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LINEAR AND $\mathbb{R}$-LINEAR BETWEENNESS SPACES

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ABSTRACT. We will prove that the theory of the class of all betweenness spaces embeddable into some vector space over the reals does not coincide with the theory of the class of all betweenness spaces embeddable into some vector space over an arbitrary ordered field. This solves a problem raised by Mendris and Zlatoš in [*Axiomatization and undecidability results for metrizable betweenness relations*, Proc. Amer. Math. Soc. **123** (1995), 873–882].

Let V be a vector space over an arbitrary ordered field F. The ternary betweenness relation T_V on V can be defined by

$$T_V(x,y,z) \iff (\exists \lambda \in F)(0 \le \lambda \le 1 \ \& \ y - x = \lambda(z - x))$$

for $x, y, z \in V$. A first-order structure (X,T) with a single ternary relation T is called a *linear betweenness space* if (X,T) can be embedded into the ternary structure (V,T_V) for some vector space V over an arbitrary ordered field F. The class of all linear betweenness spaces is denoted by $\mathcal{L}$. Further, (X,T) will be called an $\mathbb{R}$-*linear betweenness space* if it can be embedded into a vector space over the ordered field $\mathbb{R}$ of all real numbers. The class of all $\mathbb{R}$-linear betweenness spaces will be denoted by $\mathcal{L}_0$.

As proved by R. Mendris and P. Zlatoš in [2], $\mathcal{L}$ is an elementary class, recursively axiomatizable by a set of universal sentences. On the other hand, being not closed under elementary extensions, the class $\mathcal{L}_0$ is not an elementary one. However, the question whether $\mathcal{L}$ is the least elementary class containing $\mathcal{L}_0$, or, equivalently, the question whether $\operatorname{Th}\mathcal{L} = \operatorname{Th}\mathcal{L}_0$, remained open. In this note we will answer that question negatively.

Let $\alpha(a,b,u,v,w,x,z)$ denote the formula in the first-order language with equality and a single ternary relational symbol T, expressing that a,b,u,v,w,x and z are seven distinct elements and $T(u,x,v)$, $T(u,a,b)$, $T(a,b,w)$ are the

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only nontrivial betweenness relations among them. Denote by φ the following sentence of the same language

$$\begin{aligned}&(\exists\, a,b,u,v,w,x,z)\alpha(a,b,u,v,w,x,z)\\ \&\ &(\forall\, a,b,u,v,w,x,z)\Big[\alpha(a,b,u,v,w,x,z) \implies (\exists\, y,x')\big(T(w,y,z)\ \&\ T(x,x',v)\\ &\qquad\qquad \&\ T(x,a,y)\ \&\ T(y,b,x')\big)\Big].\end{aligned}$$

PROPOSITION. *Let V be a vector space over some ordered field F, $X \subseteq V$ and $T = T_V \cap X^3$. If (X,T) satisfies φ, then F is not Archimedean.*

Proof. Let $a,b,u,v,w,x,z \in X$ be such that $\alpha(a,b,u,v,w,x,z)$ holds. Denote by P the line $\overline{uv}$ in V going through the points u, v, and by Q the line $\overline{wz}$. Since $(X,T) \vDash \varphi$, there exist $y,x' \in X$ such that $y \in Q$, $x' \in P$ and $T(x,a,y)$, $T(y,b,x')$ hold. Then we have again that $\alpha(a,b,u,v,w,x',z)$, thus, using this construction repeatedly, there will arise a sequence $(x_n)_{n<\omega}$ of points of P such that $x_0 = x$ and $x_{n+1} = x'_n$; more precisely, $\alpha(a,b,u,v,w,x_n,z)$ and $T(x_n,x_{n+1},v)$ hold for each $n < \omega$. Obviously, all the points of the set $Y = \{a,b,u,v,w,z\} \cup \{x_n :\ n < \omega\} \subseteq V$ are co-planar in V, thus, locating the origin into some point of this plane, Y can be considered as a subset of a two-dimensional subspace D of V.

Now, assume that F is Archimedean. Then F can be regarded as a subfield of $\mathbb{R}$, and D can be embedded into $\mathbb{R}^2$ via an injective F-linear map, and the betweenness relation $T \cap Y^3 = T_V \cap Y^3$ coincides with the linear betweenness relation on $\mathbb{R}^2$ restricted to Y. Identifying Y with its copy in $\mathbb{R}^2$ we will prove the following claim.

Claim 1. The lines $P = \overline{uv}$ and $Q = \overline{wz}$ are parallel.

Assume the contrary. As the betweenness relation T_V, as well as the parallelness of lines obviously are invariant under affine transformations, we can regard, without loss of generality, P and Q as the co-ordinate axes in D, and assign to the vectors u, w the co-ordinates $u = (1,0)$, $w = (0,1)$. Then the points a,b,u,v,w,z,x have to be arranged in either of the ways shown in Diagram 1, depending on the position of x on P relative to u.

We will show that, in contradiction to $(X,T) \vDash \varphi$, in either case the sequence $(x_n)_{n<\omega}$ cannot be constructed. Consider, first, the situation on Diagram 1(a). Denote $A = \{(\zeta,0) \in \mathbb{R}^2 :\ \zeta > 1\} \subseteq P$. Let S_1 be the line parallel to P going through b. Put $d_1 = S_1 \cap Q$, $R_1 = \overline{ad_1}$ and $c_1 = R_1 \cap P$ (see Diagram 2). Assume that c_1 has the co-ordinates $(\zeta_1,0) \in \mathbb{R}^2$. Denote $A_1 = \{(\zeta,0) \in \mathbb{R}^2 :\ \zeta > \zeta_1\} \subseteq P$.

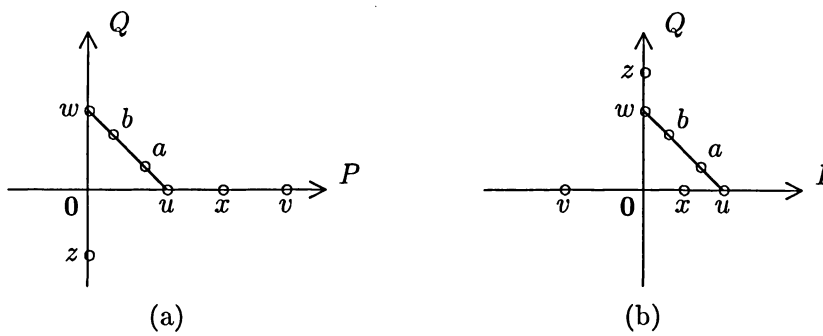

DIAGRAM 1.

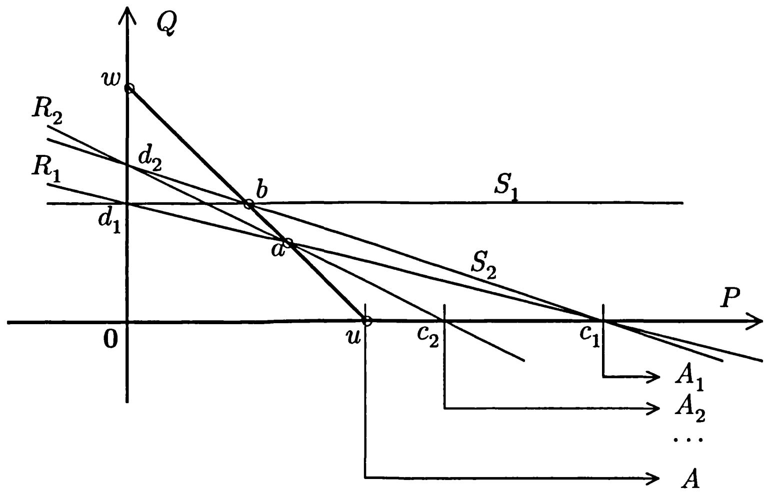

DIAGRAM 2.

Further, let $S_n = \overline{bc_{n-1}}$, $d_n = S_n \cap Q$, $R_n = \overline{ad_n}$, and $c_n = R_n \cap P$ have the co-ordinates $(\zeta_n, 0)$. We put $A_n = \{(\zeta, 0) \in \mathbb{R}^2 : \zeta > \zeta_n\}$ for $n = 2, 3, \ldots$. Translating the above construction into the language of analytic geometry, it can be shown that the sequence $(\zeta_n)_{n<\omega}$ converges to 1, hence $A = \bigcup_{n<\omega} A_n$. By induction we can prove that for any $x \in A_k$ it is not possible to construct the kth term of the sequence $(x_n)_{n<\omega}$. In other words, for any $x \in A$ only a finite part of the sequence $(x_n)_{n<\omega}$ can be constructed.

The dual situation based on Diagram 1(b) can be treated in a similar way.

We will need one more auxiliary result.

Claim 2. Let d denote the Euclidean metric on $\mathbb{R}^2$. Then $d(x_{n+1}, x_n) \geq d(x_n, x_{n-1})$ for each $0 < n < \omega$.

We can suppose that P coincides with the x-axis in $\mathbb{R}^2$, Q is parallel to P, $u = (0,0)$ and $v = (\nu_1, 0)$, $\nu_1 > 0$. Let $g : \mathbb{R}^+ \to \mathbb{R}$ denote the function defined by $g(\sigma) = \sigma' - \sigma$, where $\sigma > 0$, $s = (\sigma, 0)$ and $s' = (\sigma', 0)$ is the point assigned to s by the construction from the beginning of the proof. By means of elementary calculus it can be shown that the derivative $g'(\sigma)$ is positive for every $\sigma > 0$. As $d(x_{n+1}, x_n) = g(\xi_n)$, where $x_n = (\xi_n, 0)$, this concludes the proof of Claim 2.

Now, we are able to conclude the proof of the proposition. We have

$$d(v,x) = d(v,x_n) + d(x_n, x_{n-1}) + \cdots + d(x_1, x) \geq (n+1) \cdot d(x_1, x)$$

for any $0 < n < \omega$. However, as $d(x_1, x) > 0$ and F is Archimedean, this is the desired contradiction. □

THEOREM 1. *$\neg\varphi \in \mathrm{Th}\,\mathcal{L}_0 \setminus \mathrm{Th}\,\mathcal{L}$; consequently, the least elementary class containing $\mathcal{L}_0$ is a proper subclass of $\mathcal{L}$.*

Proof.

(1) $\neg\varphi \in \mathrm{Th}\,\mathcal{L}_0$:
Suppose (X,T) is a linear betweenness space embedded into a vector space V and $(X,T) \models \varphi$. According to the previous proposition, V cannot be a vector space over $\mathbb{R}$, hence $(X,T) \notin \mathcal{L}_0$.

(2) $\neg\varphi \notin \mathrm{Th}\,\mathcal{L}$:
We will provide an example of an $(X,T) \in \mathcal{L}$ satisfying φ. Let F be any non-Archimedean ordered field, and $V = F^2$. Identify $n < \omega$ with $n \cdot 1 \in F$ and ω with an arbitrary element of F bigger that every n. Put $X = \{a, b, u, v, w, z, x_0, x_1, \dots, y_0, y_1, \dots\} \subseteq V$, where

$$\begin{array}{llll} a = (0,0)\,, & w = (0,2)\,, & u = (0,-2)\,, & x_i = (3^i, -2)\,,\ i < \omega\,, \\ b = (0,1)\,, & z = (-\omega, 2)\,, & v = (\omega, -2)\,, & y_i = (-3^i, 2)\,,\ i < \omega\,, \end{array}$$

and $T = T_{\mathrm{V}} \cap X^3$ (see Diagram 3). One can easily check that $(X,T) \in \mathcal{L}$ and $(X,T) \models \varphi$.

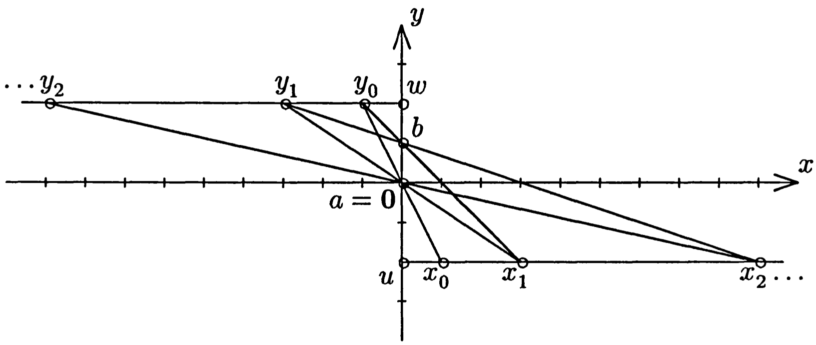

DIAGRAM 3.

□

The sentence $\neg\varphi$ is equivalent to an existential-universal one. By the next theorem, this is the simplest possible form of the "counterexamples" from $\mathrm{Th}\,\mathcal{L}_0 \setminus \mathrm{Th}\,\mathcal{L}$.

THEOREM 2. *$\mathcal{L}$ is both the least universal class and the least universal-existential class containing $\mathcal{L}_0$.*

Proof. Denote by $\mathcal{K}$ the least universal-existential class containing $\mathcal{L}_0$, i.e., $\mathrm{Th}\,\mathcal{K} = \mathrm{Th}_{\forall\exists}\,\mathcal{L}_0$. Obviously, $\mathcal{K} \subseteq \mathcal{L}$. We will show, that every countable $(X,T) \in \mathcal{L}$ belongs to $\mathcal{K}$, thus $\mathrm{Th}\,\mathcal{K} = \mathrm{Th}\,\mathcal{L}$. Indeed, every countable (X,T) can be written as the union of a chain of its finite substructures. However, $\mathcal{L}$ and $\mathcal{L}_0$ contain the same finite substructures. The result now follows from the fact that the universal-existential class $\mathcal{K}$ is closed under unions of chains. □

The situation with linear betweenness spaces is, in some sense, similar to that with the metrizable ones, studied by Mendris and Zlatoš in [1]. A metrizable betweenness space is a structures of the form (X,T_d), where d is a metric on X with values in some ordered field F, and T_d is the ternary betweenness relation on X induced by d, defined by

$$T_d(x,y,z) \iff d(x,z) = d(x,y) + d(y,z)$$

for $x,y,z \in X$. As proved in [2], the class $\mathcal{M}$ of all metrizable betweenness spaces is an elementary one, while its subclass $\mathcal{M}_0$ of all betweenness spaces metrizable by real-valued metrics is not. However, the question whether the smallest elementary class $\overline{\mathcal{M}_0}$ containing $\mathcal{M}_0$ coincides with $\mathcal{M}$ was left open in [1]. Using similar methods, in [3] we have answered this question negatively, as well, providing an existential-universal counterexample $\neg\theta \in \mathrm{Th}\,\mathcal{M}_0 \setminus \mathrm{Th}\,\mathcal{M}$.

Again, as every universal-existential sentence valid in $\mathcal{M}_0$ is true in $\mathcal{M}$, this counterexample is of the simplest possible form as far as the complexity of its quantifier prefix is concerned (cf. [3]).

It is just a matter of skill to verify that the structure $(X, T) \in \mathcal{L}$, defined in the final part of the proof of Theorem 1, is $\mathbb{R}$-metrizable. Therefore, in addition to the known proper inclusions $\mathcal{L}_0 \subset \mathcal{M}_0$ and $\mathcal{M} \subset \mathcal{L}$ (cf. [2]), we have also the following.

THEOREM 3. *The least elementary class $\overline{\mathcal{L}}_0$ containing $\mathcal{L}_0$ is a proper subclass of $\overline{\mathcal{M}}_0 \cap \mathcal{L}$.*

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