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DARBOUX PROPERTY OF THE WRONSKI DETERMINANT

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ABSTRACT. We investigate the Darboux property of the Wronski determinant of two functions being differentiable on an interval. Some related questions are also discussed.

1. Introduction

In the theory of real functions, there are notions playing very important roles for many years. For example, such notions as continuity, absolute continuity, Darboux property and differentiability are basic in many investigations of this theory ([2], [8]).

The aim of this note is to give a solution of the problem from the theory of real functions raised recently by M. Malec [7].

In order to present this problem, let us fix an interval I and assume that f and g are two real functions defined and differentiable on the interval I. Let $W(f,g)$ denote the Wronski determinant of the functions f and g, i.e.

$$W(f,g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - gf' .$$

The problem mentioned above is formulated in the following way: Is $W(f,g)$ a Darboux function on the interval I?

Recall that the function $h\colon I \to \mathbb{R}$ is said to be a Darboux function (or the function with Darboux property) if for any subinterval I_1 of I the set $h(I_1)$ is an interval.

Let us mention that the problem presented above can be formulated in a more general setting.

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For example, M a l e c [6] raised also the following question:

Let f_i $(i = 1, 2, 3)$ be twice differentiable functions on the interval I. Consider the Wronski determinant

$$W(f_1, f_2, f_3) = \det \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$$

of these functions. Is $W(f_1, f_2, f_3)$ a Darboux function on I?

2. Results

In what follows, we give two examples showing that the questions formulated before have answers in the negative.

E x a m p l e 1. Let us take the functions f and g defined on the interval $[0, 1]$ in such a way that $f(x) = x^2 \sin(x^{-4})$, $g(x) = x^2 \cos(x^{-4})$ for $x \in (0.1]$ and $f(0) = g(0) = 0$. Obviously f and g are differentiable on the interval $[0, 1]$. It can be calculated that

$$W(f, g)(x) = (fg' - gf')(x) = \frac{4}{x}$$

for $x \in (0, 1]$. Moreover, $W(f, g)(0) = 0$.

Thus, the Wronski determinant $W(f, g)$ is not a Darboux function on the interval $[0, 1]$.

E x a m p l e 2. Let f and g denote the functions from Example 1. Consider the functions f_1, f_2, and f_3 defined on the interval $[0, 1]$ in the following way

$$f_1(x) = \int_0^x f(t)\, \mathrm{d}t\,,$$

$$f_2(x) = \int_0^x g(t)\, \mathrm{d}t\,,$$

$$f_3(x) = 1\,.$$

It is easily seen that these functions are twice differentiable on the interval $[0, 1]$. Moreover, we have

$$W(f_1, f_2, f_3) = W(f_1', f_2') = W(f, g)\,.$$

Thus, in the light of Example 1, the Wronski determinant $W(f_1, f_2, f_3)$ is not a Darboux function.

In the sequel, we show that under some additional assumptions the questions considered above have positive answers.

In order to formulate a few results in this direction, let us establish some notation.

Denote by C the class of continuous functions on the interval I. Similarly, let Δ denote the class of all derivatives on I and D the class of Darboux functions on I. Finally, the symbols b and B_1 stand for the class of bounded functions and for the class of functions of Baire class 1 on I, respectively.

If we want to consider the intersection of two classes, we write them one after another. For example, $b\Delta$ will denote the collection of bounded derivatives on I.

The properties of the above introduced classes can be found for instance in [2].

In what follows, let us observe that for differentiable functions f and g on the interval I such that $f(x) \neq 0$ (or $g(x) \neq 0$), for any $x \in I$, we have that $W(f,g) \in D$ (cf. [1]). Indeed, this fact easily follows from the equality $W(f,g) = -\left(\frac{f}{g}\right)' \cdot g^2$ and from the property ensuring that $fg \in DB_1$ provided $f \in DB_1$ and $g \in C$ ([2]).

Now we are going to formulate some generalization of the result contained in the above statement.

First we state an additional notation.

Namely, for a given function $h\colon I \to \mathbb{R}$, let Z_h denote the set of all zeros of h in the interval I. Let the symbol $\overline{A}$ stand for the closure of a set A.

THEOREM 1. *Let $f,g\colon I \to \mathbb{R}$ be functions differentiable on the interval I. Assume that the set Z_f (or Z_g) has no accumulation points, and $Z_f \cap \overline{Z}_g = \emptyset$ (or $\overline{Z}_f \cap Z_g = \emptyset$). Then $W(f,g)$ has the Darboux property on I.*

P r o o f. It is easy to show that if I', I'' are intervals contained in I, $I' \cap I'' \neq \emptyset$, and if a function h has the Darboux property on each interval I' and I'', respectively, then h has the Darboux property on the union $I' \cup I''$.

Next, to fix our attention, let us assume that Z_f has no accumulation points and $Z_f \cap \overline{Z}_g = \emptyset$.

Take an interval $I_1, I_1 \subset I$. In the light of the above observation, we can assume without loss of generality that I_1 contains exactly one zero x_0 of the function f.

Denote $I_1^-(x_0) = (-\infty, x_0) \cap I_1$, $I_1^+(x_0) = (x_0, +\infty) \cap I_1$.

In view of the assertion made before Theorem 1, we deduce that $W(f,g)$ has the Darboux property on the intervals $I_1^-(x_0)$ and $I_1^+(x_0)$, respectively.

Take $\delta > 0$ such that $(x_0-\delta,\ x_0+\delta) \subset I_1$, and $(x_0-\delta,\ x_0+\delta)$ contains no zeros of the function g. Then we conclude that $W(f,g)$ has the Darboux property on the interval $(x_0-\delta,\ x_0+\delta)$.

This remark in conjunction with the properties of $W(f,g)$ established before allows us to derive that $W(f,g)$ is a Darboux function on the interval $I_1^-(x_0) \cup I_1^+(x_0) \cup (x_0-\delta,\, x_0+\delta) = I_1$.

This completes the proof. □

Let us indicate another condition ensuring that $W(f,g)$ has the Darboux property on I. Namely, it is enough to assume that $f,g\colon I \to \mathbb{R}$ are functions differentiable on the interval I and the derivative f' or g' is bounded on I. Indeed, under these assumptions the conclusion that $W(f,g) \in D$ is a consequence of the additivity of the class Δ and of the theorem of Young [9] saying that if $f \in b\Delta$ and $g \in C$, then $fg \in \Delta$.

Our next result is contained in the following theorem.

THEOREM 2. *Let f and g be functions differentiable on the interval I. Assume that at least one of these functions is of bounded variation on I. Then $W(f,g) \in \Delta$. In particular, $W(f,g)$ is a Darboux function on I.*

The proof follows immediately from the properties of the class of derivatives Δ mentioned before and from the result due to Fleissner [4] (see also [3], [5]), which has the form:

$f \in \Delta$ and g is continuous and of bounded variation on $I \implies fg \in \Delta$.

Apart from this we also use the equality

$$W(f,g) = (fg)' - 2f'g\,.$$

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