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ON INTRINSIC QUASIMETRICS PRESERVING MAPS ON NON-ABELIAN PARTIALLY ORDERED GROUPS

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ABSTRACT. In [JASEM, M.: *Intrinsic metric preserving maps on partially ordered groups*, Algebra Universalis **36** (1996), 135–140], it was proved that a stable surjective map f from an abelian directed group G_1 onto a directed group G_2 is a homomorphism if it satisfies the following condition:

(C) If $|x-y| = |z-t|$, then $|f(x)-f(y)| = |f(z)-f(t)|$ for each $x, y, z, t \in G_1$.

In this paper a stable map $f\colon G_1 \to G_2$ satisfying (C) is studied, where G_1 and G_2 are non-abelian directed groups. It is shown that a stable injective map $f\colon G_1 \to G_2$ satisfying (C) is a homomorphism in the case that G_1 is a 2-isolated directed group and G_2 is a linearly ordered group. The question whether f is a homomorphism also in the case of non-linearly ordered group G_2 remains open.

In [5], Swamy defined an *intrinsic metric* on a lattice ordered group (l-group) G as a map $d\colon G \times G \to G$ satisfying the following conditions for each $a, b, c \in G$:

(M$_1$) $d(a,b) \geq 0$ and $d(a,b) = 0$ if and only if $a = b$,
(M$_2$) $d(a,b) = d(b,a)$,
(M$_3$) $d(a,c) \leq d(a,b) + d(b,c)$,

and showed that any abelian l-group is autometrized by $d(x,y) = |x - y|$.

Holland [1] considered whether other metrics might be naturally defined on an l-group.

Rachůnek [4] generalized the notion of an intrinsic metric to any partially ordered group (po-group). He defined an intrinsic metric on a po-group G as a map $d\colon G \times G \to \exp G$ satisfying (M$_2$) and the following conditions for each $a, b, c \in G$:

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($\underline{\mathrm{M}}_1$) $d(a,b) \subseteq U(0)$ and $d(a,b) = U(0)$ if and only if $a = b$,

($\underline{\mathrm{M}}_3$) $d(a,c) \supseteq d(a,b) + d(b,c)$,

and showed that any 2-isolated abelian Riesz group is autometrized by $d(x,y) = |x-y|$.

In the case of non-abelian l-group G the map $d(x,y) = |x-y|$ need not satisfy (M_3). So, we can define an intrinsic quasimetric on an l-group G by the natural way as a map $d\colon G \times G \to G$ satisfying (M_1) and (M_2). Analogously we define an intrinsic quasimetric on a po-group G as a map $d\colon G \times G \to \exp G$ satisfying ($\underline{\mathrm{M}}_1$) and (M_2). The map $d(x,y) = |x-y|$ is an intrinsic quasimetric on a 2-isolated directed group.

In [3], P a p defined an intrinsic metrics preserving map for the case of abelian l-groups. His definition was generalized for po-groups in [2].

We use analogous definition for an intrinsic quasimetrics preserving map on po-groups.

Let G_1 and G_2 be po-groups and let d_1 and d_2 be intrinsic quasimetrics on G_1 and G_2, respectively. A map $f\colon G_1 \to G_2$ *preserves intrinsic quasimetrics* d_1 and d_2 if and only if $d_1(x,y) = d_1(z,t)$ implies $d_2(f(x), f(y)) = d_2(f(z), f(t))$ for each $x, y, z, t \in G_1$. A map $f\colon G_1 \to G_2$ is called *stable* if $f(0) = 0$.

We recall some notations and notions concerning po-groups used in the paper. Let G be a po-group. The group operation will be written additively. We denote by $U(a_1, \dots, a_n)$ the set of all upper bounds of the set $\{a_1, \dots, a_n\}$ in G. If for $a, b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a, b\}$ in G, then it will be denoted by $a \vee b$ ($a \wedge b$). For $a \in G$, $|a| = U(-a, a)$. In the case that G is an l-group, $|a| = -a \vee a$ for $a \in G$ as usual. We denote by $\exp G$ the set of all subsets of G. The set of all positive integers will be denoted by $\mathbb{N}$. A po-group G is called 2-isolated if $2a \geq 0$ implies $a \geq 0$ for each $a \in G$.

THEOREM 1. *Let G be a 2-isolated directed group, $n \in \mathbb{N}$. Let $d(a,b) = n|a-b|$ for each $a, b \in G$. Then d is an quasimetric on G.*

P r o o f. Let $a, b \in G$, $n \in \mathbb{N}$. If $x \in d(a,b)$, then $x = x_1 + \dots + x_n$, where $x_i \geq a - b$, $x_i \geq b - a$, $i = 1, \dots, n$. Thus $2x_i \geq 0$ and hence $x_i \geq 0$ for $i = 1, \dots, n$. Then $x \geq 0$. Therefore $d(a,b) \subseteq U(0)$.

If $d(a,b) = U(0)$, then $0 = z_1 + \dots + z_n$, where $z_i \geq a - b$, $z_i \geq b - a$ for $i = 1, \dots, n$. Then $2z_i \geq 0$ and hence $z_i \geq 0$ for $i = 1, \dots, n$. Then $0 = z_1 + \dots + z_n \geq z_i$ and thus $z_i = 0$ for $i = 1, \dots, n$. Then we get $0 \geq a - b$, $0 \geq b - a$. This yields $a = b$. $\square$

The following example shows that a stable map $f\colon G_1 \to G_2$ satisfying the condition (C), where G_1 and G_2 are even linearly ordered groups, need not be a homomorphism.

EXAMPLE. Let $\mathbb{Z}$ be the additive group of all integers with the natural order. Let $G_1 = G_2 = \mathbb{Z}$. For even integer $x \in G_1$ we put $f(x) = 0$, for odd integer $x \in G_1$ we put $f(x) = 1$. Then the stable map $f\colon G_1 \to G_2$ satisfies the condition (C), but f is not a homomorphism.

So it is needed to put an additional condition on a stable map satisfying (C) to be a homomorphism, for example surjectivity, injectivity.

Remark. Let G be a po-group, $a, b \in G$. If $a \geq 0$, then $|a| = |b|$ implies $a \geq b$, $a \geq -b$. If $a \geq 0$, $b \geq 0$, then $|a| = |b|$ implies $a = b$. We shall often need these assertions and we shall apply them without special references.

THEOREM 2. *Let G_1 be a 2-isolated directed group and let G_2 be a linearly ordered group. Let $f\colon G_1 \to G_2$ be a stable injective map satisfying the following condition for each $a, b, c, d \in G_1$:*

If $|a - b| = |c - d|$, then $|f(a) - f(b)| = |f(c) - f(d)|$.

Then f is a homomorphism.

Proof. First we prove that $f(-z) = -f(z)$, $f(2z) = 2f(z)$ for each $z \in G_1$. Let $z \in G_1$, $f(z) \geq 0$. Assume that $f(-z) > 0$. Since $|z - 0| = |-z - 0|$, we have $|f(z) - f(0)| = |f(-z) - f(0)|$. Then $f(z) = f(-z)$. This implies $z = -z$. Since G_1 is 2-isolated, we have $-z = 0$. Thus $f(-z) = 0$, a contradiction. Therefore $f(-z) \leq 0$. Then from $|f(z) - f(0)| = |f(-z) - f(0)|$ we obtain $f(-z) = -f(z)$. Since $|2z - z| = |z - 0|$, we have $|f(2z) - f(z)| = |f(z) - f(0)|$. This yields $f(z) \geq f(z) - f(2z)$. Therefore $f(2z) \geq 0$. From $|2z - 0| = |z - (-z)|$ we obtain $|f(2z) - f(0)| = |f(z) - f(-z)| = |2f(z)|$. Hence $f(2z) = 2f(z)$.

Let $z \in G_1$, $f(z) < 0$. Assume that $f(-z) < 0$. Since $|z - 0| = |-z - 0|$, we have $|f(z) - f(0)| = |f(-z) - f(0)|$. Thus $f(z) = f(-z)$ and hence $z = -z$. This yields $z = 0$. Then $f(-z) = 0$, a contradiction. Therefore $f(-z) \geq 0$. Then from $|f(z) - f(0)| = |f(-z) - f(0)|$ it follows that $f(-z) = -f(z)$. Since $|2z - z| = |z - 0|$, we have $|f(2z) - f(z)| = |f(z) - f(0)|$. This implies $-f(z) \geq f(2z) - f(z)$. Thus $0 \geq f(2z)$. From $|2z - 0| = |z - (-z)|$ it follows that $|f(2z) - f(0)| = |f(z) - f(-z)| = |2f(z)|$. Hence $f(2z) = 2f(z)$.

Let $x, y \in G_1$. Now we prove that $f(x + y) = f(x) + f(y)$.

a) Let $f(x) \geq 0$, $f(y) \geq 0$. Assume that $f(x + y) < 0$. From $|(x + y) - y| = |x - 0|$ we obtain $|f(x + y) - f(y)| = |f(x) - f(0)|$. Since $f(x + y) - f(y) \leq 0$, we have $f(y) - f(x + y) = f(x)$. Hence $-f(x + y) = -f(y) + f(x)$. From $|(x + y) - 0| = |x - (-y)|$ it follows that $|f(x + y) - f(0)| = |f(x) - f(-y)| = |f(x) + f(y)|$. Hence $-f(x + y) = f(x) + f(y)$. Then $-f(y) + f(x) = f(x) + f(y) \geq f(x)$. This yields $0 \geq f(y)$. Thus $f(y) = 0$ and hence $y = 0$. Then $f(x + y) = f(x) \geq 0$, a contradiction. Therefore $f(x + y) \geq 0$. From $|(x + y) - 0| = |x - (-y)|$ it follows that $|f(x + y) - f(0)| = |f(x) - f(-y)| = |f(x) + f(y)|$. This implies $f(x + y) = f(x) + f(y)$.

b) Let $f(x) \geq 0$, $f(y) < 0$. From $|(x+y)-y| = |x-0|$ we obtain $|f(x+y) - f(y)| = |f(x) - f(0)|$. Then $f(x) \geq f(x+y) - f(y)$. This implies $f(x) \geq f(x+y)$. Further we have $f(-y) = -f(y) \geq 0$. In view of a) we have $f(x-y) = f\bigl(x+(-y)\bigr) = f(x) + f(-y) = f(x) - f(y)$. Since $|x-(x+y)| = |(x-y)-x|$, we have $|f(x) - f(x+y)| = |f(x-y) - f(x)| = |f(x) - f(y) - f(x)|$. Clearly $f(x) - f(y) - f(x) \geq 0$. Hence $f(x) - f(x+y) = f(x) - f(y) - f(x)$. Therefore $f(x+y) = f(x) + f(y)$.

c) Let $f(x) < 0$, $f(y) \leq 0$. Since $f(-x) = -f(x) \geq 0$ and $f(-y) = -f(y) \geq 0$, in view of a) we have $f(-y-x) = f\bigl((-y)+(-x)\bigr) = f(-y) + f(-x) = -f(y) - f(x) \geq 0$. Then $f(x+y) = f\bigl(-(-y-x)\bigr) = -f(-y-x) = f(x) + f(y)$.

d) Let $f(x) < 0$, $f(y) > 0$. Then $f(-y) = -f(y) \leq 0$. In view of c) we get $f(x-y) = f\bigl(x+(-y)\bigr) = f(x) + f(-y) = f(x) - f(y)$. From $|(x+y)-y| = |x-0|$ it follows that $|f(x+y) - f(y)| = |f(x) - f(0)|$. Thus $-f(x) \geq f(y) - f(x+y)$. This implies $f(x+y) \geq f(x)$. Since $|x-(x+y)| = |(x-y)-x|$, we have $|f(x) - f(x+y)| = |f(x-y) - f(x)| = |f(x) - f(y) - f(x)|$. Clearly $f(x) - f(y) - f(x) \leq 0$. Then we get $f(x+y) - f(x) = f(x) + f(y) - f(x)$. Therefore $f(x+y) = f(x) + f(y)$. □

Remark. It is clear from the proof of Theorem 2 that a stable injective map $f\colon G_1 \to G_2$ (G_1, G_2 as in Theorem 2) preserving intrinsic quasimetrics $d_1(a,b) = n|a-b|$ ($n \in \mathbb{N}$) and $d_2(a,b) = |a-b|$ on G_1 and G_2 is also a homomorphism.

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