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ON DOMATIC NUMBERS OF GRAPHS

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In [1] E. J. Cockayne and S. T. Hedetniemi have proposed (among others) the following two problems:

I. *Characterize the class of graphs whose domatic number is* 2.

II. *Prove or disprove: If K(G) is an odd cycle, then*

$$c(G) \leqq d(G).$$

Here we shall solve these problems. We consider finite undirected graphs without loops and multiple edges. Sometimes we shall speak also about directed and mixed graphs, but then we shall always use the corresponding adjectives. In undirected graphs we shall use the term "circuit" instead of "cycle", used by the authors of the problems. By the term "cycle" we shall denote a directed circuit, i.e. a circuit C in a directed or mixed graph such that all edges of C are directed and each vertex of C is an initial vertex of exactly one edge and a terminal vertex of exactly one edge of C.

A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that each vertex of $V(G)-D$ is adjacent to at least one vertex of D. A domatic partition of G is a partition of $V(G)$ into dominating sets. The maximal number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

Note that the union of two dominating sets of G is a dominating set of G and the whole $V(G)$ is also a dominating set of G. Therefore in a graph G there exist domatic partitions of all cardinalities among 1 and $d(G)$. Such partitions can be constructed from a partition of the cardinality $d(G)$ by taking unions of some of its classes.

Sometimes it is advantageous to speak about domatic colourings instead of domatic partitions. A domatic colouring of a graph G is a colouring of the vertices of G with the property that each vertex is adjacent to vertices of all the colours different from its own one. (Two vertices of the same colour may be adjacent.) Then the domatic number of G is the maximal number of colours of a domatic colouring of G. The equivalence of both the definitions of the domatic number is evident.

We turn to the first problem. We shall characterize the graphs with the domatic number 2 "negatively", i.e. we shall characterize graphs with the domatic number different from 2.

According to a result of O. Ore quoted in [1], $d(G)=1$ if and only if G has at least one isolated vertex. Therefore it suffices to characterize graphs G for which $d(G)\geqq 3$.

A partial result was proved by F. Jaegar and C. Payan [3]: If G has diameter less than or equal to two and every edge of G is in a triangle, then $d(G)\geqq 3$.

(It is only a sufficient condition, not necessary; an example of a graph with the domatic number 3 not fulfilling this condition is a circuit of the length 6.)

We shall use directed graphs and mixed ones. A mixed graph is a graph in which some edges are directed and some undirected. Cycles and directed paths in mixed graphs are defined in the same way as in directed graphs (note that all edges of a cycle or of a directed path must be directed).

Now we prove a theorem.

Theorem 1. *A graph G has a domatic number greater than 2 if and only if some of its edges can be directed so that a (mixed or directed) graph G^* is obtained with the following properties:*

(a) *Each vertex of G^* is the initial vertex of a directed edge and a terminal vertex of a directed edge.*

(b) *The lengths of any two directed paths in G^* outgoing from the same vertex and incoming into the same vertex are congruent modulo* 3.

(c) *Each cycle in G^* has a length divisible by* 3.

Proof. Suppose $d(G)>2$. Then there exists a domatic partition $\{D_1, D_2, D_3\}$ of the vertex set $V(G)$ of G. If e is an edge of G joining vertices u *and* v, we direct it if and only if u and v are contained in different classes of the partition $\{D_1, D_2, D_3\}$. We direct it from u into v if and only if either $u\in D_1$, $v\in D_2$, or $u\in D_2$, $v\in D_3$, or $u\in D_3$, $v\in D_1$. As D_2, D_3 are dominating sets in G, at least one edge e_1 joins u with a vertex of D_2 and at least one edge e_2 joins u with a vertex of D_3. In G^* *the vertex* u is the initial vertex of e_1 and the terminal vertex of e_2. Analogously for $u\in D_2$ or for $u\in D_3$ we can prove that it is the initial vertex of a directed edge and the terminal vertex of a directed edge. Therefore (a) holds. If u, v are two vertices of G, then the edge $\overrightarrow{uv}$ may exist only if $u\in D_i$, $v\in D_j$, $\{i, j\} \subset \{1, 2, 3\}$, $j-i\equiv 1 \pmod 3$. By induction we obtain that a directed path from u *into* v of the length k, where k is a positive integer, may exist only if $u\in D_i$, $v\in D_j$, $\{i, j\} \subset \{1, 2, 3\}$, $j-i\equiv k \pmod 3$. If there are two directed paths from $u\in D_i$ into $v\in D_j$, one of the length k_1 and the other of the length k_2, then $j-i\equiv k_1 \pmod 3$, $j-i\equiv k_2 \pmod 3$, therefore $k_1\equiv k_2 \pmod 3$ and (b) is proved. By a similar argument (considering a cycle as a directed path in which the initial and the terminal vertex coincide) we prove (c).

Now let G be a graph whose edges can be directed so that a graph G^* with the described properties is obtained. We choose a vertex u in G and number it by 1. Then we continue the numbering of vertices by the numbers 1, 2, 3 so that if a vertex v has the number i, then all terminal (or initial) vertices of edges outgoing from (or incoming into) v are numbered by the number congruent with $i+1$ (or $i-1$ respectively) modulo 3. In this way we can number all vertices of the connected component of G which contains u and, according to (b) and (c), the numbers of all vertices are uniquely determined. Then we choose a vertex in another connected component of G (if anyone exists) and proceed in the same way. Thus we continue until all the vertices of G are numbered by the numbers 1, 2, 3. If by D_i we denote the set of all vertices of G numbered by i, we obtain a partition $(D_1, D_2, D_3\}$ of $V(G)$. The condition (a) implies that each vertex of one of the classes D_1, D_2, D_3 is adjacent to vertices of both other classes and hence $\{D_1, D_2, D_3\}$ is a domatic partition and $d(G) \geqq 3$.

Now we shall solve the second problem; it is mentioned also in [2]. First we shall explain the used notions.

A clique of a graph G is a subgraph of G which is complete and is not a proper subgraph of a complete subgraph of G. (Sometimes by a clique each complete subgraph of G is meant.) Each vertex of G is contained in at least one clique of G. The clique graph $K(G)$ of G is the graph whose vertices are in a one-to-one correspondence with cliques of G and in which two vertices are adjacent if and only if the corresponding cliques have at least one common vertex. By $c(G)$ the minimal number of vertices of a clique of G is denoted.

If $K(G)$ is a circuit of an even length, then $c(G) \leqq d(G)$; this was proved in [1]. We shall present a counterexample showing that the analogous assertion for circuits of odd lengths is false.

Theorem 2. *There exists a graph G for which $K(G)$ is a circuit of an odd length and $c(G) > d(G)$.*

Proof. Let G be the graph with the following structure. The vertex set of G is $V(G) = \bigcup_{i=1}^{5} X_i \cup Y_i$, where $|X_i| = 3$, $|Y_i| = 1$ for each $i \in \{1, 2, 3, 4, 5\}$, $X_i \cap Y_j = \emptyset$ for each $\{i, j\} \subset \{1, 2, 3, 4, 5\}$, $X_i \cap X_j = Y_i \cap Y_j = \emptyset$ for each $\{i, j\} \subset \{1, 2, 3, 4, 5\}$ such that $i \neq j$. For each $i \in \{1, 2, 3, 4, 5\}$ the set $X_i \cup X_{i+1} \cup Y_i$ induces a clique denoted by C_i (the sum $i+1$ is taken modulo 5). The graph G is the union of the cliques C_1, C_2, C_3, C_4, C_5. Evidently the clique graph $K(G)$ is a pentagon. The unique element of Y_i will be denoted by y_i for each $i \in \{1, 2, 3, 4, 5\}$. We have $c(G) = 7$ and we shall show that $d(G) < 7$. For simplicity we shall speak about a domatic colouring. Suppose that there exists a domatic colouring of G by the colours 1, 2, 3, 4, 5, 6, 7. The degree of y_i (for each i) is equal to 6. This implies that the vertices of C_i must be coloured by pairwise different colours; otherwise there would exist a colour by which neither y_i, nor a vertex adjacent to y_i would be coloured. Let A_i be the set of colours by thich the vertices of X_i are coloured. For each i we have obviously $|A_i|=3$ and $A_i \cap A_{i+1}=\emptyset$, because X_i and X_{i+1} are both contained in C_i (the sum $i+1$ is taken modulo 5). Without loss of generality let $A_1=\{1, 2, 3\}$. The set A_2 is a three-element subset of the set $\{4, 5, 6, 7\}$. The set A_3 is a three-element subset of the set $\{1, 2, 3, 4, 5, 6, 7\}-A_2$; it can contain at most one of the colours 4, 5, 6, 7 and therefore at least two of the colours 1, 2, 3. We see that $|A_1 \cap A_3| \geqq 2$ and analogously we can prove $|A_3 \cap A_5| \geqq 2$. As A_5 has at least two common elements with A_3 and at most one element of A_3 does not belong to A_1, we have $A_1 \cap A_5 \neq \emptyset$. Therefore in the clique C_5 at least two vertices are coloured by the same colour (one from X_1 and one from X_5), which is a contradiction. The domatic number of G is less than 7 and thus $c(G)>d(G)$.

Now we show that by a suitable strengthening of conditions we obtain a true assertion.

Theorem 3. *Let G be a graph whose clique graph $K(G)$ is a circuit of an odd length and let G contain a clique with the property that each of its vertices is contained also in another clique of G. Then $c(G) \leqq d(G)$.*

Proof. As $K(G)$ is a circuit of an odd length (denote this length by k), the graph is the union of cliques $C_1, \ldots, C_k$ such that for each $i=1, \ldots, k$ the clique C_i has non-empty intersections with C_{i-1} and C_{i+1} (the subscripts are taken modulo k) and with no other clique of G. According to the assumptions of this theorem at least one of the cliques $C_1, \ldots, C_k$ has the property that each of its vertices belongs also to another clique. Without loss of generality let this clique be C_1. Therefore, if $k \neq 3$, the vertex set of C_1 is the union of two disjoint sets X, Y, where X (or Y) is the vertex set of the intersection of C_1 with C_k (or C_2 respectively). Let G_0 be the graph obtained from G by deleting all edges joining vertices of X with vertices of Y. The cliques of G_0 are exactly the cliques $C_2, \ldots, C_k$ and the graph $K(G_0)$ is a snake (the graph consisting of one path). Hence $K(G_0)$ is a tree. By Theorem 4.11 from [1] we have $c(G_0) \leqq d(G_0)$. Further $c(G) \leqq c(G_0)$, because each clique of G_0 is a clique of G, and $d(G_0) \leqq d(G)$, because G_0 is a spanning subgraph of G. We have proved $c(G) \leqq d(G)$. If $k=3$, let V_i be the vertex set of C_i for $i \in \{1, 2, 3\}$. Let C_1 have the property that each of its vertices belongs also to C_2 or C_3. Consider the set $(V_1 \cap V_2) \cup (V_1 \cap V_3) \cup (V_2 \cap V_3)$. Let x, y be two distinct vertices of this set. Then each of these vertices belongs to at least two of the sets V_1, V_2, V_3 and thus at least one of these sets contains both x and y. This means x and y both belong to one of the cliques C_1, C_2, C_3, hence they are adjacent. As x, y were chosen arbitrarily, this implies that the subgraph of G induced by $(V_1 \cap V_2) \cup (V_1 \cap V_3) \cup (V_2 \cap V_3)$ is a complete graph. As $V_1 = (V_1 \cap V_2) \cup (V_1 \cap V_3)$, the clique C_1 *is a subgraph of this graph. As* C_1 is a clique, it cannot be a proper subgraph of another complete subgraph of G, therefore it is equal to the mentioned complete subgraph of G and we have $V_1 = (V_1 \cap V_2) \cup (V_1 \cap V_3) = (V_1 \cap V_2) \cup (V_1 \cap V_3) \cup (V_2 \cap V_3)$, which implies $V_2 \cap V_3 \subset V_1$. If $V_2 - (V_1 \cup V_3) = \emptyset$, then $V_2 \subset V_1 \cup V_3$ and, as $V_2 \cap V_3 \subset V_1$, we have $V_2 \subset V_1$, which is a contradiction with the assumption that V_1, V_2 are vertex sets of different cliques of G. Therefore $V_2 - (V_1 \cup V_3) \neq \emptyset$ and analogously $V_3 - (V_1 \cup V_2) \neq \emptyset$. Now let G_1 be the graph obtained from G by deleting all edges joining vertices of $V_2 - (V_1 \cup V_3)$ with vertices of $V_3 - (V_1 \cup V_2$. The graph G_1 is the union of the cliques C_2 and C_3 and $K(G_1)$ is a tree (consisting of one edge with its end vertices). We have again $c(G) \leqq c(G_1) \leqq d(G_1) \leqq d(G)$.

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О ДОМАТИЧЕСКИХ ЧИСЛАХ ГРАФОВ

Богдан Зелинка

Резюме

Доматическое число $d(G)$графа G есть максимальное число классов разбиения множества вершин графа G на доминирующие множества. В статье решены две проблемы Э. Дж. Кокейна и С. Т. Хедетниеми. Характеризованы графы с доматическим числом 2. Построен граф G, граф клик которого является контуром нечетной длины и доматическое число которого меньше минимального числа вершин клики графа G.