Mathematica Slovaca

Ryszard Jerzy Pawlak

On functions with the set of discontinuity points belonging to some $sigma$-ideal

*Mathematica Slovaca*, Vol. 35 (1985), No. 4, 327--341

Persistent URL: http://dml.cz/dmlcz/132774

Math. Slovaca 35, 1985, No. 4, 327—341

# ON FUNCTIONS WITH THE SET OF DISCONTINUITY POINTS BELONGING TO SOME $\sigma$-IDEAL

RYSZARD JERZY PAWLAK

## Preliminaries

Various properties of classes of functions in connection with $\sigma$-ideals were studied e.g. by K. Kuratowski [5], R. D. Mauldin [6, 7], Z. Semadeni [13]. The results of this paper are also connected with the mentioned problem.

Obviously, $f^{-1}(U)\setminus\operatorname{Int} f^{-1}(U)=\emptyset$ for any open $U\subset Y$ is a characterization of the continuity of $f: X\to Y$. Consider a class of functions $f$ for which $f^{-1}(U)\setminus\operatorname{Int} f^{-1}(U)$ belongs to a fixed $\sigma$-ideal $J$ for any open $U\subset Y$. Such a function will be called *J-continuous.*

In the first part of the paper we give a characterization of $J$-continuous functions for a given $\sigma$-ideal $J$. The second part deals with the relation of $J$-continuity and some other type of continuities. The third part gives some characterization of the Baire space. It is related to papers of J. C. Bradford, C. Goffman [2] and R. C. Haworth, R. A. McCoy [4].

The symbol $f: X\to Y$ denotes as usually a mapping of $X$ to $Y$. The undefined notions are used according to R. Engelking [3].

Moreover, we use $C_f(D_f)$ to denote the set of continuity (discontinuity) points of $f$. $R$, $Q$, $N$ denote the sets of reals, rationals and positive integers respectively. card$A$ stands for the cardinality of $A$, $(a, b)$ $([a, b])$ denotes open (closed) intervals respectively. Throughout the paper, the nonempty open sets are excluded as the elements of the $\sigma$-ideals in consideration.

## 1. $J$-continuity

**Definition 1.1.** *Let $X$ and $Y$ be arbitrary topological spaces and let $J$ be some $\sigma$-ideal in $X$. We say that a function $f: X\to Y$ is $J$-continuous if for every open set $U\subset Y$ we have $f^{-1}(U)\setminus\operatorname{Int} f^{-1}(U)\in J$.*

Let us recall our standard hypothesis that considered $\sigma$-ideals does not include nonempty open sets.

The definition is a generalization of the notion of the continuity. So, a natural problem is to characterize $D_f$ for a $J$-continuous function $f$.

**Lemma 1.2.** *Let $X$ be an arbitrary topological space, $Y$- a second-countable space and let $J$ be some $\sigma$-ideal in $X$. Then $f\colon X\to Y$ is a $J$-continuous function if and only if $D_f\in J$.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a countable base in $Y$. Since

$$f^{-1}(U_n)\backslash \operatorname{Int} f^{-1}(U_n)\in J \quad \text{for} \quad n=1,2,\dots$$

the necessity follows from the inclusion

$$D_f\subset\bigcup_{n=1}^{\infty}(f^{-1}(U_n)\backslash \operatorname{Int} f^{-1}(U_n)).$$

The sufficiency is implied by the fact that $f^{-1}(U)\backslash \operatorname{Int} f^{-1}(U)\subset D_f\in J$.

**Definition 1.3.** *Let $X$ be an arbitrary topological space. We say that a neighbourhood system $\{W(x)\}_{x\in X}$ is regular if*

1° *$W(x)=\{V_n(x)\}_{n=1}^{\infty}$ for every $x\in X$, and*

2° *for every $n=1,2,\dots$ and for every two elements $x$, $x'\in X$: if $x'\in V_{n+1}(x)$, then $V_{n+1}(x)\subset V_n(x')$.*

Remark. Observe, that every metric space $(X,\varrho)$ possesses a regular neighbourhood system (for any $n=1,2,\dots$ and for any $x\in X$, it is suffices to put

$V_n(x)=K(x,{}^1/_{2^n})$

**Theorem 1.4.** *Let $X$ be an arbitrary space, $Y$ — a second-countable space and let $Y$ possess a regular neighbourhood system. Then a function $f\colon X\to Y$ is $J$--continuous with respect to some $\sigma$-ideal $J$ of subsets of the space $X$ if and only if $D_f$ is a boundary set of the first category.*

Proof. Necessity. In fact, $D_f$ is boundary set according to Lemma 1.2 and by our supposition that $\sigma$-ideal $J$ does not include nonempty open sets.

Now, we shall show that $D_f$ is of first category.

Let $\{W(y)\}_{y\in Y}$, where $W(y)=\{V_n(y)\}_{n=1}^{\infty}$ for $y\in Y$, be the regular neighbourhood system of the space $Y$. Moreover, for $n=1,2,\dots$ let $D_f^{(n)}$ denote the set of such $x\in X$ that for any neighbourhood $U_x$ of $x$ there exists $x'\in U_x$ such that $f(x')\notin V_n(f(x))$.

Then

$$D_f=\bigcup_{n=1}^{\infty}D_f^{(n)}. \tag{1}$$

It suffices to show that $D_f^{(n)}$ is a nowhere dense set for $n=1,2,\dots$.

Let $n_0$ be an arbitrary natural number and let $V$ be an arbitrary open set in $X$. We infer that the collection $\{V_{n_0+1}(y)\}_{y\in Y}$ covers $Y$ i.e.

$$Y=\bigcup_{y\in Y} V_{n_0+1}(y).$$

By Theorem of Lindelöf, there exists a countable subcover $\{V_{n_0+1}(y_k)\}_{k=1}^{\infty}$ of $\{V_{n_0+1}(y)\}_{y\in Y}$. We have that:

$$V=f^{-1}(Y)\cap V=\left(\bigcup_{k=1}^{\infty} f^{-1}(V_{n_0+1}(y_k))\right)\cap V. \tag{2}$$

Since $f$ is a $J$-continuous function, then

$$f^{-1}(V_{n_0+1}(y_k))=\operatorname{Int} f^{-1}(V_{n_0+1}(y_k))\cup T_k, \tag{3}$$

where $T_k\in J$ for $k=1, 2, \dots$.

According to (2) and (3) we have

$$V=\left(\bigcup_{k=1}^{\infty} (\operatorname{Int} f^{-1}(V_{n_0+1}(y_k))\cup T_k)\right)\cap V=$$

$$=\left[\left(\bigcup_{k=1}^{\infty} \operatorname{Int} f^{-1}(V_{n_0+1}(y_k))\right)\cap V\right]\cup\left[\left(\bigcup_{k=1}^{\infty} T_k\right)\cap V\right].$$

On the other hand, we infer that $\bigcup_{k=1}^{\infty} T_k\in J$, and so $V\setminus\bigcup_{k=1}^{\infty} T_k\neq\emptyset$ and consequently for some $k^*$ we have

$$(\operatorname{Inf} f^{-1}(V_{n_0+1}(y_{k^*})))\cap V\neq\emptyset.$$

We put $W=(\operatorname{Inf} f^{-1}(V_{n_0+1}(y_{k^*})))\cap V$. Hence $W$ is a nonempty open set included in $V$.

We shall show that

$$W\cap D_f^{(n_0)}=\emptyset. \tag{4}$$

First, we observe that $f(W)\subset V_{n_0+1}(y_{k^*})$. Let $x_0\in W$, then $f(x_0)\in V_{n_0+1}(y_{k^*})$. Since $\{W(y)\}_{y\in Y}$ is a regular neighbourhood system, then

$$V_{n_0+1}(y_{k^*})\subset V_{n_0}(f(x_0)),$$

this means that for $x_0$ there exists such a neighbourhood $W$ of $x_0$ that $f(W)\subset V_{n_0}(f(x_0))$ and consequently $x_0\notin D_f^{(n_0)}$. This proves (4).

From (4) we have that $D_f^{(n_0)}$ is nowhere dense.

Sufficiency. Let us put $J=2^{D_f}$. Thus $J$ is the $\sigma$-ideal, such that $d_f\in J$ and, according to Lemma 1.2, $f$ is a $J$-continuous function.

As a consequence of the above Theorem we obtain the following corollary. We omit the easy proof.

**Corollary 1.5** (see [11, p. 61, Theorem 7.4]). *Let $X$ be a Baire space, $Y$ — a second countable space and let $Y$ possess a regular neighbourhood system. Let $f$: $X \to Y$. Then the set $D_f$ is of the first category if and only if the set $C_f$ is dense in $X$.*

Note that the notion of the Baire space is defined as follows.

**Definition 1.6** [4]. *A topological space $X$ is called a Baire space if every nonempty open set in this space is of the second category.*

It is well known that the usual continuity of $f$ may be defined by the condition: $\overline{f^{-1}(F)} \setminus f^{-1}(F) = \emptyset$, where $F$ is any closed set. One can show that the notion of $J$-continuity may be also formulated by means of closed sets. We state without proof the following theorem.

**Theorem 1.7.** *Let $X$, $Y$ be arbitrary topological spaces and let $J$ be some $\sigma$-ideal in $X$. Then a function $f$: $X \to Y$ is $J$-continuous if and only if $\overline{f^{-1}(F)} \setminus f^{-1}(F) \in J$ for every closed set $F$ in $Y$.*

## 2. Connections between J-continuity, quasi-continuity and Baire class 1

**Definition 2.1.** *We say that a function $f$: $X \to Y$, where $X$ and $Y$ are topological spaces, is in Baire class 1 if, for every nonempty closed set $F \subset X$, $f_{|F}$ possesses a point of continuity.*

We put

$$B_1(X, Y) = \{f: X \to Y: f \text{ is in Baire class } 1\},$$
$$J(X, Y) = \{f: X \leftarrow Y: f \text{ is } J\text{-continuous function, where } J \text{ is a } \sigma\text{-ideal in } X\}.$$

**Theorem 2.2.** *Let $X$ be a complete metric space, let $Y$ be a second countable $T_2$-space which is not singleton and let $J$ be a $\sigma$-ideal in $X$. Then $J(X, Y) \subset B_1(X, Y)$ if and only if $J$ does not include perfect sets.*

Proof. Assume that $J$ includes some perfect subset $P$. We shall show that there exists such a function $f \in J(X, Y)$ that $f \notin B_1(X, Y)$.

Let $\mathcal{R}$ be an arbitrary base in $X$. Let $\mathcal{R}^* = \{U \in \mathcal{R}: U \cap P \neq \emptyset\}$. Let $\{U_\alpha\}_{\alpha < \Xi}$ be a transfinite sequence consisting of all sets of the collection $\mathcal{R}^*$.

Let $V_0 = U_0$ and let $x_0 \in V_0 \cap P$. We suppose that we have defined $x_\alpha$ for $\alpha < \beta < \Xi$. We put $V_\beta = U_\beta \setminus \overline{\bigcup_{\alpha<\beta} \{x_\alpha\}}$ and

$$x_\beta = \begin{cases} x_0 & \text{for} \quad V_\beta \cap 0 = \emptyset, \\ y & \text{for} \quad V_\beta \cap P \neq \emptyset, \end{cases}$$

where $y$ is an arbitrary element of $V_\beta \cap P$.

Moreover we put

$$A = \bigcup_{\alpha<\Xi} \{x_\alpha\} \subset P.$$

We shall show that $P \subset \bar{A}$. In fact, let $x_0' \in P$ and let $U_\gamma$ be an arbitrary element of the collection $\mathcal{R}^*$ that $x_0' \in U_\gamma$ (we may assume that $U_\gamma$ is an arbitrary neighbourhood of $x_0'$).

Let us consider the two possible cases:

1° $x_0' \notin V_\gamma$. Then $x_0' \in \overline{\bigcup_{\alpha<\gamma} \{x_\alpha\}} \subset \bar{A}$.

2° $x_0' \in V_\gamma$. Thus $V_\gamma \cap P \neq \emptyset$ (because $x_0' \in V_\gamma \cap P$). Then from the set $V_\gamma \cap P$ we may select $x_\gamma \in A$ and so $V_\gamma \cap A \neq \emptyset$, consequently $U_\gamma \cap A \neq \emptyset$, it means that $x_0' \in \bar{A}$.

We put

$$P_1 = P \setminus A.$$

We shall show that

$$(*) \qquad U_\alpha \cap P_1 \neq \emptyset, \quad \text{for every} \quad \alpha < \Xi.$$

In fact, let $\alpha < \Xi$. Since $P$ is a perfect set, then infinitely many points from $P$ belong to $U_\alpha$. We denote by $y_1$ an arbitrary point of the set $U_\alpha \cap P$, different from $x_\alpha$ and $x_0$. If $y_1 \notin A$ then the condition $(*)$ is true. Otherwise $y_1 = x_{\delta_1} \in A$. In this case we put $K_1 = K(y_1, \varepsilon_1)$, where $\varepsilon_1$ is such a number that $\bar{K}_1 \subset U_\alpha \cap V_{\delta_1}$ and $\varepsilon_1 < \frac{1}{4} \min(\varrho(y_1, x_\alpha), \varrho(y_1, x_0))$.

Now we suppose that we have defined the elements $y_1, \ldots, y_{n-1} \in P$, $y_i \neq y_j$ for any $i \neq j$, and the corresponding sequence of balls $\bar{K}_{n-1} \subset \bar{K}_{n-2} \subset \ldots \subset \bar{K}_1 \subset U_\alpha$, such that $\bar{K}_i \subset K_{i-1} \cap V_{\delta_i}$ (where $K_0 = U_\alpha$) for $i = 1, 2, \ldots, n-1$. Thus if for some $k \in \{1, 2, \ldots, n-1\}$ $y_k \notin A$ then $(*)$ is true. Otherwise $y_1, \ldots, y_{n-1} \in A$. We remark that infinitely many elements from $P$ belong to $K_{n-1}$. We denote by $y_n$ any one from them, different from $y_1, \ldots, y_{n-1}$ (since $K_{n-1} \subset K_1$ then $y_n \neq x_0$). If $y_n \notin A$ then $(*)$ is true, otherwise $y_n = x_{\delta_n} \in A$. We put $K_n = K(y_n, \varepsilon_n)$, where $\varepsilon_n > 0$ is such a number that $\bar{K}_n \subset K_{n-1} \cap V_{\delta_n}$ and $\varepsilon_n < \frac{1}{4} \delta(y_n, y_{n-1})$ (obviously $V_{\delta_n} \cap P \neq \emptyset$ because $y_n \neq x_0$).

If there exists such $l$ that $y_l \notin A$, then the proof of $(*)$ is finite. Otherwise we have an infinite, decreasing sequence of closed balls, with the centres belonging to $P$ and diameters converging to zero. Let

$$\{y_0\} = \bigcap_{n=1}^{\infty} \bar{K}_n.$$

We shall show that

$$y_0 \notin A. \tag{1}$$

We first remark that

$$x_0 \neq y_0 \neq y_n \neq x_0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{2}$$

Observe that

$$y_0 = \lim_{n\to\infty} y_n. \tag{3}$$

We assume, to the contrary, that $y_0 \in A$, it means that $y_0 = x_{\delta_0} \in V_{\delta_0} \cap P$. According to (2), we have two cases:

1°° $\delta_0 < \delta_{n_0}$, for some $n_0$. According to (2), we have $y_0 = x_{\delta_0} \in V_{\delta_{n_0}} = U_{\delta_{n_0}} \setminus \overline{\bigcup_{\alpha<\delta_{n_0}} \{x_\alpha\}}$, it means that $x_{\delta_0} \notin V_{\delta_{n_0}}$. On the other hand $x_{\delta_0} = y_0 \in \bar{K}_{n_0} \subset V_{\delta_{n_0}}$, which is impossible.

2°° $\delta_0 > \delta_n$ for $n = 1, 2, \ldots$. Then according to (2), we have $y_0 = x_{\delta_0} \in V_{\delta_0} = U_{\delta_0} \setminus \overline{\bigcup_{\alpha<\delta_0} \{x_\alpha\}}$, thus $V_{\delta_0}$ is the neighbourhood of $y_0$ and $V_{\delta_0}$ does not include every point of $\{x_{\delta_n}\} = \{y_n\}$, this contradicts (3). Thus we have proved (1).

According to (3) and the fact, that for every $n$, $y_n \in P$, it is not difficult to observe that $y_0 \in P$ and since $\bar{K}_n \subset U_\alpha$ (for $n = 1, 2, \ldots$) then $y_0 \in U_\alpha$, this completes the proof of (*).

Now it is obvious that

$$P \subset \bar{A} \quad \text{and} \quad P \subset \bar{P}_1.$$

Since $Y$ is a $T_2$-space and it is not a singleton, there exist two different elements $z_1$, $z_2$ and two open sets $U_1$ and $U_2$ such that $z_1 \in U_1$, $z_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Now, we define the function $f\colon X \to Y$ in the following way:

$$f(x) = \begin{cases} z_1 & \text{for} \quad x \in A, \\ z_2 & \text{for} \quad x \notin A. \end{cases}$$

We shall show that $f$ is a $J$-continuous function.

We observe that

$$D_f \in J. \tag{4}$$

In fact, it is sufficient to show that $D_f \subset P$. Let $x \notin P$. Then there exists such $\varepsilon > 0$ that $K(x, \varepsilon) \cap P = \emptyset$. Thus $f(K(x, \varepsilon)) = \{z_2\} = \{f(x)\}$ and consequently $x \notin D_f$.

According to (4) and Lemma 1.2 we may infer that $f \in J(X, Y)$.

Of cource, $f \notin B_1(X, Y)$ because $f_{|P}$ is a function discontinuous at every point.

Sufficiency. Let $f \in J(X, Y)$. Let $F$ be an arbitrary closed set in $X$. Thus if $F$ includes an isolated point, then $f_{|F}$ is continuous at this point. Otherwise $F$ is a perfect set and consequently $F \notin J$. Hence (according to Lemma 1.2) $F$ includes some point $x_0$ of continuity of $f$ and so $x_0$ is a continuity point of $f_{|F}$.

We shall discuss some connections between $J$-continuous and some other types of functions.

**Definition 2.3.** *We say that a function $f\colon [0, 1] \to [0, 1]$ possesses the property of Świątkowski if for every two points $x$, $y$ such that $f(x) \neq f(y)$, there exists a point $z$ of continuity of $f$ such that $z \in (x, y)$ and $f(z) \in (f(x), f(y))$.*

**Definition 2.4.** *We say that a function $f$: $X \to Y$, where $X$, $Y$ are topological spaces, is quasi-continuous at $x_0$ if for every neighbourhood $V$ of $f(x_0)$ and for every neighbourhood $U$ of $x_0$ we have* $\mathrm{Int}(f^{-1}(V) \cap U) \neq \emptyset$. *We say that a function $f$ is quasi-continuous if it is quasi-continuous at every point of its domain.*

**Theorem 2.5.** *Let $f$: $[0, 1] \to [0, 1]$. Let us consider the following properties of the function $f$:*

(α) *$f$ is a Darboux function,*
(β) *$f$ possesses the property of Świątkowski,*
(γ) *There exists a σ-ideal $J$ such that $f$ is a $J$-continuous function,*
(δ) *$f$ is a quasi-continuous function,*
(η) *$f$ is in Baire class 1.*

*Then the following true:*

(a) $(\alpha) \wedge (\delta) \Rightarrow (\beta)$
(b) $(\beta) \Rightarrow (\gamma)$
(c) $(\delta) \Rightarrow (\gamma)$
(d) $(\eta) \Rightarrow (\gamma)$
(e) $(\alpha) \wedge (\beta) \Rightarrow (\delta)$
(f) $(\alpha) \wedge (\gamma) \wedge (\eta) \not\Rightarrow (\beta)$
(g) $(\gamma) \wedge (\delta) \wedge (\eta) \not\Rightarrow (\beta)$
(h) $(\beta) \wedge (\gamma) \wedge (\delta) \wedge (\eta) \not\Rightarrow (\alpha)$
(i) $(\alpha) \not\Rightarrow (\gamma)$
(j) $(\alpha) \wedge (\gamma) \wedge (\eta) \not\Rightarrow (\delta)$
(k) $(\beta) \wedge (\gamma) \wedge (\eta) \not\Rightarrow (\delta)$
(l) $(\alpha) \wedge (\beta) \wedge (\gamma) \wedge (\delta) \not\Rightarrow (\eta)$.

Proof.

(a) Proof of this implication can be found in the paper [12].
(b) Observe that if $f$ possesses the property of Świątkowski, then $D_f$ is a boundary set. Let us put $J = 2^{D_f}$. Hence $D_f \in J$ and, according to Lemma 1.2, $f$ is a $J$-continuous function.
(c) According to the proof of (b), it is sufficient to show that $C_f$ is dense in [0, 1]. But this is known from [9].
(d) The proof of this implication is a simple consequence of Theorem 1.4 and the well-known theorem saying that the set of discontinuity points of a real function in Baire class 1 is of the first category.
(e) The proof of this fact can be found in the paper [12].
(f) T. Mańk and T. Świątkowski in [8, Theorem 3] have proved that $(\alpha) \wedge (\eta) \not\Rightarrow (\beta)$. Thus according to (d) we infer that condition (f) is satisfied.
(g) For the proof of this condition we put

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in [0, \frac{1}{2}), \\ 1 & \text{for} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

(h) For the proof of this condition we put

$$f(x) = \begin{cases} x & \text{for} \quad x \in [0, \frac{1}{2}), \\ 1 & \text{for} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

(*i*) If we take a Darboux function discontinuous at every point, then, according to Theorem 1.4, $f$ cannot be $J$-continuous with respect to every $\sigma$-ideal $J$.

(j) T. Mańk and T. Świątkowski in [8, Theorem 3] have shown that $(\alpha)\wedge(\eta) \not\Rightarrow (\beta)$. Hence according to (a) and (d) we may infer that (j) is true.

(k) For the proof of this condition we put

$$f(x)=\begin{cases} x & \text{for } x\in\left[0,\frac{1}{2}\right), \\ \frac{2}{3} & \text{for } x=\frac{1}{2}, \\ \frac{1}{2}x+\frac{1}{2} & \text{for } x\in\left(\frac{1}{2},1\right]. \end{cases}$$

(l) Theorem 2 in paper [8] shows that $(\alpha)\wedge(\beta) \not\Rightarrow (\eta)$. Thus according to (b) and (e) we have that (l) is true.

We give now a necessary and sufficient condition for the quasi-continuity of a $J$-continuous function.

**Theorem 2.6.** *Let $X$, $Y$ be two topological spaces and let $J$ be a $\sigma$-ideal in $X$. Then the $J$-continuous function $f\colon X\to Y$ is quasi-continuous at $x_0\in X$ if and only if, for every neighbourhood $G$ of $f(x_0)$ and for every neighbourhood $V$ of $x_0$, $V\cap f^{-1}(G)\notin J$.*

Proof. Necessity. Since $f$ is quasi-continuous at $x_0$, then $\mathrm{Int}\,(V\cap f^{-1}(G))\neq\emptyset$ and so $V\cap f^{-1}(G)\notin J$.

Sufficiency. We assume, to the contrary, that $f$ is not quasi-continuous at $x_0$. Then there exists some neighbourhood $V$ of $x_0$ and some neighbourhood $G$ of $f(x_0)$ such that

$$\mathrm{Int}(f^{-1}(G)\cap V)=\emptyset. \tag{1}$$

We remark that

$$(V\cap f^{-1}(G))\backslash \mathrm{Int}\, f^{-1}(G)=(V\cap f^{-1}(G))\backslash \mathrm{Int}(f^{-1}(G)\cap V). \tag{2}$$

Since $f$ is $J$-continuous, then $f^{-1}(G)\backslash \mathrm{Int}\, f^{-1}(G)\in J$, and therefore $(V\cap f^{-1}(G))\backslash \mathrm{Int}\, f^{-1}(G)\in J$, then according to (2)

$$(V\cap f^{-1}(G))\backslash \mathrm{Int}(f^{-1}(G)\cap V)\in J. \tag{3}$$

In view of (1) and (3), $V\cap f^{-1}(G)\in J$, this contradicts our supposition.

## 3. Baire spaces and $\sigma$-ideals

H. Blumberg in the paper [1] has showed that for every real function $f$ of real variable there exists a set $B$ dense in $R$ and such that $f_{|B}$ is a continuous function (in this paper for the set $B$ possessing above properties we assume the term *Blumberg set*).

J. C. Bradford and C. Goffman in the paper [2] have shown that a metric space $X$ is a Baire space if and only if every real function defined on $X$ possesses a Blumberg set.

In this part a certain characterization of topological Baire spaces will be given.

**Definition 3.1.** *We say that a function $f\colon X \to Y$, where $X$ and $Y$ are arbitrary topological spaces, possesses the property $(H)$, if there exists a $\sigma$-ideal $J$ in $X$ such that, for every $\alpha \in Y$, $f^{-1}(\alpha) \setminus \operatorname{Int}\overline{(f^{-1}(\alpha))} \in J$.*

Note that $J$ does not include nonempty open sets.

**Theorem 3.2.** *A topological space $X$ is a Baire space if and only if every real function defined on $X$ possesses the property $(H)$.*

Proof. Necessity. Let $f$ be an arbitrary real function. Let $J$ denotes $\sigma$-ideal of sets of first category (since $X$, in view of the supposition, is a Baire space, then $J$ does not include nonempty open sets).

We shall show that, for $\alpha \in Y$,

$$(*)\qquad f^{-1}(\alpha) \setminus \operatorname{Int}\overline{(f^{-1}(\alpha))} \quad \text{is nowhere dense.}$$

In fact. Let $U \neq \emptyset$ be an arbitrary open set. Then two cases can arise.

1° $U \subset \overline{f^{-1}(\alpha)}$ and so $U \cap (f^{-1}(\alpha) \setminus \operatorname{Int}\overline{(f^{-1}(\alpha))}) = \emptyset$.

2° $U \not\subset \overline{f^{-1}(\alpha)}$. Then $V = U \setminus \overline{f^{-1}(\alpha)}$ is a nonempty open set such that $V \subset U$ and $V \cap f^{-1}(\alpha) = \emptyset$. Therefore

$$V \cap (f^{-1}(\alpha) \setminus \operatorname{Int}\overline{(f^{-1}(\alpha))}) = \emptyset.$$

This completes the proof of $(*)$ and the proof of the necessity.

Sufficiency. We assume to the contrary, that $X$ is not a Baire space. Then there exists such an open set $V \neq \emptyset$ that $V = \bigcup_{n=1}^{\infty} K_n$, where $K_n$ is nowhere dense for $n = 1, 2, \ldots$. We may assume that for $n \neq m$ $K_n \cap K_m = \emptyset$.

Let $f\colon X \to R$ as follows:

$$f(x) = \begin{cases} 0 & \text{for } x \in X \setminus V, \\ m & \text{for } x \in K_m. \end{cases}$$

We shall show that $f$ does not possess the property $(H)$. In fact, we suppose, on the contrary, that $f$ has the property $(H)$, then there exists a $\sigma$-ideal $J$ such that

$$f^{-1}(k) \setminus \operatorname{Int}\overline{f^{-1}(k)} \in J, \quad \text{for} \quad k = 1, 2, \ldots.$$

We infer that

$$f^{-1}(k)=K_k \quad \text{for} \quad k=1, 2, \ldots,$$

and so $\overline{f^{-1}(k)}=\bar{K}_k$ and since $\bar{K}_k$ is nowhere dense, then $\operatorname{Int}\bar{K}_k=\emptyset$ for $k=1, 2, \ldots$; this means that

$$V=\bigcup_{k=1}^{\infty} K_k=\bigcup_{k=1}^{\infty}(f^{-1}(k)\backslash\operatorname{Int}\overline{f^{-1}(k)})\in J$$

which is impossible because $J$ does not include nonempty open sets. Hence $f$ does not possess the property $(H)$.

The following examples show that there such a continuous function (it possesses a Blumberg set), that $f$ does not possess the property $(H)$ and also there exists such a function $f$, that $f$ has the property $(H)$ and $f$ does not possess a Blumberg set. The above functions are real functions defined on some metric spaces.

Example 3.3. Let $X=Q\cap[0, 1]$, $\varrho$ be the natural metric in $X$ and let $f\colon X\to R$ be the identical function (i.e. $f(x)=x$). Then $f$ is a continuous function. Moreover, $f$ does not possess the property $(H)$. In fact, let $J$ be an arbitrary $\sigma$-ideal and let $\alpha\in Q\cap[0, 1]\subset R$. Then

$$\overline{f^{-1}(\alpha)}=f^{-1}(\alpha)=\{\alpha\}$$

so $\operatorname{Int}\overline{f^{-1}(\alpha)}=\emptyset$, this means that $f^{-1}(\alpha)\backslash\operatorname{Int}\overline{f^{-1}(\alpha)}=f^{-1}(\alpha)$. We have

$$X=\bigcup_{\alpha\in Q\cap[0, 1]} f^{-1}(\alpha).$$

If now $f$ possesses the property $(H)$, then $f^{-1}(\alpha)\in J$ and so $X\in J$, which is impossible.

Example 3.4. Let $X=[0, 1]\times(Q\cap[0, 1])$, $\varrho$ be the natural metric in the plane restricted to $X$. Let $h$ denote the 1—1 function mapping $Q$ on $N$. Let $f\colon X\to R$ be defined in the following way:

$$f((x, q))=h(q)+x.$$

We first show that $f$ possesses the property $(H)$. Let $J$ denote the $\sigma$-ideal of all denumerable subsets of $X$. For every $\alpha\in R$, $\operatorname{card}f^{-1}(\alpha)\leqslant 2$, then $f^{-1}(\alpha)\in J$ and therefore $f^{-1}(\alpha)\backslash\operatorname{Int}\overline{f^{-1}(\alpha)}\in J$.

Now, we show that $f$ does not possess a Bluberg set. Let $B$ be an arbitrary set dense in $X$ and let $(b_0, q_0)\in B$. There exists a sequence $\{(b_n, q_n)\}$, $(b_n, q_n)\in B$ such that $\lim_{n\to\infty}(b_n, q_n)=(b_0, q_0)$ and $q_n\neq q_0$ for $n=1, 2, \ldots$.

This means that $f((b_n, q_n))\not\to f((b_0, q_0))$ and so $f_{|B}$ is a discontinuous function. This proves that $B$ is not a Blumberg set of $f$.

We introduce a new class of functions which may be of interest in connection with Baire spaces.

**Definition 3.5.** *We say that a function $f$: $X \to Y$, where $X$ and $Y$ are arbitrary topological spaces, possesses the property $(H^*)$ if there exists a $\sigma$-ideal $J$ in $X$ such that $f^{-1}(\alpha) \setminus \operatorname{Int} \overline{f^{-1}(U_\alpha)} \in J$ for every $\alpha \in Y$ and for every neighbourhood $U_\alpha$ of $\alpha$.*

**Theorem 3.6.** *Let $X$ and $Y$ be arbitrary topological spaces. Then if a function $f$: $X \to Y$ possesses a Blumberg set then $f$ possesses the property $(H^*)$.*

Proof. Let $B$ be a Blumberg set of $f$. Let, for $A \subset X$, $\operatorname{Int}_B(A)$ denote the set of all points $x \in B$ for which there exists a neighbourhood $U_x$ of $x$ such that $U_x \cap B \subset A$.

Of course

$$\operatorname{Int}_B(A) \subset \operatorname{Int} A \quad \text{for every closed set } A. \tag{1}$$

It is easy to see that

$$\operatorname{Int}_B(A) \subset \operatorname{Int}_B(\bar{A}) \quad \text{for every} \quad A \subset X. \tag{2}$$

We put $J = 2^{X-B}$. Since $B$ is dense in $X$, then $J$ does not include nonempty open sets. Let $\alpha \in Y$ and $U_\alpha$ be an arbitrary neighbourhood of $\alpha$. $f_{|B}$ is a continuous function, so

$$\operatorname{Int}_B(f_{|B}^{-1}(U_\alpha)) = f_{|B}^{-1}(U_\alpha) = f^{-1}(U_\alpha) \cap B,$$

this means that

$$f^{-1}(\alpha) \setminus \operatorname{Int}_B(f_{|B}^{-1}(U_\alpha)) = f^{-1}(\alpha) \setminus (f^{-1}(U_\alpha) \cup B) \subset f^{-1}(\alpha) \setminus (f^{-1}(\alpha) \cap B) \in J.$$

According to (2) we have

$$f^{-1}(\alpha) \setminus \operatorname{Int}_B(\overline{f_{|B}^{-1}(U_\alpha)}) \in J.$$

This means, according to (1), that $f^{-1}(\alpha) \setminus \operatorname{Int} \overline{f_{|B}^{-1}(U_\alpha)} \in J$ and consequently

$$f^{-1}(\alpha) \setminus \operatorname{Int} \overline{f^{-1}(U_\alpha)} \in J,$$

this ends the proof.

Of course, if a function $f$ possesses the property $(H)$, then $f$ possesses the property $(H^*)$. Theorem 3.6 shows that the function described in Example 3.3 possesses the property $(H^*)$ but does not possess the property $(H)$. On the other hand, Example 3.4 shows that the inverse theorem to Theorem 3.6 is false.

**Theorem 3.7.** *Topological space $X$ is a Baire space if and only if every real function defined on $X$ possesses the property $(H^*)$.*

Proof. Necessity is obvious.

Sufficiency. We shall show that the function described in the proof of the sufficient condition of Theorem 3.2 does not possess the property $(H^*)$. Suppose, on the contrary, that there exists a $\sigma$-ideal $J$ in $X$ such that

$$f^{-1}(k)\backslash\operatorname{Int}\overline{f^{-1}((k-\tfrac{1}{2},\, k+\tfrac{1}{2}))}\in J.$$

We infer that

$$f^{-1}((k-\tfrac{1}{2},\, k+\tfrac{1}{2}))=K_k \quad \text{(see proof of Theorem 3.2).}$$

Similarly as in the proof of Theorem 3.2, we may show that $\emptyset \neq V\in J$, where $V$ is some open set, which is impossible.

Now we may ask: what connections are there between the class of functions possessing a Blumberg set and the class of functions $f$ for which there exists such a dense set $B$ that the restriction $f_{|B}$ is a continuous function with respect to some $\sigma$-ideal (see Definition 1.1).

**Definition 3.8.** *Let $f$: $X\to Y$, where $X$ and $Y$ are arbitrary topological spaces. We say that a set $B\subset X$ is a weak Blumberg set of $f$ if $B$ is dense in $X$ and in $B$ (we understand $B$ as the subspace of $X$) there exists a $\sigma$-ideal $J(B)$ such that $f_{|B}$ is $J(B)$-continuous.*

**Theorem 3.9.** *Let $X$ be an arbitrary topological space, let $Y$ be a second-countable space. Then $f$: $X\to Y$ possesses a Blumberg set if and only if $f$ possesses a weak Blumberg set.*

Proof. Necessary condition is obvious.

Sufficiency. Let $B$ be a weak Blumberg set of $f$ and let $J(B)$ denotes such $\sigma$-ideal in $B$ that $f_{|B}$ is $J(B)$-continuous. Thus according to Lemma 1.2 $D_{f_B}\in J(B)$ and because $J(B)$ does not include open sets (in $B$) then $C_{f_B}$ is dense in $B$ and so it is dense in $X$. Moreover, we infer that $f_{|C_{f|B}}=f_{|B|C_{f|B}}$, because $C_{f_B}\subset B$, and consequently $C_{f|B}$ is the Blumberg set of $f$.

**Corollary 3.10.** *A metric space $X$ is a Baire space if and only if every real function defined on $X$ possesses a weak Blumberg set.*

**Corollary 3.11.** *A metric space $X$ is a Baire space if and only if for every real function defined on $X$ there exists a set $B_f$ dense in $X$ such that the set of discontinuity points of $f_{|B_f}$ is of the first category and a boundary set in $B_f$.*

**Definition 3.12.** *We say that a function $f$: $X\to Y$, where $X$ and $Y$ are arbitrary topological spaces, is strongly $J$-continuous ($J$ — some $\sigma$-ideal in $X$) if*

1° $\overline{f^{-1}(U)\backslash\operatorname{Int}f^{-1}(U)}\in J$ *and* $f^{-1}(U)\notin J$, *for every open set* $U\subset Y$,
*and*
2° $f^{-1}(A)\in J$, *for every nowhere dense set* $A\subset Y$.

**Theorem 3.13.** *Let $X$ be a Baire space and let $f$ be a strongly $J$-continuous function mapping $X$ onto $Y$. Then $Y$ is a Baire space.*

Proof. According to [4, Theorem 1.13] it is sufficient to show that for every sequence $\{V_n\}$ of open and dense sets in $Y$, $\bigcap_{n=1}^{\infty} V_n$ is dense in $Y$.

Let $W^*$ be an arbitrary open set in $Y$. It is sufficient to prove that

$$(*) \qquad W^* \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset.$$

First, we infer that

$$f^{-1}(V_n) \quad \text{is dense in } X, \text{for } n = 1, 2, \dots. \tag{1}$$

In fact, assume, to the contrary, that there exists a nonempty open set $W \subset X$ such that $W \cap f^{-1}(V_n) = \emptyset$, for some $n$. Thus $f(W) \cap V_n = \emptyset$ and so $f(W)$ is nowhere dense, but $f$ is strongly $J$-continuous function, this means that

$$W \subset f^{-1}(f(W)) \in J$$

which is impossible because $J$ does not include nonempty open sets.

Now we shall show that

$$f^{-1}(W^*) \cap \bigcap_{n=1}^{\infty} f^{-1}(V_n) \neq \emptyset. \tag{2}$$

In fact, suppose, on the contrary, that $f^{-1}(W^*) \cap \bigcap_{n=1}^{\infty} f^{-1}(V_n) = \emptyset$. Then $\operatorname{Int} f^{-1}(W^*) \cap \bigcap_{n=1}^{\infty} f^{-1}(V_n) = \emptyset$ and so

$$\operatorname{Int} f^{-1}(W^*) = \bigcup_{n=1}^{\infty} (\operatorname{Int} f^{-1}(W^*) \setminus f^{-1}(V_n)). \tag{3}$$

Let $n$ be an arbitrary positive integer and let $V$ be an arbitrary nonempty open set. It is easy to see that

$$V \subset \overline{V \cap f^{-1}(V_n)}. \tag{4}$$

Hence from (4) and Definition 3.12 we deduce that

$$V \cap \operatorname{Int} f^{-1}(V_n) \neq \emptyset. \tag{5}$$

In virtue of (5) we infer that $\operatorname{Int} f^{-1}(W^*) \setminus f^{-1}(V_n)$ is nowhere dense, this according to (3) means that

$$\operatorname{Int} f^{-1}(W^*) \quad \text{is of the first category.} \tag{6}$$

On the other hand

$$\operatorname{Int} f^{-1}(W^*) \neq \emptyset. \tag{7}$$

In fact, suppose, on the contrary, that $\operatorname{Int} f^{-1}(W^*) = \emptyset$. Hence, according to Definition 3.12,

$$f^{-1}(W^*) \subset \overline{f^{-1}(W^*)} \subset \overline{f^{-1}(W^*) \setminus \operatorname{Int} f^{-1}(W^*)} \in J$$

which is impossible because $f^{-1}(W^*) \notin J$.

Conditions (6) and (7) contradict the supposition that $X$ is a Baire space. Hence (2) holds true.

According to (2) we have

$$f^{-1}\left(W^* \cap \bigcap_{n=1}^{\infty} V_n\right) \neq \emptyset,$$

this proves ($*$) and ends the proof of this theorem.

REFERENCES


[1] BLUMBERG, H.: New properties of all real functions, Trans. Amer. Math. Soc. 24, 1922, 113—128.

[2] BRADFORD, J. C. and GOFFMAN, C.: Metric spaces in which Blumberg's theorem holds, Proc. Amer. Math. Soc. 11, 1960, 667—670.

[3] ENGELKING, R.: Topologia ogólna, PWN, Warszawa, 1975.

[4] HAWORTH, R. C. and McCOY, R. A.: Baire spaces, Dissert. Math. CXLI, 1977, 1—77.

[5] KURATOWSKI, K.: Sur les fonctions representables analytiquement et les ensembles de premiere catégorie, Fund. Math. 5, 1924, 75—86.

[6] MAULDIN, R. D.: $\sigma$-ideals and related Baire systems, Fund. Math. LXXI, 1971, 171—177.

[7] MAULDIN, R. D.: Some examples of $\sigma$-ideals and related Baire systems, Fund. Math. LXXI, 1971, 179—184.

[8] MAŃK, T. and ŚWIĄTKOWSKI, T.: On some class of functions with Darboux's characteristic, Zesz. Nauk. P.L. 301, Mat. z. 11, 1977, 5—10.

[9] NEUBRUNNOVA, A.: On quasicontinuous and cliquish functions, Čas. Pest. Mat. 99, 1974, 109—114.

[10] NEUGEBAUER, C. J.: Blumberg sets and quasi-continuity, Math. Zeitschr. 79, 1962, 451—455.

[11] ОКСТОБИ, Д.: Мера и категория, Москва, 1974.

[12] PAWLAK, H. and PAWLAK, R. J.: On some conditions equivalent to the condition of Świątkowski for Darboux functions of one and two variables, Zesz. Nauk. P.L. 413, Mat. z. 16, 1983, 33—40.

[13] SEMADENI, Z.: Functions with sets of continuity belonging to a fixed ideal, Fund. Math. 52, 1963, 25—39.


Received May 23, 1983

*Institute of Mathematics*
*Łódź University*
*Banacha 22, 90-238 Łódź*
*Poland*

## ОБ ФУНКЦИЯХ МНОЖЕСТВО ТОЧЕК РАЗРЫВА КОТОРЫХ ПРИНАДЛЕЖИТ К НЕКОТОРОМУ $\sigma$-ИДЕАЛУ

Ryszard Jerzy Pawlak

Резюме

В этой статье мы рассматриваем класс функции, связаных с $\sigma$-идеалами. В первой части мы говорим об необходимых и достаточных условиях для $J$-непрерывности функции $f$. Теоремы, доказанные во второй части, представляют связь между понятием $J$-непрерывности и другими понятиями, похожими на непрерывность. Последняя часть содержит, между прочем, необходимые и достаточные условия для того, чтобы топологическое пространство $X$ было пространством Бера.