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TRIPLE CONSTRUCTION OF SEMILATTICES WITH 1 ADMITTING NEUTRAL p-CLOSURE OPERATORS

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Introduction

T. Katriňák [5] characterized distributive pseudocomplemented semilattices by means of triples. In line with Katriňák, P. Mederly [6] has generalized the triple construction to modular pseudocomplemented semilattices. William H. Cornish [1] has obtained triple construction for modular semilattices with 1, possessing neutral p-closure operators. To main aim of the present paper is to obtain characterization of semilattices with 1, admitting neutral p-closure operators by means of triples, thus generalizing the triple construction of Cornish.

In § 1, some interesting properties concerning closure operators on semilattices with 1 are obtained. In theorem 1 it is shown that a p-closure operator on a semilattice with 1 is standard (see definition 1) from which it follows as a corollary that a p-closure operator on a semilattice with 1 is neutral if and only if it is semineutral (see definition 1). In [6] Mederly has proved that the filter of dense elements in a modular pseudocomplemented semilattice is neutral. Corollary 2 of the present paper shows that the same is true even in a more general class of modular semilattices (see also example 2). Further, it can be seen from the same corollary that if S is a modular semilattice with 1, any p-closure operator on S is neutral so that the word neutral in the statement of Theorem 2.3 of Cornish [1] can be deleted. In § 2 triple constructions are obtained. Also a necessary and sufficient condition for the existence of a join of two elements of a semilattice with 1, having a $(p\text{-}\vee)$-closure operator (see definition 8) is obtained (see Theorem 6).

In § 3 results similar to the result of Mederly [6] are obtained for semilattices with 1, admitting neutral p-closure operators. In [6] Mederly has proved that a modular pseudocomplemented semilattice is distributive if and only if its dense filter is distributive. In fact in the interesting theorem 12 of this paper it is shown that even a stronger result is true in a more general class of semilattices with 1.

§ 1

Let $(S; \wedge)$ be a meet semilattice and F be a filter of S Then the relation $\theta(F)$ defined by $x = y(\theta(F))$ if and only if $x \wedge f = y \wedge f$ for some $f \in F$ is a congruence relation on S called the filter congruence induced by F. For $a \in S$, $[a)$ stands for $\{n \in S | n \geqslant a\}$ and is a filter of S called the filter generated by a. The set $F(S)$ of al filters of S is partially ordered under set-inclusion. S is directed above if and only if $F(S)$ is a lattice, and for any $F_1, F_2 \in F(S)$ we have inf $\{F_1, F_2\} = F_1 \cap F_2$, where $\cap$ denotes the set-intersection, sup $\{F_1, F_2\} = \{t \in S | t \geqslant f \wedge f_2$ for some $f_1 \in F_1$ and $f_2 \in F_2\}$ denoted by $F_1 \vee F_2$

The following definitions and results can be found in [1]. However, for the sake of completeness we give them here.

Let $(S; \wedge)$ be a meet semilattice with the largest element 1. A mapping $\pi. S \to S$ is called a closure operator on S if 1) $s \leqslant \pi s$ 2) $\pi(\pi s) = \pi s$ and 3) $s \leqslant t$ implies that $\pi s \leqslant \pi t$; for all $s, t \in S$. Also $C(S) = \{s \in S | \pi s - s\}$ and $D_\pi(S) = \{d \in S | \pi d = 1\}$ are called the set of π-closed elements and π-dense elements, respectively. A closure operator π is called normalized if S has the smallest element '0' and '0' is π closed. A closure operator π is called multiplicative if $\pi(s \wedge t) = \pi s \wedge \pi t$ for all s, $t \in S$. If π is multiplicative, then one can verify that $C_\pi(S)$ is a subsemilattice with 1, and $D_\pi(S)$ is a filter of S. A p-closure operator π on S is a multiplicative closure operator such that for each $s \in S$ there exist $c \in C(S)$ and $d \in D_\pi(S)$ with $s = c \wedge d$. It is easy to check that this is equivalent to saying that there is a dense element $d \in D_\pi(S)$ such that $s = \pi s \wedge d$

Suppose $(S; \wedge)$ is a meet semilattice with the smallest element '0'. The pseudocomplement a^* of an element $a \in S$ is defined by $a \wedge x = 0$ if and only if $x \leqslant a^*$. If every element of S has a pseudocomplement, then S is called a pseudocomplemented semilattice. Define $B(S) = \{x \in S | x^{**} = x\}$ and $D(S) = \{n \in S | n^{**} = 1\}$ $(B(S), \cup, \cap, *, 0, 1)$ is a Boolean algebra, where for $a, b \in B(S)$ $a \cup b = (a^* \wedge b^*)^*$ and $1 = 0^*$. $D(S)$ is a filter of S, called the dense filter of S. For standard results on pseudocomplemented semilattices see [2] and [3]. In a pseudocomplemented semilattice, the mapping π: $S \to S$ defined by $\pi(x) = x^{**}$ is a multiplicative normalized closure operator and $C_\pi(S) = B(S)$, $D_\pi(S) = D(S)$.

We now begin with the following

Definition 1. *Let $(S; \wedge)$ be a meet semilattice with 1. A multiplicative closure operator π on S is called semi-neutral if the filter $D_\pi(S)$ satisfies $(A \vee C) \cap D_\pi(S) = (A \cap D_\pi(S)) \vee (B \cap D_\pi(S))$ for all $A, B \in F(S)$. π is called standard (neutral), if $D_\pi(S)$ is a standard (neutral) element in the lattice of filters of S (see [4]).*

Theorem 1. *A p-closure operator on a semilattice S with 1 is standard.*

Proof. Let $A, B \in F(S)$ and let $b \in (A \vee D_\pi(S)) \cap B$ so that $b \in B$ and $b \geqslant a \wedge d$ for some $a \in A$ and $d \in D_\pi(S)$ and hence $\pi b \geqslant \pi(a \wedge d) = \pi a \wedge \pi d = \pi a \wedge 1$

$= \pi a \geqslant a$. We have $\pi b \geqslant b$. Let $b = \pi b \wedge e$, for some $e \in D_\pi(S)$. This shows that $b \in (A \cap B) \vee (D_\pi(S) \cap B)$ and hence $(A \vee D_\pi(S)) \cap B = (A \cap B) \vee (D_\pi(S) \cap B)$.

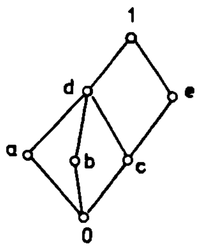
q.e.d.

Corollary 1. *A p-closure operator on a semilattice S with 1 is neutral if and only if it is semineutral.*

Proof. By the above Theorem 3 of § 3 on page 26 in [4].

Remark 1. A semi-neutral closure operator need not be standard because of the following.

Example 1.



Define

$$\pi\colon S \to S \quad \text{by} \quad \pi(n) = \begin{cases} b & \text{if } n \neq c, 1 \\ 1 & \text{otherwise,} \end{cases}$$

so that $D_\pi(S) = [c)$, which is not standard.

Mederly [6] has proved that the filter of dense elements in a modular pseudocomplemented semilattice is neutral. Now in the following corollary 2 it can be observed that the filter of dense elements is neutral even in a more general class of modular semilattices, as can be seen from the following example.

Example 2. The standard five element modular non-distributive lattice with identity mapping as closure operator is an example of a modular semilattice with 1 admitting p-closure operator which is not pseudocomplemented.

Further in [1] William H. Cornish actually stated that if S is a modular semilattice with 1 (respectively 0 and 1) possessing a p-closure operator π (normalized p-closure operator), then $\psi_\pi(S)\colon C_\pi(S) \to F(D_\pi(S))$ defined by $\psi_\pi(S)(c) = \{d \in D_\pi(S) \mid d \geqslant c\}$, for each $c \in C_\pi(S)$ is a 1-dual homomorphism ((0-1) dual homomorphism) if and only if π is a neutral closure operator. However, from the following corollary 2 it can be seen that if S is a modular semilattice with 1, then every p-closure operator on S is automatically neutral so that the word 'neutral' in the statement of theorem 2.3 of Cornish [1] can be deleted.

Corollary 2. *Let S be a modular semilattice with 1. If π is a p-closure operator on S, then π is neutral.*

Proof. By the above theorem 1 π is standard and since S is modular, it is neutral (Theorem 7 on page 48 of [4]).

Corollary 3. *In a modular pseudocomplemented semilattice, the filter of dense elements is neutral.*

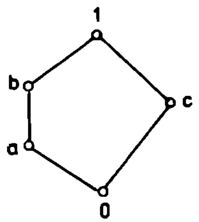
Definition 2. *A pseudocomplemented semilattice S is said to be neutral if the filter of dense elements of S is a neutral element in the lattice of filters of S.*

Theorem 2. *If π is a p-closure operator on a semilattice S with 1, then the map α: $S|\theta\ ((D_\pi(S))\to C_\pi(S)$ defined by $\alpha(\theta(D_\pi(S))[s]=\pi s$ is an isomorphism. Conversely, if α is an isomorphism and π is standard, then π is a p-closure operator.*

Proof. For the proof of the first part see Proposition 2.1 of [1]. Let $s\in S$. Since $(s,\pi s)\in\theta(D_\pi(S))$, we have $s\wedge d=\pi s\wedge d$ so that $[s)\vee[d) = [\pi s)\vee[d)$. Thus $[s)\subseteq[\pi s)\vee D_\pi(S)$ so that $[s)=[s)\cap([\pi s)\vee D_\pi(S)) = ([s)\cap[\pi s))\vee([s)\cap D_\pi(S))$. Thus $s\geqslant s_1\wedge d_1$ where $s_1\geqslant s$, $s_1\geqslant\pi s$ and $d_1\geqslant s$, $d_1\in D_\pi(S)$. Thus $s\geqslant s_1\wedge d_1\geqslant \pi s\wedge d_1\geqslant s$ so that $s=\pi s\wedge d_1$. q.e.d.

Remark 2. In general in a modular semi-lattice S with 1, one may be tempted to hope that a multiplicative closure operator is standard. However, this is not true because of the following

Example 3. S:



Define

$$\pi\colon S\to S \quad\text{by}\quad \pi(n)=\begin{cases} d & \text{if } n\neq e,1\\ 1 & \text{otherwise.}\end{cases}$$

Definition 3. *A pseudocomplemented semilattice is said to be a strong pseudocomplemented semilattice if for each $x\in S$ there is a dense element $d\in D(S)$ such that $x=x^{**}\wedge d$.*

Remark 3. Every modular pseudocomplemented semilattice is a strong pseudocomplemented semilattice but not necessarily conversely.

Now as a consequence of Theorem 2 we have the following

Corollary 4. *If S is a strong pseudocomplemented semilattice, then the mapping α: $S|\theta(D(S))\to B(S)$ defined by $\alpha(\theta(D(S))[x]=x^{**}$ is an isomorphism. Conversely, if α is an isomorphism and $D(S)$ is standard, then S is a strong pseudocomplemented semilattice.*

Proposition 1.8 of Cornish [1], which is proved for modular semilattices, can be generalized to semilattices with 1 as in the following

Theorem 3. *Let S be a semilattice with* 1, *C be a subsemilattice of S and D be a filter of S such that for each $s \in S$ there exist $c \in C$ and $d \in D$ with $s = c \wedge d$. Let ψ be a mapping from C* into $F(D)$ defined by $\psi(c) = \{d \in D | d \geq c\} = [c) \cap D$ and for $a \in C$, let θ_a denote the congruence relation on D given by $\theta_a = \{(d, e) \in D \times D | d \wedge a = e \wedge a\}$. Then the following statements are equivalent.

(1) $(A \vee B) \cap D = (A \cap D) \vee (B \cap D)$ for all principal filters A, B of S.
(2) $(A \vee B) \cap D = (A \cup D) \vee (B \cap D)$ for all filters A, B of S.
(3) $\psi(a \wedge b) = \psi(a) \vee \psi(b)$ and $\theta(\psi(a)) = \theta_a$ for all a, $b \in C$.
(4) $\theta_{a \wedge b} = \theta_a \vee \theta_b$ and $\theta(\psi(a)) = \theta_a$ for all a, $b \in c$.

Proof. $1 \Rightarrow 2$. The proof is straightforward.
$2 \Rightarrow 3$. $\psi(a \wedge b) = [a \wedge b) \cap D = ([a) \vee [b)) \cap D = ([a) \cap D) \vee ([b) \cap D)$ (by 2) $= \psi(a) \vee \psi(b)$. It is easy to verify that $\theta(\psi(a)) \subseteq \theta_a$. Now let $(d, e) \in \theta_a$ so that $d \wedge a = e \wedge a$ and hence $d \geq e \wedge a$. Thus $[d) \subseteq ([e) \vee [a)) \cap D = [e) \vee ([a) \cap D)$ so that there exist $a_1 \geq a$, $a_1 \in D$ such that $d \geq e \wedge a_1$. Similarly there exist $a_2 \geq a$ and $a_2 \in D$ such that $e \geq d \wedge a_2$. Thus $d \wedge a_1 \wedge a_2 = e \wedge a_1 \wedge a_2$ and $a_1 \wedge a_2 \geq a$, $a_1 \wedge a_2 \in D$. Hence $(d, e) \in \theta(\psi(a))$.
$3 \Rightarrow 4$. $\theta_{a \wedge b} = \theta(\psi(a \wedge b)) = \theta(\psi(a) \vee \psi(b)) = \theta(\psi(a)) \vee \theta(\psi(b)) = \theta_a \vee \theta_b$.
$4 \Rightarrow 1$. Let $t \in ([x) \vee [y)) \cap D$ so that $t \in D$ and $t \geq x \wedge y$. Let $x = a \wedge d$ and $y = b \wedge e$ where a, $b \in C$ and d, $e \in D$. Thus $t \geq a \wedge d \wedge b \wedge e$ so that $(t \wedge d \wedge e, d \wedge e) \in \theta_{a \wedge b} = \theta_a \vee \theta_b = \theta(\psi(a)) \vee \theta(\psi(b)) = \theta(\psi(a) \vee \psi(b))$ and hence $t \wedge d \wedge e \wedge \alpha = d \wedge e \wedge \alpha$ for some $\alpha \geq \beta \cap \gamma$ where $\beta \in \psi(a)$ and $\gamma \in \psi(b)$. Now $d \wedge \beta \in [x) \cap D$, $e \cap \gamma \in [y) \cap D$ and $t \geq d \wedge \beta \wedge e \wedge \gamma$ and hence $t \in ([x) \cap D) \vee ([y) \cap D)$. Thus $([x) \vee [y)) \cap D = ([x) \cap D) \vee ([y) \cap D)$. q.e.d.
Remark 4. It can be seen that proposition 1.8 of Cornish [1] is a corollary of the above theorem 3.

§ 2

In [1] William H. Cornish characterized modular semilattices with 1, possessing neutral p-closure operators, by means of triples. In this section characterization of semilattices with 1, admitting neutral p-closure operators, is obtained. The following definitions 4 and 5 can be found in [1].

Definition 4. *Let S be a meet semilattice with* 1 *and T be a join semilattice with* 0. *A mapping ψ: $S \to T$ is called a* 1*-dual homomorphism if $\psi(a \wedge b) = \psi(a) \vee \psi(b)$ for all a, $b \in S$ and $\psi(1) = 0$. It is a* (0-1) *dual homomorphism if S has* 0, *T has* 1, *ψ is a* 1*-dual homomorphism such that $\psi(0) = 1$.*

Definition 5. *By a closure isomorphism σ. $S \to T$ where S and T are semilattices with the largest element admitting the closure operators π and ϱ, respectively, we mean an isomorphism from S into T satisfying $\sigma(\pi s) = \varrho(\sigma s)$ for all $s \in S$.*

Definition 6. *(C, D, ψ) is said to be a generalized triple if C and D are semilattices with 1 and ψ is a 1-dual homomorphism from C into $F(D)$. It is a generalized 0-triple if in addition C has 0 and ψ is a (0-1) dual homomorphism. (B, D, ψ) is said to be a generalized B-triple if B is a Boolean algebra, D is a semilattice with 1, and ψ is a (0-1) dual homomorphism from B into $F(D)$.*

Definition 7. *Two generalized triples (C, D, ψ) and (C_1, D_1, ψ_1) are said to be isomorphic if there is a pair $(f\ g)$ where f is an isomorphism of C onto C_1, g is an isomorphism of D onto D_1 such that for each $c \in C$, $F(g)(\psi(c)) = \psi_1(f(c))$, where $F(g)$ denotes the isomorphism from $F(D)$ onto $F(D_1)$ induced by g.*

Theorem 4. *A semilattice S with 1 and a neutral p-closure operator π on S is such that the semilattice itself and the closure operator are determined up to a closure isomorphism by the generalized triple*

$$(C_\pi(S), D_\pi(S), \psi_\pi(S))$$

Proof. It is easy to check that $C_\pi(S)$ and $D_\pi(S)$ are semilattices with 1 and $\psi_\pi(S)$: $C_\pi(S) \to F(D_\pi(S))$ defined by $\psi_\pi(S)(c) = [c)\cap D_\pi(S)$ is a 1-dual homomorphism. This means that $(C_\pi(S), D_\pi(S), \psi_\pi(S))$ is a generalized triple. The set $S_1\{\langle c, \theta(\psi_\pi(S))(c)[d]\rangle | c \in C_\pi(S), d \in D_\pi(S)\}$. Define π_1: $S_1 \to S_1$ by $\pi_1(\langle c, \theta(\psi_\pi(S))(c)[d]\rangle) = \langle c, \theta(\psi_\pi(S))(c)[1]\rangle$. A similar proof as that of Cornish [1] shows that S_1 is a semilattice with 1, π_1 is a neutral p-closure operator on S_1 such that (S, π) and $(S_1 \pi_1)$ are closure isomorphic. q e.d.

Corollary 5. *A semilattice with 0 and 1 and a normalized neutral p-closure operator π is such that the semilattice itself and the closure operator are determined up to a closure isomorphism by ghe generalized 0-triple*

$$(C_\pi(S), D_\pi(S), \psi_\pi(S)).$$

Proof. The proof is by the above theorem together with a routine verification.

Corollary 6. *A neutral strong pseudocomplemented semilattice is determined up to an isomorphism by the generalized B-triple*

$$(B(S), D(S), \psi(S)).$$

Proof. It is easy to see that $\psi(S)$: $B(S) \to F(D(S))$ defined by $\psi(S)(a) = [a)\cap D(S)$ is a (0-1) dual homomorphism so that $(B(S), D(S), \psi(S))$ is a generalized B-triple. Let S_1 be the constructed semilattice as in theorem 4. For $x = \langle c, \theta(\psi(c))[d]\rangle \in S_1$ define $x^* = \langle c', \theta(\psi(c'))[1]\rangle$ where c' is the complement of c in $B(S)$. It is straightforward to verify that S_1 is a neutral strong pseudocomplemented semilattice such that S and S_1 are isomorphic.

Corollary 7. *A modular semilattice S with* 1 *(respectively* 0 *and* 1*) and a neutral p-closure operator* π *is such that S itself and the closure operator are determined up to a closure isomorphism by the triple*

$$(C_\pi(S), D_\pi(S), \psi_\pi(S)).$$

Remark 5. Observe that there are neutral strong pseudocomplemented semilattices which are not even modular.

Theorem 5. *If* (C, D, ψ) *is a generalized triple, then there is a semilattice S with* 1 *and a neutral p-closure operator* π *on S such that there are isomorphisms* σ: $C \to C_\pi(S)$ *and* ϱ: $D \to D_\pi(S)$ *and* 1*-dual homomorphism* $\psi_\pi(S)$: $C_\pi(S) \to F(D_\pi(S))$ *satisfying* $F(\varrho)(\psi(c)) = \psi_\pi(S)(\sigma(c))$ *for each* $c \in C$, *where* $F(\varrho)$ *is the isomorphism of* $F(D)$ *onto* $F(D_\pi(S))$ *induced by* ϱ. *Further, if* (C, D, ψ) *is a generalized* 0*-triple, then* π *is normalized.*

Proof. Consider $S = \{\langle c, \theta(\psi(c))[d]\rangle | c \in C$ and $d \in D\}$. If $x = \langle a, \theta(\psi(a))[d]\rangle$ and $y = \langle b, \theta(\psi(b))[e]\rangle$, define $x \leqslant y$ if and only if $a \leqslant b$ and $\theta(\psi(a))[d] \leqslant \theta(\psi(a))[e]$ in $D|\theta(\psi(a))$. It is easy to check '$\leqslant$' is well defined and S becomes a semilattice with 1 under this ordering. Define π: $S \to S$ by $\pi(x) = \langle a, \theta(\psi(a))[1]\rangle$. It is routine to verify that $C_\pi(S) = \{\langle a, \theta(\psi(a))[1]\rangle | a \in C\}$. $D_\pi(S) = \{\langle 1, \theta(\psi(1))[d] | d \in D\}$ and that π is a p-closure operator on S. Now we claim that $(A \vee B) \cap D_\pi(S) = (A \cap D_\pi(S)) \vee (B \cap D_\pi(S))$ for A, $B \in F(S)$. Let $\langle 1, \theta(\psi(1))[t]\rangle \in (A \vee B) \cap D_\pi(S)$ so that $\langle 1, \theta(\psi(1))[t]\rangle \geqslant \langle a, \theta(\psi(a))[d]\rangle \wedge \langle b, \theta(\psi(b))[e]\rangle$ and hence $(t \wedge d \wedge e, d \wedge r) \in \theta(\psi(a \wedge b)) = \theta(\psi(a) \vee \psi(b))$. Thus there exist $\alpha \geqslant \beta \wedge \gamma$, $\beta \in \psi(a)$ and $\gamma \in \psi(b)$ such that $t \wedge d \wedge e \wedge \alpha = d \wedge e \wedge \alpha$. Now $\langle 1, \theta(\psi(1))[d \wedge \beta]\rangle \in A \cap D_\pi(S)$ and $\langle 1, \theta(\psi(1))[e \wedge \gamma]\rangle \in B \cap D_\pi(S)$ and $\langle 1, \theta(\psi(1))[t]\rangle \geqslant \langle 1, \theta(\psi(1))[\alpha \wedge \beta]\rangle \wedge \langle 1, \theta(\psi(1))[e \wedge \gamma]\rangle$. Define $\psi_\pi(S)$: $C_\pi(S) \to F(D_\pi(S))$ by $\psi_\pi(S)\ (\langle a, \theta(\psi(a))[1]\rangle) = [\langle a, \theta(\psi(a))[1]\rangle) \cap D_\pi(S)$. Since ψ is a 1-dual homomorphism it follows that $\psi_\pi(S)$ is a 1-dual homomorphism. Clearly the map σ: $C \to C_\pi(S)$ defined by $\sigma(a) = \langle a, \theta(\psi(a))[1]\rangle$ is an isomorphism and ϱ: $D \to D_\pi(S)$ defined by $\varrho(d) = \langle 1, \theta(\psi(1))[d]\rangle$ is an isomorphism and (C, D, ψ), $(C_\pi(S), D_\pi(S), \psi_\pi(S))$ are isomorphic generalized triples. The proof of the last statement is straightforward. q.e.d.

Corollary 8. *If* (B, D, ψ) *is a generalized B-triple, then there is a neutral strong pseudocomplemented semilattice S such that there are isomorphisms* σ: $B \to B(S)$, ϱ: $D \to D(S)$ *and* $\psi(S)$ *a* (0-1) *dual homomorphism from* $B(S)$ *into* $F(D(S))$ *satisfying* $F(\varrho)(\psi(c)) = \psi(S)(\sigma(c))$ *for each* $c \in B$, *where* $F(\varrho)$ *denotes the extension of* ϱ *to* $F(D)$.

Proof. Let S be the constructed semilattice as in the proof of the theorem 5. A routine verification shows that for $x=\langle a, \theta(\psi(a))[d]\rangle \in S$, $x^*=\langle a', \theta(\psi(a'))[1]\rangle$ is a pseudocomplement of x, and thus it is a pseudocomplemented semilattice. Now the proof of the corollary follows by observing the fact that the closure operator defined in the proof of the theorem 5 is precisely the closure operator $x \to x^{**}$ on this pseudocomplemented semilattice S.

Definition 8. *A p-closure operator π on a semilattice S with 1 is said to be a $(p\text{-}\vee)$ closure operator if $G_\pi(S)$ is a lattice.*

Lemma 1. *Let π be a $(p\text{-}\vee)$ closure operator on a semilattice S with 1. If $x \vee y$ exists in S, then $\pi(x \vee y) = \pi x \vee \pi y$.*

Proof. The proof is straightforward.

In the following theorem a necessary and sufficient condition for the existence of a join of two elements in a semilattice with 1, admitting a $(p\text{-}\vee)$ closure operator, is obtained.

Theorem 6. *Let π be a $(p\text{-}\vee)$ closure operator on a semilattice S with 1. Let x, $y \in S$. Then $x \vee y$ exists in S if and only if there exist a π-dense element $t \geq x$, y and $t \wedge (\pi x \vee \pi y) \leq f$, for every π-dense element f such that $f \geq x$, $f \geq y$. In this case $x \vee y = (\pi x \vee \pi y) \wedge t$.*

Proof. First assume the condition. We show that $x \vee y = (\pi x \vee \pi y) \wedge t$, where t satisfies the condition stated in the statement of the theorem. Clearly $(\pi x \vee \pi y) \wedge t$ is an upper bound of x and y. Let $x \leq z$ and $y \leq z$ and $z = \pi z \wedge f$, where f is a π-dense element so that $\pi x \leq \pi z$, $\pi y \leq \pi z$ and hence $\pi x \vee \pi y \leq \pi z$. We have $x \leq z \leq f$ and $y \leq z \leq f$ so that $(\pi x \vee \pi y) \wedge t \leq f$. Thus $(\pi x \vee \pi y) \wedge t \leq \pi z \wedge f = z$. Conversely, assume that $x \vee y$ exists in S so that $x \vee y = \pi(x \vee y) \wedge t = (\pi x \vee \pi y) \wedge t$ (by Lemma 1). Thus t is a π-dense element such that $t \geq x$ and $t \geq y$. Now if f is a π-dense element such that $f \geq x$, $f \geq y$, then $f \geq x \vee y = (\pi x \vee \pi y) \wedge t$. q.e.d.

Corollary 9. Let S be a strong pseudocomplemented semilattice. Let x, $y \in S$. Then $x \vee y$ exists in S if and only if there is a dense element $t \geq x$, y and $t \wedge (x^{**} \vee y^{**}) \leq f$, for every dense element $f \geq x$, y. In this case $x \vee y = (x^{**} \vee y^{**}) \wedge t$.

Remark 6. In [5] Katriňák has obtained a necessary and sufficient condition for the existence of a join of two elements in a pseudocomplemented distributive semilattice in terms of triples (see Corollary 5.5 of [5]), which, however, is equivalent to the following "if S is a distributive pseudocomplemented semilattice, and x, $y \in S$, then $x \vee y$ exists in S if and only if there is a dense element $t \geq x$, y such that if f is a dense element, $f \geq x$, $f \geq y$, then $(x^{**} \vee y^{**}) \wedge t \leq f$. In this case $x \vee y = (x^{**} \vee y^{**}) \wedge t$".

In line with Katriňák, Mederly has generalized this in [6] to modular pseudocomplemented semilattices. But the above theorem and corollary show that this is true even in a more general class, namely in semilattices with 1 admitting $(p\text{-}\vee)$ closure operators.

Theorem 7. *Let C and D be semilattices with 1. If C has more than one element, then there is a 1-dual homomorphism from C into $F(D)$ so that (C, D, ψ) is a generalized triple. If C has 0, then it can be chosen so that (C, D, ψ) is a generalized 0-triple.*

Proof. Similar to the proof of theorem 2.6 of Cornish [1].

Corollary 10. *Let B be a Boolean algebra and D be a semilattice with 1. Then there is a 1-dual homomorphism ψ from B into $F(D)$ so that (B, D, ψ) is a B-triple.*

§ 3

In this article results similar to the result of Mederly [6] are obtained for semilattices with 1, admitting neutral p-closure operators. However, the proofs of the theorems in this article are straightforward and are similar to the proof of Mederly in [6]. Hence in the following we just state the results. However in theorem 12 of this article we prove an interesting result, namely that if π is a neutral p-closure operator on a semilattice S with 1, then S is distributive (modular) if and only if $C_\pi(S)$ and $D_\pi(S)$ are distributive (modular), which is a generalization of Theorem 7.3 of [6].

Theorem 8. *Let (S, π) and (S_1, π_1) be semilattices with 1, admitting p-closure operators. Let h be a homomorphism from S into S_1 (i.e. a '$\wedge$' homomorphism, preserving π and 1). Then the restriction $h|C_\pi(S)$ is a homomorphism from $C_\pi(S)$ into $C_{\pi_1}(S_1)$ and the restriction $h|D_\pi(S)$ is a homomorphism of $D_\pi(S)$ into $D_{\pi_1}(S_1)$ that preserves 1. Moreover h is onto if and only if $h|C_\pi(S)$ and $h|D_\pi(S)$ are onto.*

Corollary 11. *Let S and S_1 be strong pseudocomplemented semilattices and let h be a homomorphism of S into S_1. Then the restriction $h|B(S)$ is a homomorphism of $B(S)$ into $B(S_1)$ and the restriction $h|D(S)$ is a homomorphism of $D(S)$ into $D(S_1)$ that preserves 1. Moreover h is onto if and only if $h|B(S)$ and $h|D(S)$ are onto.*

Definition 9. *Let (C, D, ψ) and (C_1, D_1, ψ_1) be generalized triples. A homomorphism of the generalized triples (C, D, ψ) and (C_1, D_1, ψ_1) is a pair $(f-g)$, where f is a homomorphism of s into S_1, g is a homomorphism of D into D_1 such that for every $c \in C$, $g(\psi(c)) \subseteq \psi_1(f(c))$. A similar definition can be given in the case of generalized B-triples.*

Theorem 9. *Let (S, π) and (S_1, π_1) be semilattices with 1 admitting neutral p-closure operators and (C, D, ψ), (C_1, D_1, ψ_1) the associated generalized triples,* *respectively. Let h be a homomorphism of S into S_1 and h_C, h_D be the restrictions of h to C and D, respectively. Then (h_C, h_D) is a homomorphism of the generalized triples. Conversely, every homomorphism $(f-g)$ of the generalized triples uniquely determines a homomorphism h of S into S_1 with $h_C=f$ and $h_D=g$.*

Corollary 12. *Let S and S_1 be neutral strong pseudocomplemented semilattices and (B, D, ψ), (B_1, D_1, ψ_1) the associated triples, respectively. Then (h_B, h_D) is a homomorphism of the generalized B-triples, where h_B and h_D are the restrictions of h to B and D, respectively. Conversely, every homomorphism $(f-g)$ of the generalized B-triples uniquely determines a homomorphism h of S into S_1 with $h_B=f$, $h_D=g$.*

Theorem 10. *Let π be a neutral p-closure operator on a semilattice S with 1 and let S_1 be a subalgebra of S. Then $C_1=S_1\cap C_\pi(S)$ is a subalgebra of $C_\pi(S)$ and $D_1=S_1\cap D_\pi(S)$ is a subalgebra of $D_\pi(S)$. The triple associated with S_1 is (C_1, D_1, ψ_1), where ψ_1 is given by $\psi_1(a)=\psi(a)\cap D_1$ for $a\in C_1$.*

Corollary 13. *Let S_1 be a subalgebra of a neutral strong pseudocomplemented semilattice S. Then $B_1=S_1\cap B(S)$ is a subalgebra of $B(S)$, and $D_1=S_1\cap D(S)$ is a subsemilattice of $D(S)$ containing 1. The triple associated with S_1 is (B_1, D_1, ψ_1), where ψ_1 is given by $\psi_1(a)=\psi(a)\cap D_1$ for $a\in B_1$.*

Definition 10. *Let π be a multiplicative closure operator on a semilattice S with 1. Let α be a congruence on $C_\pi(S)$ and β be a congruence on $D_\pi(S)$. (α, β) is said to be a congruence pair if, whenever $a\equiv 1(\alpha)$ and $d\in D_\pi(S)$, $d\geqslant a$ implies that $(d, 1)\in\beta$.*

Theorem 11. *Let β be a congruence relation on a semilattice S with 1, admitting a p-closure operator π. Then $(B\cap(C_\pi(S)\times C_\pi(S)), \beta\cap(D_\pi(S)\times D_\pi(S)))$ is a congruence pair. Conversely, if (β_C, β_D) is a congruence pair, then there is a congruence relation β on (S, π) such that $\beta\cap(C_\pi(S)\times C_\pi(S))=\beta_C$ and $\beta\cap(D_\pi(S)\times D_\pi(S))=\beta_D$.*

Proof. The proof of the first part is straightforward. Conversely, let (β_C, β_D) be a congruence pair and g the natural mapping from $D_\pi(S)$ into $D_\pi(S)|\beta_D$ defined by $g(d)=\beta_D(d)$. Now define $\beta=\{(x, y(\in S\times S|(\pi x, \pi y)\in\beta_C$ and $(g(d), g(e))\in\theta$ $(g([\pi x)\cap D))\cap\theta(g[\pi y)\cap D))$, where $x=\pi x\wedge d$ and $y=\pi y\wedge e\}$. It is straightforward to verify that β has the required properties.

Corollary 14. *If β is a congruence relation on a strong pseudocomplemented semilattice S, then $(\beta\cap(B(S)\times B(S), \beta\cap(D(S)\times D(S)))$ is a congruence pair. Conversely, if (β_B, β_D) is a congruence pair, then there is a congruence relation β on S such that $\beta\cap(B(S)\times B(S))=\beta_B$ and $\beta\cap(D(S)\times D(S))=\beta_D$.*

Proof. By the above theorem 11, together with a routine verification, the proof follows.

Lemma 2. *Let π be a multiplicative closure operator on a semilattice S with 1. If S is distributive (modular), $C_\pi(S)$ and $D_\pi(S)$ are distributive as well.*

Proof. The proof is routine.

Theorem 12. *Let π be a neutral p-closure operator on a semilattice S with 1. S is a distributive (modular) semilattice if and only if $C_\pi(S)$ and $D_\pi(S)$ are distributive (modular).*

Proof. First suppose that $C_\pi(S)$ and $D_\pi(S)$ are distributive; let $x=\pi x\wedge d$, $y=\pi y\wedge e$, $z=\pi z\wedge f\in S$ and $z\geqslant x\wedge y$ so that $\pi z\geqslant\pi(x\wedge y) = \pi x\wedge\pi y$ and hence $\pi z=x_1\wedge y_1$ where $x_1\geqslant\pi x\geqslant x$, $y_1\geqslant\pi y\geqslant y$ and x_1, $y_1\in C_\pi(S)$. There is also $f\geqslant \pi x\wedge\pi y\wedge d\wedge e$ so that

$$[f)\subseteq([\pi x)\vee[\pi y)\vee[d)\vee[e))\cap D=([\pi x)\cap D)\vee([\pi y)\cap D)\vee[d)\vee[e).$$

Thus $f\geqslant\alpha\wedge\beta\wedge d\wedge e$ where $\alpha\geqslant\pi x$, $\alpha\in D$ and $\beta\geqslant\pi y$, $\beta\in D$. Since $D_\pi(S)$ is distributive $f=\alpha_1\wedge\beta_1\wedge d_1\wedge e_1$, $\alpha_1\geqslant\alpha$, $\beta_1\geqslant\beta$, $d_1\geqslant d$, $e_1\geqslant e$. Put $x'=x_1\wedge\alpha_1\wedge d_1$ and $y'=y_1\wedge\beta_1\wedge c_1$. Thus $x'\geqslant x$, $y'\geqslant y$ and $x'\wedge y' = \pi z\wedge f=z$. Thus S is a distributive semilattice. By a similar proof one can show that S is modular whenever $C_\pi(S)$ and $D_\pi(S)$ are modular. Since the converse follows from Lemma 2, the proof is complete. Q.E.D.

Corollary 15. *Let S be a neutral strong pseudocomplemented semilattice Then S is a distributive (modular) semilattice if and only if $D(S)$ is one.*

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КОНСТРУКЦИЯ ТРОЕК ДЛЯ ПОЛУСТРУКТУР С 1 И С НЕЙТРАЛЬНЫМ p-ЗАМЫКАНИЕМ

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Резюме

Известно, что всякую модулярную (дистрибутивную) полуструктуру S с псевдодополнениями можно охарактеризовать ей принадлежащей тройкой $(B(S), D(S), \psi(S))$, где $B(S)$-алгебра Буля замкнутых элементов из S, $D(S)$-фильтр плотных элементов из S и ψ-конъективное отображение из $B(S)$ в $F(D(S))$, структуру всех фильтров из $D(S)$. Авторы обобщают этот результат для полуструктур с 1 и с нейтральным p-замыканием.