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# ON THE UNION OF MATCHING MATROIDS

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Let $\mathscr{H} = (H_i;\ i \in I)$ be a family of finite graphs. The $\mathscr{H}$ — ***packing problem*** consists, for a given graph $G$, in finding maximum subsets of vertices of $G$ that can be covered by vertex disjoint copies of graphs from $\mathscr{H}$. For example, if $\mathscr{H} = \{K_2\}$, then the $\mathscr{H}$ — packing problem consists in finding maximum subsets of vertices saturated by a matching.

Set $M_{\mathscr{H}}(G) = \{X \subset V(G)$; *there are vertex disjoint subgraphs* $W_1, \ldots, W_s$ *such that* $X \subset \bigcup_{i \le s} V(W_i)$ and each $W_i$ *is isomorphic to some* $H_j \in \mathscr{H}\}$.

$S_k$ will be a star on $k + 1$ vertices.

P. Tomasta in his lecture given at the conference "Combinatorics and graph theory", Luhačovice 85, called attention to connections between the $\mathscr{H}$ — packing problem and the matroids. In fact, for certain families $\mathscr{H}$ the positive solution is already known. Edmonds and Fulkerson proved that the subsets of vertices of a graph that can be saturated by a matching form a matroid. It has been proved recently that $M_{\mathscr{H}}(G)$ is a matroid when

i. $\mathscr{H} = \{K_2\} \cup \{H_1, \ldots, H_r\}$, $H_i$ *hypomatchable graphs* (*see* [3], [4], [7]).

ii. $\mathscr{H}$ *is a sequential set of stars* (*see* [1], [2], [6], [8], [9]).

Some further families $\mathscr{H}$ with the property that $M_{\mathscr{H}}(G)$ is a matroid are given in [10].

In the following theorem we answer three questions formulated by P. Tomasta.

**Theorem 1.**

A. *For every graph* $G$, $M_{\{S_1, S_2, \ldots, S_r\}}(G)$ *is a representable matroid.*

B. *Let F, G be connected graphs and* $\mathscr{H} = \{H$; *H is a connected* (*noninduced*) *subgraph of F with at least two vertices*$\}$. *Then* $M_{\mathscr{H}}(G)$ *is a matroid.*

C. *Let F, G be connected graphs and* $\mathscr{H} = \{H$; *H is a connected induced subgraph of F with at least two vertices*$\}$. *Then* $M_{\mathscr{H}}(G)$ *is a matroid.*

The case of packing by sequential set of stars (i.e. $\mathscr{H} = \{S_1, S_2, \ldots, S_r\}$) was studied extensively. The following theorem was observed first in another setting by M. LasVergnas.

**Theorem 2.** [9] *Let G be a graph and r be an integer. Then* $M_{\{S_1, S_2, \ldots, S_r\}}(G)$ *is a matroid union of r matching matroids* $M(G)$.

Proof of A. The matching matroid is transversal [5]. A union of transversal matroids is transversal as well. Every transversal matroid is representable [11]. □

We generalize Theorem 2 for packings with additional constraints.

**Theorem 3.** [10] *Let $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_r \geqslant 0$ be integers and $G$ be a graph. Let $\mathscr{H} = \{K_2 = S_1, \ldots, S_{r+1}\}$ be a sequentional set of stars. For an $\mathscr{H}$ packing $Q$, let $f_i(Q)$ denote the number of $S_{i+1}$'s used by $Q$. We call a packing $Q$ admissible if $f_i(Q)$, $i = 1, \ldots, r$, satisfies the system of inequalities*

$$\sum_{i=K}^{r} (i - K + 1) f_i(Q) \leq \sum_{i=K}^{r} b_i \qquad K = 1, \ldots, r.$$

*Then the system of subsets of $V(G)$ that can be saturated by an admissible packing forms a matroid.*

Proof will appear in [10].

The following well-known lemma simply holds by induction.

**Lemma 4.** *Let $F = (V, E)$ be a connected graph with maximum degree $r$. Then $V \in M_{\{S_1, S_2, \ldots, S_r\}}(F)$.*

Proof of B. Let $r$ be a maximum degree of $F$. Then $M_{\mathscr{H}}(G) = M_{\{S_1, \ldots, S_r\}}(G)$ by Lemma 4. □

**Lemma 5.** *Let $H = (V, E)$ be a connected graph. Then there exists a family of vertex disjoint induced subgraphs $(H_1, \ldots, H_n)$ of $H$ such that*

1. $V = \bigcup_{i \leq n} V(H_i)$,
2. *each $H_i$ is a triangle or a star.*

Proof. Proceed by induction on $|V|$. Let $x$ be a vertex of $H$ and $(G_1, \ldots, G_m)$ be a complete covering of $H \setminus \{x\}$ by triangles and induced stars. Let $y$ be a neighbour of $x$. Without loss of generality assume $y \in V(G_1)$.

1. If $G_1$ is a triangle, then replace $G_1$ by a perfect matching of $G_1 \cup \{x\}$.
2. Let $G_1$ be a star on at least three vertices. If there exists an end vertex $z$ of $G_1$ such that $\{x, z\} \in E(H)$, then replace $G_1$ by the edge $\{x, z\}$ and the star $G_1 \setminus \{z\}$. Otherwise $y$ is the centre of $G_1$ and then replace $G_1$ by the induced star $G_1 \cup \{x\}$.
3. If $G_1$ is an edge, then $G_1 \cup \{x\}$ is a triangle or a star $S_2$. □

The packing by triangles and edges (i.e. $\mathscr{H} = \{K_2, K_3\}$) is a special case of packing by edges and a set of hypomatchable graphs.

**Theorem 6.** [3] $M_{\{K_2, K_3\}}(G)$ is a matroid.

Proof of C. If follows from Lemma 5 that if $F$ has no induced $S_2$, then $M_{\mathscr{H}}(G) = M_{\{K_2, K_3\}}(G)$, otherwise $M_{\mathscr{H}}(G) = M_{\{S_1, S_2, \ldots, S_r\}}(G)$, where $r$ is the maximum degree of an induced star in $F$. □

Further results concerning matroids induced by packing subgraphs will appear in [10].

## REFERENCES

[1] AKIYAMA, J. AVIS, D.—ERA, H.: On a 1,2-factor of a graph, TRU Math. 16, 1980, 97 102.
[2] AMAHASHI, A. KANO, M.: On factors with given components, Discrete Math. 42, 1982, 7 26.
[3] CORNUEJOLS, A.—HARTVIGSEN, D.: An Extension of Matching Theory, J. Comb. Th. Ser. B40, 1986, 285 296.
[4] CORNUEJOLS, A.—HARTVIGSEN, D. PULLEYBLANK, W. R.: Packing Subgraphs in a Graph, O. R. Letters 1, 1982, 139—143.
[5] EDMONDS, J. FULKERSON, D. R.: Transversals and matroid partition, J. Res. Nat. Bur. Stand. Sec. B69, 1965, 147—153.
[6] HELL, P. KIRKPATRICK, D.: Star factors and star packings, Tr82-6, Dept. Computing Science, Simon Fraser University 1982.
[7] HELL, P. KIRKPATRICK, D.: Packings by cliques and by finite families of graphs, Discrete Math. 49, 1984, 118—133.
[8] HELL, P. KIRKPATRICK, D.: Packings by complete bipartite graphs, SIAM J. Alg. Disc. M. 7, 1986, 199—209.
[9] LAS VERGNAS, M.: An extension of Tutte's 1-factor theorem, Discrete Math. 23, 1987, 241 255.
[10] LOEBL, M.—POLJAK, S.: On matroids induced by packing subgraphs, JCT-B 44, 1988, 338 354.
[11] PIFF, J.—WELSH, P. J. A.: On the vector representation of matroids, J. London Math. Soc. 2, 1970, 284—288.


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## ОБ ОБЪЕДИНЕНИИ МАТРОИДОВ ПАРОСОЧЕТАНИЯ

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Резюме

В работе показано, что матроиды, нарожденные системами вершинно-непересекающихся звезд, являются объединением матроидов паросочетания.