Ivan Dobrakov; Jana Dobrakovová

Daniell type extensions of L_1-gauges and integrals

Mathematica Slovaca, Vol. 47 (1997), No. 3, 267--281

Persistent URL: `http://dml.cz/dmlcz/132796`

Math. Slovaca, **47** (1997), No. 3, 267–281

DANIELL TYPE EXTENSIONS OF L_1-GAUGES AND INTEGRALS

IVAN DOBRAKOV† * — JANA DOBRAKOVOVÁ **

(Communicated by Miloslav Duchoň)

ABSTRACT. We show that a Daniell type extension procedure is possible if the subadditivity is replaced by a much weaker requirement of subadditive continuity. Particullary, we prove that the Fatou and Daniell properties are preserved by this completion of the elementary L_1-gauge.

Introduction

An approach to vector integration based on application of the Daniell scheme, using outer L_1-pseudonorms induced by vector measures, named L_1-gauges, was elaborated by K. Bichteler in [3]. Its generalization to non locally convex spaces outlined in [4] led in [5] to substantial progress in stochastic integration, particularly in the L^p-theory for $p \in [0,1)$. A similar investigation of subadditive Daniell gauges and the integrals they dominate was done by M. Wilhelm in [20], [21] and [22] in an abstract setting. In [8], we will show that L_1-gauges of a non linear integration corresponding to Riesz type representations of non linear Hammerstein operators on $C(S,E)$ from [1] and [2] are only subadditively continuous, see (3) in Definition 1 below. Hence, it is of interest to investigate the Daniell scheme of integration in such a general setting. At the same time, we continue our program announced already in [9], see also [10] – [13].

Elementary L_1-gauges introduced by Definition 1 usually have an additional property connected with multiplication by scalars which determines the linear topological structure they induce, see Theorems 1 and 2. In Section 2, we show that an elementary L_1-gauge J on a function lattice $\mathcal{F}$ always has an extension to a function lattice $\mathcal{L}$ such that $\mathcal{L}$ is complete in the pseudometric induced by this extension.

AMS Subject Classification (1991): Primary 28C05, 28C99.
Key words: vector lattice, subadditive continuity, elementary L_1-gauge, Fatou property, Daniell property.

In the main Section 3 of this paper, we prove that our extension preserves the Fatou and Daniell properties, see Theorems 8 and 9 respectively.

Finally, in Theorem 11, we show that, by applying our extension procedure again, we obtain no further extension. The proofs of these results are subtle and more involved then in the subadditive case treated in [3] and [20]. At the same time they show how far subadditivity can be replaced by the much weaker requirement of subadditive continuity.

We mentioned earlier that L_1-gauges of our extra generality occur in the non linear integration theory ([8]) corresponding to Riesz type representation of non linear Hammerstein operators on $C(S,E)$ from [1] and [2]. Namely, having a non linear measure m on a δ-ring P of subsets of a set T, in [8], we show that

$$J(f)=J(f,T)=\sum_{k=1}^{\infty} k^{-2}\cdot \tilde{m}(kf,T)\cdot\bigl(1+\tilde{m}(kf,T)\bigr)^{-1}$$

is in general only a subadditively continuous L_1-gauge with the Daniell property on $\tilde{L}_1(m)$.

Here $\tilde{m}(g,E)$ is defined as in part II of [7], and

$$\tilde{L}_1(m)=\bigl\{g\,,\ \ g \text{ is } P\text{-measurable and } J(g,\cdot)\colon \sigma(P)\to\bigl[0,\tfrac{\Pi^2}{6}\bigr) \text{ is continuous}\bigr\}\,.$$

In other words, the analog $J\colon \tilde{L}_1(m)\to\bigl[0,\frac{\Pi^2}{6}\bigr)$ of the L_1-pseudonorm $\int\limits_T |f|\,\mathrm{d}\mu$ of the classical $L_1(\mu)$ in non linear integration theory is in general only a subadditively continuous L_1-gauge. Let us note that here $J(f)$ must in general depend on the multiples kf, $k=1,2,\dots$, since, otherwise, $\tilde{L}_1(m)$ will not be a linear space.

1. Elementary L_1-gauges

Suppose T is a non-empty set. A collection $\mathcal{F}$ of functions on T with values in $\mathbb{R}=(-\infty,+\infty)$ (in $\mathbb{R}^*=[-\infty,+\infty]$) is called an $\mathbb{R}$- ($\mathbb{R}^*$-) function lattice if $af+bg$, $f\vee g$, $f\wedge g\in\mathcal{F}$ provided $f,g\in\mathcal{F}$ and $a,b\in\mathbb{R}$. $S(\mathbb{R})$ and $C_{00}(T)$ are the standard examples of $\mathbb{R}$-function lattices. For more information about function lattices, see [12].

DEFINITION 1. By *elementary L_1-gauge* we mean a couple $(\mathcal{F},J)$, where $\mathcal{F}$ is an $\mathbb{R}$-function lattice on T, $J\colon\mathcal{F}\to\mathbb{R}^+$ and has the following properties:

(1) $J(0)=0$ and $J(f)=J\bigl(|f|\bigr)$ for each $f\in\mathcal{F}$.

(2) $0\le f\le g \implies J(f)\le J(g)$.
We say that J is *monotone* on $\mathcal{F}^+=\{f, f\in\mathcal{F},\ \ f\ge 0\}$.

(3) $f_n \in \mathcal{F}$, $n = 1,2,\dots$, and $J(f_n - f_m) \to 0$ as $n,m \to \infty \implies J(f_n) - J(f_m) \to 0$ as $n,m \to \infty$.
We say that J is *extendable subadditively continuous.*

(4) $f, f_n \in \mathcal{F}^+$, $f_n \nearrow f \implies J(f_n) \nearrow J(f)$.
We say that J *has the Fatou property.*

(5) $f_n, g_n \in \mathcal{F}$, $n = 1,2,\dots$ and $J(f_n) + J(g_n) \to 0$ as $n \to \infty \implies J(f_n + g_n) \to 0$ as $n \to \infty$.
We say that J *has the pseudometric generating property.*

Clearly, (3) implies the subadditive continuity, or autocontinuity of J on $\mathcal{F}$, i.e.,

$$f, f_n \in \mathcal{F},\ n = 1,2,\dots,\ \text{and } J(f - f_n) \to 0 \implies J(f_n) \to J(f)$$
$$\iff \forall f{\in}\mathcal{F}\ \forall\varepsilon{>}0\ \exists\delta{>}0 \text{ such that}$$
$$g \in \mathcal{F} \text{ and } J(g) < \delta \implies J(f) - \varepsilon \le J(f+g) \le J(f) + \varepsilon\,.$$

If $\mathcal{F}$ is complete in ϱ, $\varrho(f,g) = J(f-g)$, then the converse is also true.

We now list subsequent strengthenings of (3).

- J is uniformly subadditively continuous, or uniformly autocontinuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that
  $f, g \in \mathcal{F}\ J(f-g) < \delta \implies |J(f) - J(g)| < \varepsilon$.
  This property clearly implies (5).
- J is K-subadditive if there is a $K \in [1,\infty)$ such that
  $J(f+g) \le J(f) + K \cdot J(g)$ for each $f, g \in \mathcal{F}$.
- J is subadditive if $J(f+g) \le J(f) + J(g)$ for each $f, g \in \mathcal{F}$.
- J is additive on $\mathcal{F}^+$-clear.

Note that a subadditive (additive) J is, in fact, countably subadditive (additive).

We say that J is a *Daniell elementary L_1-gauge* if $f_n \in \mathcal{F}^+$, $n = 1,2,\dots$, and $f_n \searrow 0 \implies J(f_n) \searrow 0$. This Daniell property implies that J is monotonically continuous on $\mathcal{F}^+$, and particularly the Fatou property of J.

As we will see in Theorems 1 and 2 below, the following additional properties of J concerning multiplication by scalars from $\mathbb{R}$ determine the linear topological structure induced by J on $\mathcal{F}$.

- J is scalarly continuous if $J(a_n f) \to J(af)$ whenever $f \in \mathcal{F}$, $a_n, a \in \mathbb{R}$, $n = 1,2,\dots$, and $a_n \to a$, or equivalently, $\lim\limits_{n\to\infty} J\left(\frac{f}{n}\right) = 0$ for each $f \in \mathcal{F}$. Obviously, each Daniell J has this property.
- J is locally bounded if there exists $a > 0$, and for each $n = 1,2,\dots$ there is a positive integer $N(n)$ such that $J\left(\frac{f}{N(n)}\right) < \frac{1}{n}$ whenever $f \in \mathcal{F}$ and $J(f) < a$.

- J is p-homogeneous if there is a $p \in (0,1]$ such that $J(af) = |a|^p \cdot J(f)$ for each $a \in \mathbb{R}$ and each $f \in \mathcal{F}$.
- J is subhomogeneous if $J(af) \leq |a| \cdot J(f)$ for each $a \in \mathbb{R}$, $f \in \mathcal{F}$.
- J is homogeneous if $J(af) = |a| \cdot J(f)$ for each $a \in \mathbb{R}$, $f \in \mathcal{F}$.

For $E \subset T$ and $f\colon T \to \mathbb{R}$ put $\|f\|_E = \sup\limits_{t \in E} |f(t)|$. Assertions (1) and (2) of the next lemma are immediate, while (3) follows from the well-known theorem of Dini.

LEMMA 1.

(1) *Let $1 \in \mathcal{F}$, and let each $f \in \mathcal{F}$ be a bounded function. Then an elementary L_1-gauge $J\colon \mathcal{F} \to \mathbb{R}^+$ is scalarly continuous if and only if $J(f_n) \to 0$ whenever $f_n \in \mathcal{F}$, $n = 1, 2, \dots$, and $\|f_n\|_T \to 0$.*

(2) *An elementary L_1-gauge $J\colon S(\mathbb{R}) \to \mathbb{R}^+$ is scalarly continuous if and only if $J(f_n \cdot \chi_E) \to 0$ whenever $E \in \mathbb{R}$, $f_n \in S(\mathbb{R})$, $n = 1, 2, \dots$, and $\|f_n\|_E \to 0$.*

(3) *An elementary L_1-gauge $J\colon C_{00}(T) \to \mathbb{R}^+$ is scalarly continuous if and only if it is a Daniell L_1-gauge.*

THEOREM 1. *Suppose $J\colon \mathcal{F} \to \mathbb{R}^+$ has the properties (1), (2) and (5) of Definition 1. Then:*

(1) *There are δ_k, $0 < \delta_k \leq 2^{-k}$, $k = 1, 2, \dots$, such that $\delta_k \searrow 0$, $J(f+g) < \delta_k$ whenever $f, g \in \mathcal{F}$ and both $J(f), J(g) < \delta_{k+1}$, and $J\Big(\sum\limits_{i=k+1}^{k+p} f_i\Big) < \delta_k$ for each $k, p = 1, 2, \dots$ whenever $f_k \in \mathcal{F}$ and $J(f_k) < \delta_k$ for each $k = 1, 2, \dots$.*

(2) *There is an invariant pseudometric $\rho_c\colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}^+$ such that $\rho_c(f,g) \to 0$ if and only if $J(f-g) \to 0$, $f, g \in \mathcal{F}$. $(\mathcal{F}, \rho_c)$ is a pseudometric linear topological space if and only if J is scalarly continuous.*

Proof.

(1) According to (5) of Definition 1, for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that $J(f+g) < \varepsilon$ whenever $f, g \in \mathcal{F}$ and both $J(f), J(g) < \delta(\varepsilon)$. Take δ_1 so that $0 < \delta_1 < 2^{-1}$, and put $\delta_2 = 2^{-1}\delta_1 \wedge \delta(2^{-1}\delta_1), \dots, \delta_{n+1} = 2^{-1}\delta_n \wedge \delta(2^{-1}\delta_n), \dots$.

(2) Take δ_k, $k = 1, 2, \dots$, from (1), and put $V_k = \big\{(f,g) \in \mathcal{F} \times \mathcal{F},\ J(f-g) < \delta_k\big\}$. Clearly, V_k, $k = 1, 2, \dots$, is a countable base of an invariant uniformity. It remains for us to use the metrization Lemma 12 and Problem N in [14; Chapter VI]. □

THEOREM 2. *Suppose $J\colon \mathcal{F} \to \mathbb{R}^+$ is a Daniell elementary L_1-gauge. Then there is a subadditive Daniell L_1-gauge $J_c\colon \mathcal{F} \to \mathbb{R}^+$ such that $J(f) \to 0$ if and only if $J_c(f) \to 0$, $f \in \mathcal{F}$.*

If J is subhomogeneous (locally bounded), then J_c may be chosen so as to be homogeneous (p-homogeneous).

Proof. Take ρ_c as in Theorem 1, and put $\|f\| = \rho_c(f,0)$. Then clearly $f_n, g_n \in \mathcal{F}$, $|g_n| \le |f_n|$, $n = 1,2,\dots$, and $\|f_n\| \to 0$ implies $\|g_n\| \to 0$. If $\lambda_n \in \mathbb{R}$, $n = 1,2,\dots$, and $\lambda_n \to 0$, then $\|\lambda_n f\| \to 0$ for each $f \in \mathcal{F}$ by scalar continuity of J.

Hence, just as in the proof of Theorem VII.1.4 in [19], we obtain that J_c, $J_c(f) = \sup\{\|g\|,\ 0 \le g \le |f|\}$, has the required property. J_c is clearly subhomogeneous (locally bounded) if and only if J is so. In that case, $V = \{f, f \in \mathcal{F},\ J(f) \le 1\}$ is bounded convex neighbourhood of 0. Hence Kolmogorov's theorem, see [16; Theorem III.2.1″], implies the existence of a homogeneous J_c.

The locally bounded case follows from Theorem III.2.1 in [16]. □

Suppose $J\colon \mathcal{F} \to \mathbb{R}^+$ is uniformly autocontinuous and scalarly continuous. Then clearly, $\sup_{f\in\mathcal{F}} |J(f+g) - J(f)| < \infty$ for each $g \in \mathcal{F}$. Moreover, if J is locally bounded, then there is an $a > 0$ such that $\sup\{|J(f+g) - J(f)|,\ f, g \in \mathcal{F},\ J(g) < a\} < \infty$.

Less obvious is the next lemma:

LEMMA 2. *Suppose $J\colon \mathcal{F} \to \mathbb{R}^+$ is uniformly autocontinuous and p-homogeneous. Then there is a $K \in [1,\infty)$ such that $J(f+g) \le J(f) + K \cdot J(g)$ whenever $f, g \in \mathcal{F}$ and $J(g) \le 1$.*

Proof. Take $g \in \mathcal{F}$ so that $0 < J(g) \le 1$, and let $\varepsilon > 0$. By the uniform autocontinuity of J, there is a δ, $0 < \delta < J(g)$, such that $|J(f) - J(h)| < \varepsilon$ whenever $f, h \in \mathcal{F}$ and $J(f-h) < \delta$.

Next take $\delta_1 > 0$ so that $J(h) < \delta$ whenever $h \in \mathcal{F}$ and $J(2^{-1}h) < \delta_1$. Since, by assumption J is scalarly continuous, there is a positive integer $N > 1$ such that $J\big((N-1)^{-1}g\big) \ge \delta$ and $J(N^{-1}g) < \delta$.

But then $J(N^{-1}g) \ge \delta_1$ since, otherwise, we have the contradiction: $J\big((N-1)^{-1}g\big) \le J(2N^{-1}g) < \delta$. Hence $\delta_1 \le J(N^{-1}g) = N^{-p} \cdot J(g)$ for some $p \in (0,1]$ by the assumed p-homogeneity of J.

But then $N \le \big(\delta_1^{-1} \cdot J(g)\big)^{\frac{1}{p}} \le \delta_1^{-\frac{1}{p}} \cdot J(g)$ since $J(g) \le 1$. Hence $J(f+g) = J(f + N \cdot N^{-1} \cdot g) \le J(f) + N \cdot \varepsilon \le J(f) + \varepsilon \cdot \delta_1^{-\frac{1}{p}} \cdot J(g)$ for each $f \in \mathcal{F}$. □

2. Extension of L_1-gauges

In what follows, $(\mathcal{F}, J)$ will be a given elementary L_1-gauge. All functions considered are defined on T and have values in $[-\infty, \infty]$. Unless otherwise specified, the arrows $\to$ ($\nearrow$ or $\searrow$) denote pointwise (monotone) convergence of functions considered. Denote

$$\mathcal{F}^o = \{f\,,\ \exists f_n \in \mathcal{F}\,,\ n = 1, 2, \dots\,,\ \text{such that } f_n \nearrow f\}\,,$$
$$\mathcal{F}_u = \{f\,,\ \exists f_n \in \mathcal{F}\,,\ n = 1, 2, \dots\,,\ \text{such that } f_n \searrow f\}\,.$$

For $f \in \mathcal{F}^{o^+} = \{f\,,\ f \in \mathcal{F}^o \text{ and } f \geq 0\}$ we put $J^o(f) = \lim\limits_{n\to\infty} J(f_n)$, where $f_n \in \mathcal{F}^+$, $n = 1, 2, \dots$, and $f_n \nearrow f$. The monotonicity and the Fatou property of $J\colon \mathcal{F} \to \mathbb{R}$ imply that $J^o\colon \mathcal{F}^{o^+} \to [0, \infty]$ is uniquely defined. Evidently, J^o extends J, is monotone and has the pseudometric generating property. In a standard way, see [18; 6.2.III.d], it follows that J^o has the Fatou property. Hence J^o is countably subadditive provided J is subadditive. Finally, J^o has any of the properties: uniformly subadditively continuous, positively additive, locally bounded, p-homogeneous, subhomogeneous, homogeneous, provided J has the corresponding property. Note that, except for the Fatou property and positive additivity, the analogs hold for the outer L_1-gauge J^* which we now define.

Put

$$\mathcal{F}^* = \{f\,,\ \exists g \in \mathcal{F}^{o^+} \text{ such that } |f| \leq g\}\,,$$

and for $f \in \mathcal{F}^*$ we define its outer L_1-gauge $J^*(f)$ by

$$J^*(f) = \inf\{J^o(g)\,,\ g \in \mathcal{F}^{o^+}\,,\ |f| \leq g\}\,.$$

We define the null functions $\mathcal{N}$ and the null sets by:

$$\mathcal{N} = \{f\,,\ f \in \mathcal{F}^* \text{ and } J^*(f) = 0\}\,, \qquad \text{and} \qquad \boldsymbol{N} = \{N\,,\ N \subset T\,,\ \chi_N \in \mathcal{N}\}\,.$$

$\mathcal{F}^*$ is clearly a hereditary $\mathbb{R}^*$-function lattice, J^* extends J^o, and $J^*(\mathcal{F}) = J^*(|f|)$ for each $f \in \mathcal{F}^*$.

THEOREM 3.

(1) *Let δ_k, $k = 1, 2, \dots$, be as in Theorem 1. If $f, g \in \mathcal{F}^*$ and $J^*(f), J^*(g) < \delta_{k+1}$, then $J^*(f+g) < \delta_k$, and if $f_i \in \mathcal{F}^*$ and $J^*(f_i) < \delta_i$ for $i = 1, 2, \dots$, then $J^*\Big(\sum\limits_{i=k+1}^{k+p} f_i\Big) < \delta_k$ and $J^*\Big(\sum\limits_{i=k+1}^{\infty} f_i\Big) \leq \delta_k$ for each $k, p = 1, 2, \dots$.*

(2) *$\mathcal{N}$ is a hereditary σ-sublattice of $\mathcal{F}^*$, $\boldsymbol{N}$ is a hereditary σ-ring of subsets of T, and $f \in \mathcal{N}$ if and only if the set $\{t\,,\ t \in T\,,\ f(T) \neq 0\} \in \boldsymbol{N}$.*

(3) *$\mathcal{F}^*$ is complete in ρ^*, $\rho^*(f, g) = J^*(f - g)$.*

(4) *If* $f, f_n \in \mathcal{F}^*$, $n = 1, 2, \dots$, *and* $J^*(f - f_n) \to 0$ *as* $n \to \infty$, *then there is a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_{n_k} \to f$ *almost everywhere* $\boldsymbol{N}$ *(a.e.* $\boldsymbol{N}$*) as* $k \to \infty$.

Proof.

(1) If $f, g, f_i \in \mathcal{F}^o$, $i = 1, 2, \dots$, then the Fatou property of J^o implies the required assertions. From here, the general case easily follows from the definition of J^*.

(2) The first assertion follows from (1), while the equivalence is a consequence of the inequalities:

$$\begin{aligned} |f| &\le \lim_{n\to\infty} n \cdot |f| \\ &= +\infty \cdot \chi_{\{t,\, t\in T,\, f(t)\neq 0\}} = \lim_{n\to\infty} n \cdot \chi_{\{t,\, t\in T,\, f(t)\neq 0\}} \\ &\ge \chi_{\{t,\, t\in T,\, f(t)\neq 0\}}\,. \end{aligned}$$

(3) Suppose $f_n \in \mathcal{F}^*$, $n = 1, 2, \dots$, and $J^*(f_n - f_m) \to 0$ as $n, m \to 0$. Take a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $J^*(f_{n_{k+1}} - f_{n_k}) < \delta_{k+1}$, where δ_k, $k = 1, 2, \dots$, is from (1).

Then $\sum\limits_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| = h_k \searrow h$, and $J^*(h) = 0$ since $J^*(h) \le J^*(h_k) \le \delta_k$ for each $k = 1, 2, \dots$ by (1).

Define $f(t) = \lim\limits_{k\to\infty} f_{n_k}(t)$ if $h(t) = 0$, and $f(t) = 0$ if $h(t) > 0$. Obviously, $f_{n_k} \to f$ a.e. $\boldsymbol{N}$ as $k \to \infty$. Since $|f - f_{n_k}| \le \lim\limits_{p\to\infty} \sum\limits_{i=k}^{p} |f_{n_{i+1}} - f_{n_i}|$ for each $k = 1, 2, \dots$, $J^*(f - f_{n_k}) \to 0$ as $k \to \infty$, hence also $J^*(f - f_n) \to 0$ as $n \to \infty$.

(4) From the proof of (3) it is evident that a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $J^*(f - f_{n_k}) < \delta_k$ for $k = 1, 2, \dots$ has the required property. □

Definition 2. We denote by $\mathcal{L}$ the closure of $\mathcal{F}$ in $(\mathcal{F}^*, J^*)$, and for $f \in \mathcal{L}$ we put $J(f) = \lim\limits_{n\to\infty} J(f_n)$, where $f_n \in \mathcal{F}$, $n = 1, 2, \dots$, is such that $J^*(f - f_n) \to 0$ as $n \to \infty$.

The existence, finiteness and uniqueness of the above limit follow from the extendable subadditive continuity of $J\colon \mathcal{F} \to [0, \infty)$ and the pseudometric generating property of $J^*\colon \mathcal{F}^* \to [0, \infty]$. $\mathcal{L}$ is clearly an $\mathbb{R}^*$-function lattice.

Using (4) of Theorem 3 we have the following equivalent:

Definition 2′. We say that $f \in \mathcal{L}$ if $f \in \mathcal{F}^*$, and there are $f_n \in \mathcal{F}$, $n = 1, 2, \dots$, such that $f_n \to f$ a.e. $\boldsymbol{N}$ and $J(f_n - f_m) \to 0$ as $n, m \to \infty$. In that case, we put $J(f) = \lim\limits_{n\to\infty} J(f_n)$.

Using Definition 2 one easily checks the assertions of Theorem 4.

THEOREM 4. *$J\colon \mathcal{L} \to [0,\infty)$ extends $J\colon \mathcal{F} \to \mathbb{R}^+$, it shares the properties (1), (2), (3), and (5) of $J\colon \mathcal{F} \to \mathbb{R}^+$ from Definition 1, and, except for the Daniell property, it also shares any of the additional properties of J on $\mathcal{F}$ listed in Section 1.*

We note that the Fatou and Daniell properties of J on $\mathcal{L}$ will be investigated in the Section 3 below.

LEMMA 3.

(1) *$\mathcal{N} = \{f,\ f \in \mathcal{L}$ and $J(f) = 0\}$.*

(2) *Let $f \in \mathcal{L}$. Then $\{t,\ t \in T,\ |f(t)| = \infty\} \in \boldsymbol{N}$, and the factor space $\mathbb{L} = \mathcal{L}/\mathcal{N}$ is a linear lattice.*

Proof.

(1) Clearly, $\mathcal{N} \subset \mathcal{L}$, and $J(f) = 0$ for $f \in \mathcal{N}$. Suppose $f \in \mathcal{L}$ and $J(f) = 0$. Take $f_n \in \mathcal{F}$, $n = 1,2,\dots$, so that $J^*(f - f_n) \to 0$. Thus $J^*(f) = J^*(f - f_n + f_n) \to 0$ by the pseudometric generating property of J^*. Hence $f \in \mathcal{N}$.

(2) Take $f_n \in \mathcal{F}$, $n = 1,2,\dots$, so that $J^*(f - f_n) \to 0$. Since each f_n, $n = 1,2,\dots$, is finite valued, we have $J^*\big(+\infty\cdot\chi_{\{t,\ t\in T,\ |f(t)|=+\infty\}}\big) = J^*\big(|[f - f_n + (-f + f_n - f + f_n)] - [(f - f_n - f + f_n) - f + f_n]|\big) \le J^*\big(6\cdot|f - f_n|\big) \to 0$. (We are using the convention $(+\infty) + (-\infty) = (-\infty) + (+\infty) = 0$.)

The last assertion is evident. □

Using Definition 2 one also easily checks the assertions of (1) of the next theorem. Assertion (2) of Theorem 5 follows in the same way as (2) in Theorem 1, and Theorem 2.

THEOREM 5.

(1) *The analogs of assertions (1), (3), and (4) of Theorem 3 hold for $(\mathcal{L}, J)$ and ρ, $\rho(f,g) = J(f-g)$, $f,g \in \mathcal{L}$.*

(2) *There is an invariant metric ρ_c on $\mathcal{L}\times\mathcal{L}$ equivalent with ρ. $(\mathcal{L},\rho_c)$ is a complete metric space for any such control metric ρ_c.*

It is a linear topological space if J is scalarly continuous. The analog of Theorem 2 holds.

3. Fatou and Daniell properties of the extension

Theorem 6 below is of fundamental importance for showing that our extension $J\colon \mathcal{L} \to \mathbb{R}^+$ shares various convergence properties of the elementary L_1-gauge $J\colon \mathcal{F} \to \mathbb{R}^+$, particularly those mentioned above. In Theorem 11, we prove that by applying the extension procedure to $J\colon \mathcal{L} \to \mathbb{R}^+$, we obtain no further enlargement of $\mathcal{L}$.

We will need the following notions.

$$\begin{aligned}
\widehat{\mathcal{F}^o} &= \{f\,,\ \text{there are } f_n \in \mathcal{F}\,,\ n=1,2,\dots\,,\ \text{such that} \\
&\qquad\qquad f_n \nearrow f \text{ and } J^o(f-f_n) \to 0\}\,, \\
\widehat{\mathcal{F}_u} &= \{f\,,\ \text{there are } f_n \in \mathcal{F}\,,\ n=1,2,\dots\,,\ \text{such that} \\
&\qquad\qquad f_n \searrow f \text{ and } J^o(f_n-f) \to 0\}\,, \\
\hat{\mathcal{F}} &= \{f\,,\ \text{there exists } h \in \mathcal{F}^{o^+} \text{ such that } |f| \le h\}\,, \\
(\widehat{\mathcal{F}^o})_u &= \{f\,,\ \text{there are } f_n \in \widehat{\mathcal{F}^o}\,,\ n=1,2,\dots\,,\ \text{such that } f_n \searrow f\}\,, \\
(\widehat{\mathcal{F}_u})^o &= \{f\,,\ \text{there are } f_n \in \widehat{\mathcal{F}_u}\,,\ n=1,2,\dots\,,\ \text{such that } f_n \nearrow f\}\,.
\end{aligned}$$

For $f \in \hat{\mathcal{F}}$ put

$$\hat{J}(f) = \inf\{J^o(h)\,,\ h \in \mathcal{F}^{o^+}\,,\ |f| \le h\}\,,$$

and let $\hat{\mathcal{L}}$ denote the closure of $\mathcal{F}$ in $(\hat{\mathcal{F}}, \hat{J})$, i.e., $f \in \hat{\mathcal{L}}$ if $f \in \hat{\mathcal{F}}$, and there are $f_n \in \mathcal{F}$, $n=1,2,\dots$, such that $\hat{J}(f-f_n) \to 0$.

Clearly, $J^*(f) \le \hat{J}(f)$ for each $f \in \hat{\mathcal{F}}$, hence

$$\hat{\mathcal{N}} = \{f\,,\ f \in \hat{\mathcal{F}}\,,\ \hat{J}(f) = 0\} \subset \mathcal{N} \quad \text{and} \quad \hat{\boldsymbol{N}} = \{E\,,\ E \subset T\,,\ \chi_E \in \hat{\mathcal{N}}\} \subset \boldsymbol{N}\,.$$

Further, $\hat{J}\colon \hat{\mathcal{F}} \to \mathbb{R}^+ = [0,\infty)$, and $\hat{J}$ behaves similarly as $J^*\colon \mathcal{F}^* \to \mathbb{R}^*$.

THEOREM 6. *Suppose $f_n \in \mathcal{F}$, $n=1,2,\dots$, and $J(f_n - f_m) \to 0$ as $n,m \to \infty$. Then there is a subsequence $\{f_{n_k}\} \subset \{f_n\}$, a sequence $\varphi_k \in \widehat{\mathcal{F}^o}$, $k=1,2,\dots$, and a sequence $\psi_k \in \widehat{\mathcal{F}_u}$, $k=1,2,\dots$, such that:*

(1) *$\varphi_k \searrow \varphi \in (\widehat{\mathcal{F}^o})_u$, $\psi_k \nearrow \psi \in (\widehat{\mathcal{F}_u})^o$, $\psi_k \le f_{n_k} \le \varphi_k$, for each $k = 1,2,\dots$, and $J^o(\varphi_k - \psi_k) \to 0$ as $k \to \infty$,*

(2) *$f_{n_k}(t) \to \varphi(t)$ a.e. $\widehat{\boldsymbol{N}}$, and $\varphi(t) = \psi(t)$ a.e. $\widehat{\boldsymbol{N}}$,*

(3) *$\hat{J}(f_n - \varphi) = \hat{J}(f_n - \psi) \to 0$ as $n \to \infty$.*

P r o o f. Let δ_k, $k=1,2,\dots$, be the sequence of Theorem 1, and take a subsequence $\{f_{n_k}\} \subset \{f_n\}$ so that $J(f_{n_{k+1}} - f_{n_k}) < \delta_k$ for each $k = 1,2,\dots$. Then

$$J^o\left(\sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|\right) \le \delta_{k-1} < \delta_{k-2}$$

by the Fatou property of $J^o\colon \mathcal{F}^{o^+} \to [0,\infty]$. For $k = 1,2,\dots$ put

$$\varphi_k = \Big(f_{n_k} + \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|\Big) + \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \,.$$

Clearly, $\varphi_k \in \widehat{\mathcal{F}^o}$ for each $k = 1,2,\dots$. If $|\varphi_k(t)| < \infty$, then $|f_{n_i}(t)| < \infty$ and $|\varphi_i(t)| < \infty$ for $i \geq k$, hence

$$\varphi_k(t) - \varphi_{k+1}(t) = f_{n_k}(t) - f_{n_{k+1}}(t) + 2|f_{n_{k+1}}(t) - f_{n_k}| \leq 0 \,.$$

If $\varphi_k(t) = -\infty$, then $f_{n_i}(t) = -\infty$ for $i \geq k$, hence $\varphi_{k+1}(t) = -\infty$. Thus $\varphi_k \searrow \varphi \in \big(\widehat{\mathcal{F}^o}\big)_u$.

For $k = 1,2,\dots$ put

$$\psi_k = \Big(f_{n_k} - \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|\Big) - \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \,.$$

Then $\psi_k \in \widehat{\mathcal{F}_u}$ for each $k = 1,2,\dots$, and similarly as above, $\psi_k \nearrow \psi \in \big(\widehat{\mathcal{F}_u}\big)^o$.

Obviously, $\psi_k \leq f_{n_k} \leq \varphi_k$ for each $k = 1,2,\dots$. Since

$$\varphi_k - \psi_k \leq 4\sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \,, \qquad J^o(\varphi_k - \psi_k) < \delta_{k-6}$$

for $k = 7,8,\dots$. Hence $\varphi(t) = \psi(t)$ a.e. $\hat{\boldsymbol{N}}$.

Since $\psi = \liminf_k \psi_k \leq \liminf_k f_{n_k} \leq \limsup_k f_{n_k} \leq \limsup_k \varphi_k = \varphi$, $f_{n_k}(t) \to \varphi(t)$ a.e. $\hat{\boldsymbol{N}}$.

Clearly,

$$|f_{n_k} - \varphi| \leq |f_{n_k} - \varphi_k| + |\varphi_k - \varphi| \,,$$

$$|f_{n_k} - \varphi_k| \leq 2\sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \,,$$

and

$$\varphi_k - \varphi = \sum_{i=k}^{\infty} (\varphi_i - \varphi_{i+1}) \leq 3\sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \,.$$

Hence $\hat{J}(f_{n_k} - \varphi) < \delta_{k-7}$ for $k = 8,9,\dots$.

Thus $\hat{J}(f_n - \varphi) \to 0$ as $n \to \infty$ by the pseudometric generating property of $\hat{J}$ on $\hat{\mathcal{F}}$, see (5) in Definition 1. □

COROLLARY 1. *$\hat{\mathcal{L}}$ is complete in $\hat{\rho}$, $\hat{\rho}(f,g) = \hat{J}(f-g)$, and $f \in \hat{\mathcal{L}}$ if and only if $f \in \hat{\mathcal{F}}$, and there are $f_n \in \mathcal{F}$, $n = 1,2,\dots$, such that $f_n(t) \to f(t)$ a.e. $\hat{\boldsymbol{N}}$, and $J(f_n - f_m) \to 0$ as $n,m \to \infty$.*

COROLLARY 2. *$f \in \hat{\mathcal{L}}(\mathcal{L})$ if and only if $f \in \hat{\mathcal{F}}(\mathcal{F}^*)$, and for each $\varepsilon > 0$ there are $g \in \widehat{\mathcal{F}_u}$ and $h \in \widehat{\mathcal{F}^o}$ such that $g(t) \le f(t) \le h(t)$ a.e. $\hat{\boldsymbol{N}}$ ($\boldsymbol{N}$) and $J(h-g) < \varepsilon$.*

COROLLARY 3. *$\mathcal{F} \subset \widehat{\mathcal{F}^o} \oplus \widehat{\mathcal{F}_u} \oplus \mathcal{N} \subset \hat{\mathcal{L}} \subset \big((\widehat{\mathcal{F}^o})_u \oplus \mathcal{N}\big) \cap \big((\widehat{\mathcal{F}_u})^o \oplus \mathcal{N}\big)$, $\widehat{\mathcal{F}^o} \oplus \widehat{\mathcal{F}_u} \oplus \mathcal{N} \subset \mathcal{L} \subset \big((\widehat{\mathcal{F}^o})_u \oplus \mathcal{N}\big) \cap \big((\widehat{\mathcal{F}_u})^o \oplus \mathcal{N}\big)$, and $\mathcal{L} = \hat{\mathcal{L}} \oplus \mathcal{N}$, where $\oplus$ means addition of elements.*

THEOREM 7. *$\hat{J}(f) = J(f)$ for $f \in \hat{\mathcal{L}}$.*

Proof. Let $f \in \hat{\mathcal{L}}$, $f_n \in \mathcal{F}$, $n = 1, 2, \dots$, and $\hat{J}(f - f_n) \to 0$. Then $J^*(f - f_n) \to 0$, hence $J(f_n) \to J(f)$. Thus, by Theorem 6, $\hat{J}(f) = \inf\{J^o(h),\ h \in \widehat{\mathcal{F}^o}^+,\ |f| \le h\} \le \inf_k J^o(\varphi_k) = \lim_{k\to\infty} J(\varphi_k) = \lim_{k\to\infty} J(f_{n_k}) = J(f)$.

Conversely, for $\varepsilon > 0$ take $h_\varepsilon \in \widehat{\mathcal{F}^o} \subset \mathcal{L}$ so that $\varphi \le h_\varepsilon$ and $J(h) = J^o(h) \le \hat{J}(\varphi) + \varepsilon$, where φ is from Theorem 6. Then $J(f) = J(\varphi) \le J(h) = J^o(h) \le \hat{J}(\varphi) + \varepsilon = \hat{J}(f) + \varepsilon$. □

THEOREM 8. *Suppose $f, f_n \in \mathcal{L}^+$, $n = 1, 2, \dots$, and $f_n(t) \nearrow f(t)$ a.e. $\boldsymbol{N}$. Then $J(f_n) \nearrow J(f)$. In particular, $J\colon \mathcal{L} \to \mathbb{R}^+$ has the Fatou property.*

Proof. First we show that $J\colon \hat{\mathcal{L}} \to \mathbb{R}^+$ has the Fatou property. Let $f, f_n \in \hat{\mathcal{L}}^+$, $n = 1, 2, \dots$, and $f_n \nearrow f$. Take $h \in \widehat{\mathcal{F}^o}^+$ so that $h \ge f$, and let $\varepsilon > 0$. Let finally δ_k, $k = 1, 2, \dots$, be a sequence according to assertions (1) of Theorems 1 and 5.

Owing to autocontinuity of $J\colon \hat{\mathcal{L}} \to \mathbb{R}^+$, see assertion (1) of Theorem 5, for each $n = 1, 2, \dots$ there is a $k_n > k_{n-1}$, $k_0 = 1$, such that $g \in \hat{\mathcal{L}}$ and $\hat{J}(g) < \delta_{k_n}$ implies $J(f_n + g) \le J(f_n) + \varepsilon$. According to Theorem 6, for each $n = 1, 2, \dots$ there is an $h_n \in \widehat{\mathcal{F}^o}^+$ such that $h_n \ge f_n$ and $J(h_n - f_n) < \delta_{k_{n+1}}$.

Obviously, $\left(\bigvee_{i=1}^n h_i\right) \wedge h \nearrow h' \ge f$, hence

$$J\left(\left(\bigvee_{i=1}^n h_i\right) \wedge h\right) = J^o\left(\left(\bigvee_{i=1}^n h_i\right) \wedge h\right) \nearrow J^o(h') \ge \hat{J}(f) = J(f)$$

by the Fatou property of $J^o\colon \mathcal{F}^o \to [0, \infty]$ and by Theorem 7. Clearly,

$$g_n = \left(\bigvee_{i=1}^n h_i\right) \wedge h - f_n \le \sum_{i=1}^n (h_i - f_i),$$

hence, $J(g_n) < \delta_{k_n}$. Thus

$$J\left(\left(\bigvee_{i=1}^n h_i\right) \wedge h\right) = J(f_n + g_n) \le J(f_n) + \varepsilon,$$

hence,

$$J(f) = \hat{J}(f) \le J^o(h') = \lim_{n\to\infty} J\left(\left(\bigvee_{i=1}^{n} h_i\right) \wedge h\right) \le \lim_{n\to\infty} J(f_n) + \varepsilon .$$

Since $\varepsilon > 0$ was arbitrary, $J(f) \le \lim_{n\to\infty} J(f_n)$. The converse inequality follows from monotonicity of J on $\mathcal{L}^+$.

Suppose now $f, f_n \in \mathcal{L}^+$, $n = 1, 2, \dots$, and $f_n \nearrow f$ a.e. $\boldsymbol{N}$. Take $N \in \boldsymbol{N}$ so that $f_n \nearrow f$ on $T - N$, and put $f_n' = f_n \cdot \chi_{T-N} + f \cdot \chi_N$ for $n = 1, 2, \dots$.

Then $f_n' \in \mathcal{L}^+$ and $J(f_n') = J(f_n)$ for $n = 1, 2, \dots$, and $f_n' \nearrow f$ on T. Now it is easy to verify that $\mathcal{L} = \hat{\mathcal{L}}_N$, where $\hat{\mathcal{L}}_N$ corresponds to $J_N \colon \mathcal{F} \oplus \mathcal{N} \to \mathbb{R}^+$, $J_N(f + u) = J(f)$ for $f \in \mathcal{F}$ and $u \in \mathcal{N}$.

Hence $J(f_n) = J(f_n') \nearrow J(f)$ by the first part of the proof. □

THEOREM 9. *Suppose $J \colon \mathcal{F} \to \mathbb{R}^+$ is a Daniell elementary L_1-gauge. Then $J(f_n) \searrow 0$ whenever $f_n \in \mathcal{L}^+$, $n = 1, 2, \dots$, and $f_n(t) \searrow 0$ a.e. $\boldsymbol{N}$. Hence $J \colon \mathcal{L} \to \mathbb{R}^+$ has the Daniell property, $\widehat{\mathcal{F}^o} = \mathcal{F}^o \cap \mathcal{L}$ and $\widehat{\mathcal{F}_u} = \mathcal{F}_u \cap \mathcal{L}$.*

Proof. Without loss of generality, suppose $f_n \in \mathcal{L}^+$, $n = 1, 2, \dots$, $f_n \searrow 0$, and let $\varepsilon > 0$. By the pseudometric generating property of $J \colon \mathcal{L} \to \mathbb{R}^+$, see Definition 1 and Theorem 4, take $\delta > 0$ so that $J(f + g) < \varepsilon$ whenever $f, g \in \mathcal{L}$ and both $J(f), J(g) < \delta$. Take a sequence δ_k, $k = 1, 2, \dots$, as in assertions (1) of Theorems 1 and 5, and let k_0 be such that $\delta_{k_0} < \delta$. By Corollary 2 of Theorem 6, there are $h_n \in \widehat{\mathcal{F}^o}^+$, $n = 1, 2, \dots$, such that $h_n(t) \ge f_n(t)$ a.e. $\boldsymbol{N}$, and $J(h_n - f_n) \le \delta_{k_0+n}$ for each $n = 1, 2, \dots$, and $h_n \searrow h$.

Since $h = \lim_{n\to\infty} (h_n - f_n) \le \sum_{n=1}^{\infty} (h_n - f_n)$, $\hat{J}(h) \le \delta_{k_0} < \delta$. Take $h' \in \widehat{\mathcal{F}^o}$, $n = 1, 2, \dots$, such that $h' \ge h$ and $\hat{J}(h') < \delta$, and let $u_n \in \mathcal{F}^+$, $n = 1, 2, \dots$, be such that $u_n \nearrow h'$. Then $h_n \le (h_n - u_n) + h'$ for each $n = 1, 2, \dots$, and $h_n - u_n \searrow 0$.

Since $h_n - u_n \in \widehat{\mathcal{F}^o}$, $n = 1, 2, \dots$, there are $v_n \in \mathcal{F}^+$, $n = 1, 2, \dots$, such that $v_n \le h_n - u_n$ and $J(h_n - u_n - v_n) < \delta_{k_0+1+n}$ for each $n = 1, 2, \dots$.

Put $w_n = \bigwedge_{i=1}^{n} v_i$, $n = 1, 2, \dots$. Then $w_n \in \mathcal{F}^+$ for each $n = 1, 2, \dots$, and $w_n \searrow 0$. Hence, by the Daniel property of $J \colon \mathcal{F} \to \mathbb{R}^+$, there is an n_0 such that $J(w_n) < \delta_{k_0+1}$ for $n \ge n_0$.

Since $h_n - u_n - w_n = \bigwedge_{i=1}^{n} (h_i - u_i) - \bigwedge_{i=1}^{n} v_i \le \sum_{i=1}^{n} (h_i - u_i - v_i)$, $J(h_n - u_n - w_n) < \delta_{k_0+1}$ for each $n = 1, 2, \dots$. Hence

$$J(h_n - u_n) = J(h_n - u_n - w_n + w_n) < \delta_{k_0} < \delta \qquad \text{for} \quad n \ge n_0 .$$

Thus $J(f_n) \leq J(h_n) < \varepsilon$ for $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, the theorem is proved. □

We now show that by applying our extension procedure to $J\colon \mathcal{L} \to \mathbb{R}^+$ we obtain no further enlargement of $\mathcal{L}$. This is of basic importance in connection with the Stone type representation of $J\colon \mathcal{L} \to \mathbb{R}^+$.

Denote by J_1^o the extension of J to $\mathcal{L}^o \supset \mathcal{F}^o$. Clearly, $J_1^*(f) \leq J^*(f)$ for each $f \in \mathcal{F}^* = \mathcal{L}^*$, where

$$J_1^*(f) = \inf\{J_1^o(h),\ \ h \in \mathcal{L}^{o+},\ \ |f| \leq h\} \qquad \text{for} \quad f \in \mathcal{F}^*.$$

THEOREM 10. *Suppose $J\colon \mathcal{F} \to \mathbb{R}^+$ is uniformly autocontinuous. Then $J_1^*(f) = J^*(f)$ for each $f \in \mathcal{F}^* = \mathcal{L}^*$. Hence $J(f) = J^*(f)$ for $f \in \mathcal{F}$ in this case.*

P r o o f. Let $f \in \mathcal{F}^*$, $\varepsilon > 0$, and $u \in \mathcal{L}^o$ be such that $u \geq f$ and $J_1^o(u) \leq J_1^*(f) + 3^{-1}\cdot\varepsilon$. Take $u_n \in \mathcal{L}^+$, $n = 1, 2, \dots$, so that $u_n \nearrow u$ and $J(u_n) \nearrow J_1^o(u)$. Let δ_k, $k = 1, 2, \dots$, be a sequence as in assertion (1) of Theorems 1 and 5, and take k_0 so that $\delta_{k_0} < \delta(3^{-1}\cdot\varepsilon)$, where $\delta(3^{-1}\cdot\varepsilon)$ is such that $|J^*(v) - J^*(w)| < 3^{-1}\cdot\varepsilon$ whenever $v, w \in \mathcal{F}^*$ and $J^*(v - w) < \delta(3^{-1}\cdot\varepsilon)$. By Corollary 2 of Theorem 6, for each $n = 1, 2, \dots$ there exists $h_n \in \widehat{\mathcal{F}^o}$ such that $h_n(t) \geq u_n(t)$ a.e. $\boldsymbol{N}$ and $J^*(h_n - u_n) < \delta_{k_0+n}$. For $n = 1, 2, \dots$ put $h_n' = \bigvee_{i=1}^{n} h_i$. Then $h_n'(t) \nearrow h'(t) \geq f(t)$ a.e. $\boldsymbol{N}$, and $h_n' - u_n \leq \sum_{i=1}^{n} (h_i - u_i)$.

Hence $J^o(h') = \lim_{n\to\infty} J^o(h_n')$ and $J^o(h_n' - u_n) < \delta_{k_0} < \delta(3^{-1}\cdot\varepsilon)$ for each $n = 1, 2, \dots$. Since $h'(t) \geq f(t)$ a.e. $\boldsymbol{N}$, there is a $v \in \mathcal{F}^{o+}$ such that $h' + v \geq f$ and $J^o(v) < \delta(3^{-1}\cdot\varepsilon)$.

Thus

$$\begin{aligned} J^*(f) &\leq J^o(h' + v) \leq J^o(h') + 3^{-1}\cdot\varepsilon \\ &= 3^{-1}\cdot\varepsilon + \lim_{n\to\infty} J^o(h_n') = 3^{-1}\cdot\varepsilon + \lim_{n\to\infty} J^o(h_n' - u_n + u_n) \\ &\leq 2\cdot 3^{-1}\cdot\varepsilon + \lim_{n\to\infty} J(u_n) = 2\cdot 3^{-1}\cdot\varepsilon + J_1^o(u) \\ &\leq J_1^*(f) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $J_1^*(f) = J^*(f)$. □

THEOREM 11. *Suppose $f_k \in \mathcal{F}^* = \mathcal{L}^*$, $k = 1, 2, \dots$. Then $J_1^*(f_k) \to 0$ if and only if $J^*(f_k) \to 0$. Hence, by applying the extension procedure to $J\colon \mathcal{L} \to \mathbb{R}^+$, we obtain no further enlargement of $\mathcal{L}$.*

P r o o f. Since $J_1^* \leq J^*$ on $\mathcal{F}^* = \mathcal{L}^*$, consider $f_k \in \mathcal{F}^*$, $k = 1, 2, \dots$, such that $J_1^*(f_k) < \delta_k$, where δ_k, $k = 1, 2, \dots$, is a sequence as in assertion (1) of Theorems 1 and 5.

By Theorem 8, for each $k = 1, 2, \dots$ there exists $f'_k \in \mathcal{L}^{o^+}$ and a sequence $f'_{k,n} \in \mathcal{L}^+$, $n = 1, 2, \dots$, such that $f'_k \geq f_k$ and $f'_{k,n} \nearrow f'_k$ for each k, and $J(f'_{k,n}) \nearrow J_1^o(f'_k) < \delta_k$.

By virtue of to Corollary 2 of Theorem 6, for each $k, n = 1, 2, \dots$ there exists $h'_{k,n} \in \widehat{\mathcal{F}^o}^+$ such that $h'_{k,n}(t) \geq f'_{k,n}(t)$ a.e. $\boldsymbol{N}$ and $J(h'_{k,n} - f'_{k,n}) < \delta_{k+n}$.

For $k, n = 1, 2, \dots$ put $h_{k,n} = \bigvee_{i=1}^{n} h'_{k,i}$. Then $h_{k,n} - f'_{k,n} \leq \sum_{i=1}^{n} (h'_{k,i} - f'_{k,i})$, hence $J(h_{k,n} - f'_{k,n}) < \delta_k$ and $J(f'_{k,n}) < \delta_k$ for each $k, n = 1, 2, \dots$.

Thus $h_{k,n} \nearrow h_k \in \mathcal{F}^{o^+}$, $h_k(t) \geq f'_k(t)$ a.e. $\boldsymbol{N}$, and $J^o(h_k) = \lim_{n \to \infty} J(h_{k,n}) \leq \delta_{k-1} < \delta_{k-2}$ for each $k = 3, 4, \dots$.

Take $u_k \in \mathcal{F}^{o^+}$ so that $h_k + u_k \geq f'_k$ and $J^o(u_k) < \delta_{k-2}$, $k = 3, 4, \dots$. Then $J^*(f_k) \leq J^*(f'_k) \leq J(h_k + u_k) < \delta_{k-3}$ for each $k = 4, 5, \dots$, hence $J^*(f_k) \to 0$ as $k \to \infty$.

The theorem is proved. □

REFERENCES

[1] BATT, J.: *A survey of some recent results on compact mappings.* In: Vector and Operator Valued Measures and Applications (D. H. Tucker, H. B. Maynard, eds.), Academic Press, Inc., New York-London, 1973, pp. 23–32.

[2] BATT, J.: *Nonlinear integral operators on $C(S, E)$*, Studia Math. **48** (1973), 145-177.

[3] BICHTELER, K.: *Integration Theory (with Special Attention to Vector Measures).* Lecture Notes in Math. 315, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[4] BICHTELER, K.: *Measures with values in non locally convex spaces.* In: Measure Theory (Proc. Conf. Oberwolfach, 1975). Lecture Notes in Math. 541, Springer-Verlag, Berlin-New York, 1976, pp. 277–285.

[5] BICHTELER, K.: *Stochastic integration and L^p-theory of semimartingales*, Ann. Probab. **9** (1981), 49–89.

[6] BROOKS, J. K.—DINCULEANU, N.: *Lebesgue-type spaces for vector integration, linear operators, veak completenes and veak compactness*, J. Math. Anal. Appl. **54** (1976), 348–389.

[7] DOBRAKOV, I.: *On integartion in Banach spaces I; II*, Czechoslovak Math. J. **20(95)** (1970), 511–536; 680–695.

[8] DOBRAKOV, I.: *On integartion in Banach spaces XVI*, Czechoslovak Math. J. (To appear).

[9] DOBRAKOV, I.: *On submeasures I*, Dissertationes Math. (Rozprawy Mat.) **112** (1974), 1–35.

[10] DOBRAKOV, I.—FARKOVÁ, J.: *On submeasures II*, Math. Slovaca **30** (1980), 65–81.

[11] DOBRAKOV, I.: *On extension of submeasures*, Math. Slovaca **34** (1984), 265–271.

[12] DOBRAKOV, I.: *A concept of measurability for the Daniell integral*, Math. Slovaca **28** (1978), 361–378.

[13] DOBRAKOV, I.: *On Lebesgue pseudonorm on $C_0(T)$*, Math. Slovaca **32** (1982), 327–335.

[14] KELLEY, J. L.: *General Topology*, D. Van Nostrand, Toronto-New York-London, 1957.

[15] KLUVÁNEK, I.: *Integration Structures.* Proc. Centre Math. Analysis Australian National Univ., Vol. 18, Canberra, 1988.

[16] ROLEWICZ, S.: *Metric Linear Spaces.* Monografie Mat., Vol. 56, PWN, Warszawa, 1972.

[17] SZCZYPIŃSKI, T: *Non-linear operator valued measures and integration*, Comment. Math. Prace Mat. **27** (1988), 313–333.

[18] TAYLOR, A. E.: *General Theory of Functions and Integration*, Blaisdell Pub. Co., New York-Toronto-London, 1965.

[19] VULICH, B. Z.: *Introduction to the Theory of Semiordered Spaces* [Vvedenie v teoriyu pluuporyadochennykh prostranstv], Fizmatgiz, Moskva, 1961.

[20] WILHELM, M.: *Integration of functions with values in a normed group*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **20** (1972), 911-916.

[21] WILHELM, M.: *Fubini-type theorems for integrable spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **22** (1972), 257-261.

[22] WILHELM, M.: *Real integrable spaces*, Colloq. Math. **32** (1974-5), 233–248.

Received September 22, 1992
Revised November 14, 1994

* *Mathematical Institute*
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA

** *Department of Mathematics*
Faculty of Mechanical Engineering
Slovak Technical University
Nám. Slobody 17
SK–812 31 Bratislava
SLOVAKIA