Boris Rudolf

Existence theorems for nonlinear operator equation $Lu + Nu = f$ and some properties of the set of its solutions

Mathematica Slovaca, Vol. 42 (1992), No. 1, 55--63

Persistent URL: http://dml.cz/dmlcz/132764

Math. Slovaca, 42 (1992), No. 1, 55–63

Mathematica Slovaca

EXISTENCE THEOREMS FOR NONLINEAR OPERATOR EQUATION $Lu + Nu = f$ AND SOME PROPERTIES OF THE SET OF ITS SOLUTIONS

BORIS RUDOLF

ABSTRACT. We give an existence theorem and study the compactness and connectedness of the set of solutions to the operator equation $Lu + Nu = f$. The linear operator L is not necessarily invertible and the nonlinear operator N is continuous and bounded. The proofs are based on a topological degree and Leray-Schauder techniques.

1. Introduction

This paper deals with the problem of the existence of a solution to the equation

$$Lu + Nu = f, \tag{1}$$

where $L\colon D(L) \subset X \to Z$ is a *linear*, $N\colon X \to Z$ a *nonlinear operator* and $f \in Z$. X and Z are *Banach spaces*. The operator L is not necessarily invertible. The basic papers in this area are the ones by M a w h i n [5] and C e s a r i [1].

This paper confirms some results given in the article by W a r d [7], where the existence of a solution to the equation (1) is proved in a special case. We are also interested in some properties of the set of solutions to the equation (1).

The results of this paper are applied to the two point boundary value problem with a selfadjoint differential operator.

AMS Subject Classification (1991): Primary 34B15.
Key words: Nonlinear equation, Leray-Schauder degree, Krasnosel'skij's theorem.

2. Existence Theorems

Let X, Z be *real Banach spaces* with *norms* $|\cdot|_X$, $|\cdot|_Z$, S be a *real Hilbert space* with the *inner product* $\langle\cdot,\cdot\rangle$ and the *norm* $\|\cdot\|$. Let Z^* be the *dual of* Z.

We assume that

$$X \subset Z^* \subset S \subset Z \tag{2}$$

and the natural embedding of each space into those which contain it is continuous. Further, we assume that $(v,u) = \langle v,u\rangle$ for each $u \in Z^*$ and $v \in S$, where (g,f) denotes the *value of a linear functional* $f \in Z^*$ at $g \in Z$. We use the *Ward assumptions* [7, p. 233].

(H1) For some $\lambda \in (-\infty,\infty)$, $L - \lambda I$ is *one to one* and *onto* Z, $(L-\lambda I)^{-1}$ is a *continuous operator on* Z, and it is *compact* as a mapping from Z into X.

(H2) There is a number $\alpha \in (-\infty,\infty)$ such that

$$(Lu,u) \geqq \alpha\|u\|^2 \qquad \text{for all} \quad u \in D(L).$$

(H3) $N : X \to Z$ is *continuous* and maps bounded sets in X into bounded sets in Z.

(H4) There is a number $r > 0$ such that

$$(Lu,u) + (Nu,u) \geqq (f,u) \tag{3}$$

whenever $u \in D(L)$ and $\|u\| = r$.

Number α in (H2) can be assumed to be zero since the operators $L - \alpha I$, $N + \alpha I$ satisfy the same assumptions as the operators L, N.

It follows from assumption (H1) that there is a number $c > 0$ such that $-c$ is in a resolvent set of L. (See [7, p. 233].)

First we improve an existence theorem of Ward [7, Theorem 3]. We use the main idea of the proof used in [7] which is based on the following

Lemma 1. *Assume that* (H1), (H3) *hold. Then the operator* $T : X \to X$ *defined by*

$$Tu = -(L+cI)^{-1}(N-cI)u + (L+cI)^{-1}f \tag{4}$$

is completely continuous and equation (1) *is equivalent to*

$$Tu = u. \tag{5}$$

The difference between the theorem given below and the one of Ward is that we assume that the inequality (3) holds only for $u \in D(L)$, $\|u\| = r$.

THEOREM 1. *Assume that* (H1) – (H3) *hold and that there are positive numbers* r, R *such that* (3) *holds whenever either* $\|u\| = r$ *and* $|u|_X \leqq R$ *or* $\|u\| < r$ *and* $|u|_X = R$. *Then there is a solution to the equation* (1).

Proof. From Lemma 1 it follows that (1) is equivalent to (5). We use the *Leray-Schauder degree theory* for the operator T on

$$\Omega = \{u \in X,\ |u|_X < R \text{ and } \|u\| < r\}.$$

Clearly Ω is open in X. If $u \in \partial\Omega$, then $\|u\| = r$ and $\|u\|_X \leqq R$ or $\|u\| < r$ and $|u|_X = R$.

We show that the equation

$$\lambda T u = u \tag{6}$$

has no solution on $\partial\Omega$ for every $\lambda \in \langle 0, 1)$.
Suppose u is such a solution. Then

$$(L + cI)u + \lambda(N - cI)u - \lambda f = 0. \tag{7}$$

Hence

$$(Lu, u) + \lambda(Nu - f, u) = (\lambda - 1)c\|u\|^2 < 0. \tag{8}$$

Since $u \in \partial\Omega$, the inequality (3) holds and we obtain that

$$(Nu - f, u) > 0.$$

As it was mentioned above we can use the assumption (H2) with $\alpha = 0$ and obtain

$$(Lu, u) + \lambda(Nu - f, u) > 0.$$

The last inequality is in contradiction with the inequality (8).

That means there is no solution to (6) on $\partial\Omega$, for every $\lambda \in \langle 0, 1)$ and either there is a solution to (5) on $\partial\Omega$ or the Leray-Schauder degree

$$d(I - T, \Omega, 0) = d(I, \Omega, 0) \neq 0.$$

This implies that there is a solution $u \in D(L)$ to the equation (1).

We apply the following corollary in the last part of this paper.

COROLLARY 1. *Assume that* (H1) (H4) *hold and that*

(i) *there are numbers* $a > 0$, $b \geqq 0$ *such that for* $u \in D(L)$

$$(Lu, u) \geqq a|u|_X^2 - b\|u\|^2,$$

(ii) $\inf\limits_{\substack{\|u\|<r \\ u\in D(L)}} (Nu, u) > -\infty$.

Then there is a solution to the equation (1).

Proof. The result follows from Theorem 1. We show that (3) holds for every $u \in D(L)$ with $\|u\| < r$ and $|u|_X = R$. From (i), (ii) we obtain

$$(Lu, u) + (Nu - f, u) \geqq a|u|_X^2 - b\|u\|^2 + \inf_{\substack{\|u\|<r \\ u\in D(L)}} (Nu, u) - k|f|_Z|u|_X$$

where k is the norm of the embedding of X into Z.

We have

$$a|u|_X^2 - b\|u\|^2 + \inf_{\substack{\|u\|<r \\ u\in D(L)}} (Nu, u) - k|f|_Z|u|_X \geqq 0$$

for every $u \in D(L)$ with $\|u\| < r$ and $|u|_X = R$, where $R \geqq R_1$. Here R_1 is the positive solution to the quadratic equation

$$aR_1^2 - k|f|_Z R_1 + \inf_{\substack{\|u\|<r \\ u\in D(L)}} (Nu, u) - br^2 = 0. \tag{9}$$

If (9) has two positive solutions, then R_1 is the greater one.

3. The properties of the set of solutions

We denote the set of solutions to (1) as U.

THEOREM 2. *Assume that* (H1) and (H3) *hold. Then* U *is a closed,* σ*-compact set in* X.

Proof. Suppose that $U \neq \emptyset$ and $\{u_n\}$ is a sequence of solutions to (1) such that u_n converges to u. Then $u_n = Tu_n$ and the continuity of the operator T implies $Tu_n \to Tu$ in X. Then $u = Tu$ and thus the set U i closed. Moreover

$$U_n = \{u \in U,\ |u|_X \leqq n\}, \qquad n = 1, 2, \ldots$$

is closed and

$$U_n = T(U_n),$$

which implies the compactness of U_n. U is now σ-compact as

$$U = \bigcup_{n=1}^{\infty} U_n.$$

Now we deal with the connectedness of the set of solutions to the equation (1). We use the following lemma:

LEMMA 2. (*Krasnosel'skij's Theorem* [8, pp. 155-156]) *Let $\Omega \neq \emptyset$ be an open bounded subset of the Banach space X. Assume that $T_n, T\colon \Omega \to X$ are completely continuous operators for every $n \geqq n_0$,*

$$d_n = \sup_{x\in\overline{\Omega}} \|T(x) - T_n(x)\| \to 0 \qquad \textit{for} \quad n \to \infty,$$
$$Tx \neq x \qquad \textit{for every} \quad x \in \partial\Omega,$$

and the Leray-Schauder degree

$$d(I - T, \Omega, 0) \neq 0.$$

Further assume that the equation

$$x = T_n(x) + \big(T(\bar{x}) - T_n(\bar{x})\big)$$

has at most one solution for each $\bar{x} \in \Omega$, $n \geqq n_0$.

Then the set of solutions to the equation $x = Tx$ is nonempty, compact and connected.

THEOREM 3. *Assume* (H1)–(H3) *and* (H4c), *there is an $r > 0$ such that*

$$(Lu, u) + (Nu, u) > (f, u) \tag{10}$$

whenever $u \in D(L)$, $|u|_X = r$ hold. Suppose that there is a sequence of nonlinear operators $N_i\colon X \to Z$ such that N_i uniformly converges to N on $\overline{\Omega}$, where $\Omega = \{u \in X,\ |u|_X < r\}$ and each equation

$$Lu + N_i u = f', \qquad i = 1, 2, \ldots$$

has at most one solution in $\overline{\Omega}$ for every $f' \in Z$.

Then the set U of solutions to (1) *in* $\overline{\Omega}$ *is nonempty, compact and connected.*

Proof. We use the equivalence between (1) and (5) and apply Lemma 2 to the equation (5).

Let $\{T_i\}$ be the sequence of operators $T_i : X \to X$

$$T_i u = -(L + cI)^{-1}(N_i - cI)u + (L + cI)^{-1} f, \tag{11}$$

where T, T_i are completely continuous and

$$\sup |T_i u - Tu|_X \to 0 \qquad \text{for} \quad i \to \infty.$$

Obviously $Tu \neq u$ on $\partial\Omega$.
Similarly as in the proof of Theorem 1 in [7, p. 234], it can be shown that

$$d(I - T, \Omega, 0) \neq 0.$$

Let $\bar{x} \in \Omega$ and

$$T(\bar{x}) - T_i(\bar{x}) = z.$$

Assume that u is a solution to the equation

$$u = T_i u + z. \tag{12}$$

Using (11) we obtain the equation

$$Lu + N_i u = f + (L + cI)z = f',$$

which has at most one solution on $\overline{\Omega}$ under the condition of the theorem. Now by Lemma 2 it follows that the set of solutions to (1) is nonempty, compact and connected.

COROLLARY 2. *Assume that* (H1), (H3), (H4c) *hold. Further assume that* (H2) *holds with* $\alpha = 0$ *and for each* $u, v \in \overline{\Omega}$

$$(Nu - Nv, u - v) \geqq 0, \tag{13}$$

where $\Omega = \{u \in X,\ |u|_X < r\}$.
Then the set of solutions to (1) *in* $\overline{\Omega}$ *is nonempty, compact and connected.*

Proof. We consider the sequence of operators $N_i : X \to Z$ defined by $N_i x = Nx + \frac{1}{i}x$. Then

$$(N_i u - N_i v, u - v) > 0$$

for each $u, v \in \Omega$, $u \neq v$. Hence

$$\big((Lu + N_i u) - (Lv + N_i v), u - v\big) > 0 \qquad \text{for} \quad u \neq v$$

and thus the equation $Lu + N_i u = f'$ has at most one solution in $\overline{\Omega}$ for every $f' \in Z$. Further the sequence N_i converges uniformly to N in $\overline{\Omega}$. The result follows from Theorem 3.

4. An example

Let $Z = S = L_2\langle a,b\rangle$ and $X = H^{2n}\langle a,b\rangle = \{u \in C^{2n-1}\langle a,b\rangle, u^{(2n)} \in S\}$. Let $\tau = \sum\limits_{i=0}^{2n} a_i(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i$ be a symmetric differential operator of the $2n$th order with coefficients $a_i \in C^\infty\langle a,b\rangle$.

Let $B_i(u) = \sum\limits_{j=0}^{2n-1} \alpha_{ij}u^{(j)}(a) + \sum\limits_{j=0}^{2n-1} \beta_{ij}u^{(j)}(b)$ be a set of $2n$ linearly independent boundary values.

We consider the problem

$$\tau u + f\left(\int\limits_a^b u^2 \mathrm{d}t\right) g(t,u) = 0 \tag{14}$$

$$B_i u = 0, \tag{15}$$

where $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \langle a,b\rangle \times \mathbb{R} \to \mathbb{R}$ are continuous functions and

$$\text{(G)} \qquad \alpha(t) \leqq \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \leqq \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \leqq c$$

uniformly for $t \in \langle a,b\rangle$. Here $\alpha(t) \in C\langle a,b\rangle$, $0 \leqq \alpha(t) \leqq c$ and $\alpha \neq 0$. Moreover we assume that

$$\limsup_{x\to\infty} f(x) = \beta > 0, \qquad \beta \in \mathbb{R}.$$

It is shown in [7, p. 239] that the condition (i) from Corollary 1 is satisfied. We prove the conditions (ii) and (H4). Let $u \in D(L)$. Then

$$\begin{aligned}(Nu,u) &= \int_a^b f\left(\int_a^b u^2 \mathrm{d}t\right) g\big(s,u(s)\big)u(s)\,\mathrm{d}s \\ &= f\left(\|u\|^2\right)\int_a^b g\big(s,u(s)\big)u(s)\,\mathrm{d}s.\end{aligned}$$

Assumption (G) implies that for every $\varepsilon > 0$ there is an $R > 0$ such that for $|x| > R$

$$\big(\alpha(t)-\varepsilon\big)x^2 \leqq g(t,x)x \leqq (c+\varepsilon)x^2.$$

Let $E = \{t,\ |u(t)| > R\}$, $F = \langle a,b\rangle - E$. Then

$$-\varepsilon\|u\|^2 \leqq \int\limits_E g\big(t,u(t)\big)u(t)\,\mathrm{d}t \leqq (c+\varepsilon)\|u\|^2,$$

$$\left|\int\limits_F g\big(t,u(t)\big)u(t)\,\mathrm{d}t\right| \leqq m.$$

This means that

$$\inf_{\|u\|<r}(Nu,u) \geqq -\sup_{\|u\|<r}\left|f\left(\|u\|^2\right)\right| \sup_{\|u\|<r}\left|\int_a^b g\big(t,u(t)\big)u(t)\,\mathrm{d}t\right|$$

$$\geqq -\sup_{\|u\|<r}\left|f\left(\|u\|^2\right)\right|\left[(c+\varepsilon)r^2+m\right] > -\infty$$

and condition (ii) is proved.

Now let $u \in D(L)$ such that $f\left(\|u\|^2\right) \geqq \frac{\beta}{2} > 0$. Then

$$(Lu,u)+(Nu,u) \geqq (Lu,u)+\frac{\beta}{2}\int_a^b g\big(t,u(t)\big)u(t)\,\mathrm{d}t$$

$$= (Lu,u)+\frac{\beta}{2}\int_E g\big(t,u(t)\big)u(t)\,\mathrm{d}t+\frac{\beta}{2}\int_F g\big(t,u(t)\big)u(t)\,\mathrm{d}t$$

and by Lemma 1 [7, p. 237] we obtain the estimation

$$(Lu,u)+(Nu,u)$$

$$\geqq (Lu,u)+\frac{\beta}{2}\int_a^b \alpha(t)\big(u(t)\big)^2\mathrm{d}t-\frac{\beta}{2}\varepsilon\|u\|^2-\int_F \frac{\beta}{2}\alpha(t)\big(u(t)\big)^2\mathrm{d}t-\frac{\beta}{2}m(b-a)$$

$$\geqq \delta\|u\|^2-\frac{\beta}{2}\varepsilon\|v\|^2-c,$$

where δ is the least positive eigenvalue of the operator $L\big(u(t)\big)+\frac{\beta}{2}\alpha(t)u(t)$, $c>0$ is a constant. That means if we choose R such that

$$\delta-\frac{\beta}{2}\varepsilon>0,$$

$$R>\left(\frac{c}{\delta-\varepsilon\frac{\beta}{2}}\right)^{\frac{1}{2}}$$

$$\text{and} \qquad f(R^2) \geqq \frac{\beta}{2},$$

we get

$$(Lu,u)+(Nu,u)>0$$

and (H4) holds for every $u \in D(L)$, $\|u\|=R$.

Corollary 1 then implies the existence of a solution to the problem (14), (15) and by Theorem 2 we get the closedness and σ-compactness of the set of solutions to that problem.

REFERENCES

[1] CESARI, L.: *Functional analysis and Galerkin's method*, Michigan Math. J. **11** (1964), 385–414.

[2] FUČÍK, S.—KUFNER, A.: *Nelineární diferenciální rovnice*, SNTL, Praha, 1978.

[3] GREGUŠ, M.—ŠVEC, M.—ŠEDA, V.: *Obyčajné diferenciálne rovnice*, Alfa, Bratislava, 1985.

[4] KODDINGTON, E. A.—LEVINSON, N.: *Teorija obyknovennych differencial'nych uravnenij*, Izdat. Inostr. lit., Moskva, 1958.

[5] MAWHIN, J.: *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J.Differential Equations **12** (1972), 610–636.

[6] TAYLOR, A. E.: *Úvod do funkcionální analysy*, Academia, Praha, 1973.

[7] WARD, J. R.: *Existence theorems for nonlinear boundary value problems at resonance*, J.Differential Equations **35** (1980), 232–247.

[8] ZEIDLER, E.: *Vorlesungen über nichtlineare Funktionalanalysis I*, Teubner Verlag, Leipzig, 1976.

Received November 25, 1988

Katedra matematiky EF STU
Mlynská dolina
812 19 Bratislava