$ two absolutely convex bounded sets such that $I_{b_4}(S_{B_3}) \subseteq B_6$. If the measure α satisfies the $(^*, b_2)$-condition, then*

$$\lim_{n\to\infty} \|\alpha \otimes \beta\|_{B_3,B_6}\left(\bigcup_{k\geq n} U_k\right) = 0\,. \tag{5.1}$$

P r o o f . If (5.1) does not hold there exists $\varepsilon > 0$ and a sequence $(f_n) \subseteq S_{B_3}$ such that $f_n \chi_{\Omega'\times\Omega\setminus \bigcup_{k\geq n} U_k} \equiv 0$ and

$$q_{B_6}\left(\int_{\Omega'\times\Omega} f_n \, \mathrm{d}\alpha\otimes\beta\right) > \varepsilon\,.$$

If $B_5 = b_2(B_3, B_1')$, $f_n = \sum_{i=1}^{r_n} x_i^n \chi_{V_i^n}$ being $\{V_i^n\}_{i=1}^{r_n} \subseteq \Delta\otimes\Sigma$ a pairwise disjoint family such that $\bigcup_{i=1}^{r_n} V_i^n \subseteq \bigcup_{j\geq n} U_j$ and $\{x_i^n\}_{i=1}^{r_n} \subseteq B_3$, and

$$g_n(t) = \sum_{i=1}^{r_n} b_2\left[x_i^n, \alpha\big((V_i^n)^t\big)\right]$$

for every $t \in \Omega$ and $n \in \mathbb{N}$, then $I_{b_2}(B_3) \subseteq B_5$ and the inequality

$$\lim_n q_{B_5}\left[g_n(t)\right] \leq \lim_n \|\alpha\|_{B_3,B_5}\left(\bigcup_{j\geq n} U_j^t\right) = 0$$

holds for every $t \in \Omega$ and (since the measure β satisfies the $(^{**}, b_3)$-condition) there follows from Egoroff's theorem (for the Banach valued case, [7, p. 41]) the existence of a measurable set $A \in \Sigma$ such that $\|\beta\|_{B_5,B_6}(A) \leq \frac{\varepsilon}{2}$ and the sequence (g_n) converges to zero in $(X_5)_{B_5}$ uniformly on $\Omega \setminus A$, and therefore, there exists $n_0 \in \mathbb{N}$ such that

$$q_{B_5}\left[g_n(t)\right] \leq \varepsilon/2\,\|\beta\|_{B_5,B_6}(\Omega) \quad {}^2$$

and

$$q_{B_6}\left(\int_{\Omega'\times\Omega} f_n \,\mathrm{d}\alpha\otimes\beta\right) = q_{B_6}\left[\int_{\Omega}\left(\sum_{i=1}^{r_n} b_2\left(x_i^n, \alpha((V_i^n)^t)\right)\right)\mathrm{d}\beta\right]$$

$$\leq \frac{\varepsilon}{2\|\beta\|_{B_5,B_6}(\Omega)}\|\beta\|_{B_5,B_6}(\Omega) + \|\beta\|_{B_5,B_6}(A) \leq \varepsilon\,,$$

which is a contradiction.

PROPOSITION 6. *If the measure α verifies the $(^*, b_2)$-condition, then the product measure $\alpha\otimes\beta$ verifies the $(^*, b_4)$-condition. Moreover if α verifies the $(^{**}, b_2)$-condition, then the measure $\alpha\otimes\beta$ verifies the $(^{**}, b_4)$-condition and if λ, μ are two control measures of α and β respectively, then $\lambda\otimes\mu$ is a control measure of $\alpha\otimes\beta$.*

Proof. Let $B_3 \in \mathfrak{B}_3$ and $B_6 \in \mathfrak{B}_6$ be two absolutely convex sets such that $I_{b_4}(S_3) \subseteq B_6$ and $B_5 = b_2(B_3, B_1')$. If the measure α verifies the $(^*, b_2)$-condition, then there exist two measures $\lambda_{B_3,B_5}\colon \Delta \to \mathbb{R}^+$ and $\mu\colon \Sigma \to \mathbb{R}^+$ such that $\|\alpha\|_{B_3,B_5} \ll \lambda_{B_3,B_5}$ and $\|\beta\|_{B_5,B_6} \ll \mu$ (clearly $I_{b_3}(S_{B_5}) \subseteq B_6$).

If $U \in \Delta\otimes\Sigma$ verifies that $\lambda_{B_3,B_5}\otimes\mu(U) = 0$, then there follows from the Fubini theorem for scalar measures the existence of a measurable set $A \in \Sigma$ such that $\mu(\Omega\setminus A) = 0$ and $\lambda_{B_3,B_5}(U^t) = 0$ for every $t \in A$, and therefore,

$$b_4\left[x_3, \alpha\otimes\beta(V)\right] = \int_{\Omega} b_2\left[x_3, \alpha(V^t)\right]\,\mathrm{d}\beta$$

$$= \int_A b_2\left[x_3, \alpha(V^t)\right]\,\mathrm{d}\beta + \int_{\Omega\setminus A} b_2\left[x_3, \alpha(V^t)\right]\,\mathrm{d}\beta = 0$$

for every $x_3 \in B_3$ and $V \in \Delta\otimes\Sigma$ with $V \subseteq U$. From which it follows that $\|\alpha\otimes\beta\|_{B_3,B_6}(U) = 0$.

Moreover, if there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we can found $U_n \in \Delta\otimes\Sigma$ with $\lambda_{B_3,B_5}\otimes\mu(U_n) \leq \frac{1}{2}n$ and $\|\beta\|_{B_3,B_6}(U_n) > \varepsilon$, then if $V_n = \bigcup_{i\geq n} U_i$, $W_n = V_n\setminus V_{n+1}$ and $V = \bigcap_{n\in\mathbb{N}} V_n$ we have that

$$\|\beta\|_{B_3,B_6}(U_n) \leq \|\alpha\otimes\beta\|_{B_3,B_6}(W_n) + \|\alpha\otimes\beta\|_{B_3,B_6}(V)\,,$$

from which a contradiction follows since $\|\alpha\otimes\beta\|_{B_3,B_6}(V) = 0$ and it results from Proposition 5 that

$$\lim_n \|\alpha\otimes\beta\|_{B_3,B_6}(W_n) = 0\,.$$

The remainder of the proof follows now easily from the preceding.

[2] If $\|\beta\|_{B_5,B_6}(\Omega) = 0$, then the result is trivial.

THEOREM 7. *If Λ is a σ-algebra of subsets of a set Ω'', $\gamma\colon \Lambda \to X_3$ is a bornological measure such that $\gamma\colon \Lambda \to X_{B_3}$ is a countably additive vector measure for some absolutely convex set $B_3 \in \mathfrak{B}_3$ and the measure α verifies the $(**, b_2)$-condition, then the product measures $(\gamma \otimes \alpha) \otimes \beta$ and $\gamma \otimes (\alpha \otimes \beta)$ exist, they coincide and the equality*

$$\gamma \otimes (\alpha \otimes \beta)(U) = \int_{\Omega} \left[\int_{\Omega'} \gamma(U^{(s,t)}) \, \mathrm{d}\alpha \right] \mathrm{d}\beta$$

holds for every $U \in \Lambda \otimes \Delta \otimes \Sigma$ ($U^{(s,t)}$ being as usual the (s,t)-section of U for every pair $(s,t) \in \Omega' \times \Omega$).

P r o o f . It follows immediately from Theorem 2 and Proposition 6.

THEOREM 8. *Let us suppose the existence of two measures $\lambda\colon \Delta \to \mathbb{R}^+$ and $\mu\colon \Sigma \to \mathbb{R}^+$ such that $\|\alpha\|_{b_2} \ll \lambda$, $\|\beta\|_{b_3} \ll \mu$ and the $(\alpha \otimes \beta, b_4)$-null sets and the $\lambda \otimes \mu$-null sets coincide. If $f\colon \Omega' \otimes \Omega \to X_3$ is an essentially bounded $(\alpha \otimes \beta, b_4)$-integrable function, then the function $f_t\colon \Omega' \to X_3$, defined by $f_t(t) = f(s,t)$ for every $s \in \Omega'$, is (α, b_2)-integrable for almost all $t \in \Omega$ and the function $F\colon \Omega \to X_5$ such that*

$$F(t) = \int_{\Omega'} f_t \, \mathrm{d}\alpha$$

is (β, b_3)-integrable when f_t is (α, b_2)-integrable, and

$$\int_{\Omega' \times \Omega} f \, \mathrm{d}\alpha \otimes \beta = \int_{\Omega} \left(\int_{\Omega'} f_t \, \mathrm{d}\alpha \right) \mathrm{d}\beta . \tag{8.1}$$

P r o o f . In fact, there exists an absolutely convex set $B_3 \in \mathfrak{B}_3$, a $(\alpha \otimes \beta, b_4)$-null set N and a sequence of simple functions $f^{(n)}\colon \Omega' \times \Omega \to X_3$ such that $f(\Omega' \times \Omega \setminus N) \bigcup_{n \in \mathbb{N}} f^{(n)}(\Omega' \times \Omega \setminus N) \subseteq B_3$, $(f^{(n)}(s,t))$ converges to $f(s,t)$ in $(X_3)_{B_3}$ for every pair $(s,t) \in \Omega' \times \Omega \setminus N$ and

$$\lim_n \int_{\Omega' \times \Omega} f^{(n)} \, \mathrm{d}\alpha \otimes \beta = \int_{\Omega' \times \Omega} f \, \mathrm{d}\alpha \otimes \beta .$$

Since

$$\lambda \otimes \mu(N) = \int_{\Omega} \lambda(N)^t \, \mathrm{d}\mu ,$$

there exists a μ-null set $N_2 \in \Sigma$ such that $\lambda(N^t) = 0$ for every $t \in \Omega \setminus N_2$ and the sequence $(f_t^{(n)}(s))$ converges to $f_t(s)$ (in $(X_3)_{B_3}$) for every $t \in \Omega \setminus N_2$ and $s \in \Omega' \setminus N^t$. Thus, since the function f is $\alpha \otimes \beta$-essentially bounded, it results that the function f_t is (α, b_2)-integrable for every $t \in \Omega \setminus N_2$, and

$$\lim_n \int_{\Omega'} f_t^{(n)} \, d\alpha = \int_{\Omega'} f_t \, d\alpha, \tag{8.2}$$

in $(X_5)_{B_5}$ B_5 being an absolutely convex bounded subset of X_5 such that $b_2(B_3, B_1') \subseteq B_5$.

It results from Proposition 4, the bounded convergence theorem and (8.2) that the function F is (β, b_3)-integrable and

$$\int_{\Omega} F(t) \, d\beta = \lim_n \int_{\Omega} \left(\int_{\Omega'} f_t \, d\alpha \right) d\beta,$$

from where by Proposition 4 we obtain (8.1).

THEOREM 9. *With the notations of the last theorem, if the measure α verifies the $(^*, b_2)$-condition and $f\colon \Omega' \times \Omega \to X_3$ is a strong $(\alpha \otimes \beta, b_4)$-integrable function such that $f(\Omega' \times \Omega) \in \mathfrak{B}_3$, then the function f_t is (α, b_2)-integrable for almost every $t \in \Omega$, the function F is (β, b_3)-integrable and* (8.1) *holds.*

Proof. It is enough to proceed like in the proof of Theorem 8.

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