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EDGE AND VERTEX OPERATIONS ON UPPER EMBEDDABLE GRAPHS[1]

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(*Communicated by Martin Škoviera*)

ABSTRACT. A connected graph G is called upper embeddable if its maximum genus equals $\lfloor \beta(G)/2 \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the Betti number of G. In this paper, we investigate the effect of adding or deleting an edge (possibly a multi-edge or a loop) and the effect of adding a vertex (or vertices) to an upper embeddable graph. Subsequently, several new classes of upper embeddable graphs are obtained.

0. Introduction

This paper is devoted to an investigation of those graphs which are upper embeddable. Since the maximum genus is invariant under homeomorphisms, the results we obtain below obviously extend to graphs homeomorphic to these graphs. Recall that the maximum genus $\gamma_M(G)$ of a connected graph G is the largest genus of an orientable surface on which G has a 2-cell embedding. A connected graph G is called *upper embeddable* if its maximum genus equals $\lfloor \beta(G)/2 \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the *Betti number* of G. For basic information and results, we refer the reader to the book *Graphs and Digraphs* [1].

Unless explicitly stated, we shall consider graphs in which multi-edges and loops are allowed, i.e., *pseudographs*. Thus, without mentioning otherwise, "graph" stands for "pseudograph". If loops are not permitted, then the graph is a *multigraph*, and a *simple graph* is one which contains no multi-edges and no loops.

A spanning tree T of a connected graph G is a *splitting tree* of G if at most one component of $G - E(T)$ has odd size. It follows that, if $G - E(T)$ is

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connected, then T is a splitting tree. In [3], [10], Jungerman and Xuong independently gave a characterization of upper embeddable graphs.

THEOREM 1. ([3], [10]) *A graph G is upper embeddable if and only if G has a splitting tree.*

Thus, to determine whether a graph is upper embeddable, it suffices to check the existence of a splitting tree. Later, Nebeský [5] gave another characterization theorem for the upper embeddable graphs. Before stating the theorem, we need two notations. We denote the number of components in a graph H by $c(H)$. Furthermore, let $b(H)$ be the number of components C such that the Betti number of C is odd.

THEOREM 2. ([5]) *A connected graph G is upper embeddable if and only if $b(G-A)+c(G-A)-2 \leq |A|$ for every subset A of $E(G)$.*

Using either Theorem 1 or Theorem 2, many interesting families of graphs have been shown to be upper embeddable. We recall here the following two of the results:

THEOREM 3. ([9]) *Any multigraph of diameter two is upper embeddable.*

THEOREM 4. *Any 4-edge-connected graph is upper embeddable.*

In fact, Theorem 4 is a direct consequence of the following non-trivial result.

THEOREM 5. ([4]) *Every 4-edge-connected graph contains two pairwise edge-disjoint spanning trees.*

Note that the two classes of graphs mentioned in Theorems 3 and 4 have as their common property that the addition of an edge or a multi-edge results in an upper embeddable graph. This is not true in general.

In Section 1, we investigate the effect of adding or deleting an edge (or a multi-edge) and then, in Section 2, we study the operation of adding vertex (or vertices) to an upper embeddable graph; as a consequence, several new classes of upper embeddable graphs are obtained.

1. Edge operations

In [6], Nebeský gave the following definition: A simple graph G is *absolutely upper embeddable* if every simple graph which is spanned by G is upper embeddable. Thus if G is absolutely upper embeddable, then we can add any edge to G (as long as the new graph is simple) to obtain a new upper embeddable graph. He also characterized absolutely upper embeddable simple graphs.

THEOREM 1.1. ([6]) *A connected simple graph G is absolutely upper embeddable if and only if $i(G-A)+c(G-A)-2 \leq |A|$ for every subset A of $E(G)$, where $i(H)$ denotes the number of components F of H with the property that either $\beta(F)$ is odd or F is not a complete graph.*

However, if we start with a multigraph G which is upper embeddable, for example with the one in Figure 1.1, then the above theorem does not guarantee that the addition of an edge (or a multi-edge) to G will produce an upper embeddable graph.

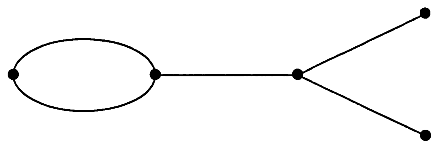


Figure 1.1.

Nevertheless, there are multigraphs which are upper embeddable and the addition of any edge (including multi-edges) will end up with a new upper embeddable graph. We will call them absolutely upper embeddable multigraphs. Thus a multigraph G is *absolutely upper embeddable* if every multigraph which is spanned by G is upper embeddable. Loopless graphs of diameter 2 and 4-edge-connected graphs are examples of absolutely upper embeddable multigraphs. The next result provides a characterization of these graphs.

THEOREM 1.2. *A connected multigraph G is absolutely upper embeddable if and only if $c(G-A)+\mathrm{nt}(G-A)-2 \leq |A|$ for every subset A of $E(G)$, where $\mathrm{nt}(H)$ denotes the number of nontrivial components of H.*

Proof.

Necessity: Assume that there exists a subset A_0 of $E(G)$ such that $c(G-A_0)+\mathrm{nt}(G-A_0)-2 > |A_0|$. Consider the graph $\tilde{G}$ obtained from G in such a way that one new edge is inserted into each nontrivial component F of $G-A_0$ whenever $\beta(F)$ is even. Clearly, $\tilde{G}$ is spanned by G and $c(\tilde{G}-A_0) = c(G-A_0)$ and $b(\tilde{G}-A_0) = \mathrm{nt}(G-A_0)$. This implies that

$$c(\tilde{G}-A_0)+b(\tilde{G}-A_0)-2 = c(G-A_0)+\mathrm{nt}(G-A_0)-2 > |A_0|.$$

By Theorem 2, $\tilde{G}$ is not upper embeddable. Thus, G is not absolutely upper embeddable.

Sufficiency: Assume that a multigraph G is not absolutely upper embeddable. Let H be a minimal spanning supergraph of G such that H is not upper embeddable. From Theorem 2, we known that $b(H-A)+c(H-A)-2 > |A|$ for some subset A of $E(G)$. Among such subsets A, let A^* be a subset of $E(H)$ with minimum number of elements such that $b(H-A^*)+c(H-A^*)-2 > |A^*|$.

Also, let $A^*_G = A^* \cap E(G)$. Since $G - A^*_G$ is a subgraph of $H - A^*$ and $V(G - A^*_G) = V(H - A^*)$, we have $c(G - A^*_G) \geq c(H - A^*)$. Now we claim that $\mathrm{nt}(G - A^*_G) \geq b(H - A^*)$, i.e., each component F of $H - A^*$ with $\beta(F)$ odd contains some edges which belong to $E(G - A^*_G)$.

Suppose the contrary. There exists a component of $H - A^*$, F' with $\beta(F')$ odd such that $V(F')$ induces an empty graph in $G - A^*_G$. Since G is connected, there exists a vertex $u \in V(F')$ which is joined to a vertex v in a component F_v with $\beta(F_v)$ odd and $uv \in A^*_G$. (If $\beta(F_v)$ is even, by letting $A' = A^* \setminus \{uv\}$, we get $|A'| < |A^*|$, $b(H - A') = b(H - A^*)$, and $c(H - A') = c(H - A^*) - 1$. This implies that $b(H - A') + c(H - A') - 2 \geq b(H - A^*) + c(H - A^*) - 1 - 2 > |A^*| - 1 \geq |A'|$, contradicting the assumption that A^* is minimal.) In fact, there are at least two vertices u and u' in F' which are adjacent to v and v' respectively, where v and v' lie in two distinct components F_v and $F_{v'}$ respectively, and $\beta(F_v)$ and $\beta(F_{v'})$ are both odd. Suppose the contrary. Since G is connected and $|V(F')| \geq 2$ (for $\beta(F')$ is odd and F' contains no loops), there exists an edge $e \in A^*_G$ which joins two vertices of F'. Also, $\beta(F')$ is odd and $E(F') \subseteq E(H) - E(G)$, therefore there exists an edge e' of $E(H) - E(G)$ which lies on a cycle of F'. Thus, by letting $H_0 = H - e'$, $A_0 = A^* - \{e\}$, we get $c(H_0 - A_0) = c(H - A^*)$, $b(H_0 - A_0) = b(H - A^*)$ and $|A_0| < |A^*|$. This implies that $c(H_0 - A_0) + b(H_0 - A_0) - 2 > |A_0|$, contradicting the minimality of $|E(H)|$. Now, let $H' = H - E(F')$, and let $A'' = A^* - \{e \mid e$ is an edge joining a vertex of F' and a vertex in a component F_u of $H - A^*$ with $\beta(F_u)$ odd$\}$. Then $c(H' - A'') \geq c(H - A^*) - 1$ and $b(H' - A'') = b(H - A^*) - 1$. But $|A''| \leq |A^*| - 2$, and this implies that

$$b(H' - A'') + c(H' - A'') - 2 \geq b(H - A^*) + c(H - A^*) - 2 - 2 > |A^*| - 2 \geq |A''|.$$

Since $|H'| < |H|$, this contradicts the minimality of H. Thus we have shown that $\mathrm{nt}(G - A^*_G) \geq b(H - A^*)$, and hence

$$c(G - A^*_G) + \mathrm{nt}(G - A^*_G) - 2 \geq c(H - A^*) + b(H - A^*) - 2 > |A^*| \geq |A^*_G|,$$

concluding the proof of sufficiency. □

As pointed out by one of the referees, Theorem 1.2 can also be derived from Corollary 2 in [7]. First a multigraph G in question is replaced by a multigraph $\hat{G}$ obtained from G in such a way that exactly one new edge joining u and v is added between every pair of distinct vertices u and v. By Theorem 1, G is an absolutely upper embeddable multigraph if and only if every multigraph spanning $\hat{G}$ and simultaneously spanned by G is upper embeddable. Now, Corollary 2 of [7] can be applied to derive Theorem 1.2.

We also remark that Theorem 1.2 can be used to give a new proof of Theorem 1 due to Škoviera, see Fu and Tsai [2]. While the original proof in [9] was based on Theorem 1, the one given in [2] employs, in fact, a variation of Theorem 2.

A connected graph is said to be *minimally non-upper embeddable* if it is not upper embeddable, but the deletion of any edge, not a bridge, yields an upper embeddable graph. The graph in Figure 1.2 is an example of a minimally non-upper embeddable graph.

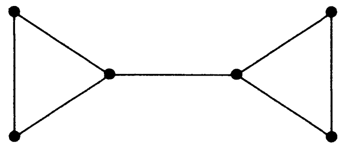


Figure 1.2.

Our next aim is to characterize minimally non-upper-embeddable graphs. First we show that the edge-connectivity of a minimally non-upper embeddable graph must be less than two.

PROPOSITION 1.3. *In every non-upper embeddable 2-edge-connected graph G, there exists an edge e such that $G - e$ is non-upper embeddable.*

Proof. Assume that G is 2-edge-connected and non-upper embeddable. By Theorem 2, there exists a subset A of $E(G)$ such that $b(G - A) + c(G - A) - 2 > |A|$. Let $e \in A$. Since $G - e$ is connected, $b\big(G - e - (A - \{e\})\big) + c\big(G - e - (A - \{e\})\big) - 2 = b(G - A) + c(G - A) - 2 > |A| > |A - \{e\}|$. Using Theorem 2 again we see that $G - e$ is not upper embeddable. □

Thus a minimally non-upper embeddable graph contains a bridge. We also need the following lemma.

LEMMA 1.4. *If G is minimally non-upper embeddable, then $\beta(G)$ is even.*

Proof. Assume that $\beta(G)$ is odd. Clearly, G is not a tree. Since G is minimally non-upper embeddable, there exists an edge $e \in E(G)$ (which is not a bridge) such that $G - e$ is upper embeddable. By Theorem 1, $G - e$ has a spanning tree T such that $G - e - E(T)$ contains at most one odd-size component. Since $\beta(G-e)$ is even, $G-e-E(T)$ contains no odd size component. This implies that T is a splitting tree for G, and G is upper embeddable, a contradiction. □

THEOREM 1.5. *A connected graph G is minimally non-upper embeddable if and only if*

(1) *G contains a bridge e such that $G - e = G_1 \cup G_2$, $\beta(G_1)$ and $\beta(G_2)$ are odd; and*
(2) *$b(G - A) + c(G - A) - 1 \le |A|$ for every subset A of $E(G)$ except A is a bridge or an empty set.*

P r o o f .

Necessity: By way of contradiction, let H be a minimally non-upper embeddable graph that does not satisfy (1) and has minimum order. Then, by Lemma 4.1, for every bridge e of H, both components H_1 and H_2 of $H - e$ have even Betti number. Suppose that $e_1, e_2, \dots, e_n$ are the bridges of H, and let $H - e_i = H_1^{(i)} \cup H_2^{(i)}$, $i = 1, 2, \dots, n$. Since $\beta\big(H_1^{(i)}\big)$ and $\beta\big(H_2^{(i)}\big)$ are even and G is non-upper embeddable, either $H_1^{(i)}$ or $H_2^{(i)}$ is non-upper embeddable. We may assume that $H_1^{(i)}$ is non-upper embeddable. Moreover, since G is minimally non-upper embeddable, $H_1^{(i)}$ must be minimally non-upper embeddable. By Proposition 1.3, $H_1^{(i)}$ must contains bridge(s). Due to the fact that $\big|V\big(H_1^{(i)}\big)\big| < |V(H)|$, there exists a bridge e_0 of $H_1^{(i)}$ such that $H_1^{(i)} - e_0 = H_a \cup H_b$, and $\beta(H_a)$ or $\beta(H_b)$ is odd. By Lemma 1.4, we conclude that both $\beta(H_a)$ and $\beta(H_b)$ are odd. Obviously, e_0 is also a bridge of H, and $H - e_0 = H_a \cup \big(H_b \cup \{e_i\} \cup H_2^{(i)}\big)$ or $H - e_0 = H_b \cup \big(H_a \cup \{e_i\} \cup H_2^{(i)}\big)$. Thus H_a or H_b is a component of H with odd Betti number. This contradicts the assumption, which proves (1).

Next, we claim that (2) holds. Suppose the contrary. Then we can find a subset A_0 of $E(G)$, where A_0 is not a bridge or an empty set, such that $b(G - A_0) + c(G - A_0) - 1 > |A_0|$. Let $e_0 \in A_0$ and e_0 be not a bridge of G. Clearly,

$$b\big(G - e_0 - (A_0 - \{e_0\})\big) + c\big(G - e_0 - (A_0 - \{e_0\})\big) - 1 > |A_0| = |A_0 - \{e_0\}| + 1 .$$

This implies that

$$b\big(G - e_0 - (A_0 - \{e_0\})\big) + c\big(G - e_0 - (A_0 - \{e_0\})\big) - 2 > |A_0 - \{e_0\}| .$$

By Theorem 2, $G - e_0$ is not upper embeddable. This contradicts the assumption that G is minimally non-upper embeddable, and we have (2).

Sufficiency: Obviously, (1) implies that G is non-upper embeddable. Now, if there exists an edge $e_0 \in E(G)$ which is not a bridge such that $G - e_0$ is not upper embeddable, then, from Theorem 2, we get that there exists a nonempty subset A_1 of $E(G - e_0)$ such that $b(G - e_0 - A_1) + c(G - e_0 - A_1) - 2 > |A_1|$. Thus $b\big(G - (A_1 \cup \{e_0\})\big) + c\big(G - (A_1 \cup \{e_0\})\big) - 1 > |A_1 \cup \{e_0\}|$, where $A_1 \cup \{e_0\}$ is not a bridge or an empty set. This contradicts (2). □

Now we shall study the effect of deleting an edge from an upper embeddable graph. We are mainly interested in those graphs which are still upper embeddable after deleting any edge (as long as the graph is still connected). For example, the graph in Figure 1.3 is upper embeddable and the deletion of any edge (not a bridge) will not affect the upper embeddability.

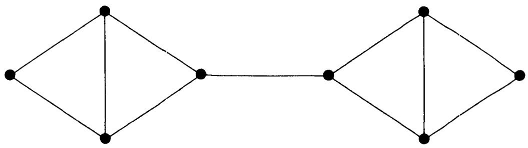


Figure 1.3.

An upper embeddable graph G is said to be *strongly upper embeddable* if for any edge e (not a bridge) in G, $G-e$ is upper embeddable. For example, the graph in Figure 1.3 is a strongly upper embeddable graph. We note that, in an upper embeddable graph G which is not a tree, we can always find an edge e such that $G-e$ is also upper embeddable. Since the proof is easy, we simply state the result.

PROPOSITION 1.6. *Let G be an upper embeddable graph which is not a tree. Then there exists an edge $e \in E(G)$ such that $G-e$ is also upper embeddable.*

The following two theorems characterize strongly upper embeddable graphs.

THEOREM 1.7. *A 2-edge-connected graph G is strongly upper embeddable if and only if $b(G-A)+c(G-A)-1 \leq |A|$ for each nonempty subset A of $E(G)$.*

Proof. Let G be a 2-edge-connected strongly upper embeddable graph. Assume that there exists a nonempty subset A_0 of $E(G)$ such that $b(G-A_0)+c(G-A_0)-1 > |A_0|$. Let e be an edge in A_0. By direct checking, we deduce that $G-e$ is not upper embeddable. Hence G is not strongly upper embeddable, which is a contradiction.

Conversely, let G be a 2-edge-connected and $b(G-A)+c(G-A)-1 \leq |A|$ for each nonempty subset A of $E(G)$. Assume that G is not strongly upper embeddable. By Theorem 2, G is upper embeddable. Since G is not strongly upper embeddable, there exists an edge $e \in E(G)$ such that $G-e$ is not upper embeddable. This implies that there exists $A_1 \subseteq E(G-e)$ such that $b\big((G-e)-A_1\big)+c\big((G-e)-A_1\big)-2 > |A_1|$. Then $b\big(G-\big(A_1 \cup \{e\}\big)\big)+c\big(G-\big(A_1 \cup \{e\}\big)\big)-1 > |A_1 \cup \{e\}|$ for the nonempty subset $A_1 \cup \{e\}$ of $E(G)$, which is a contradiction. □

THEOREM 1.8. *A connected graph G which contains bridges is strongly upper embeddable if and only if*

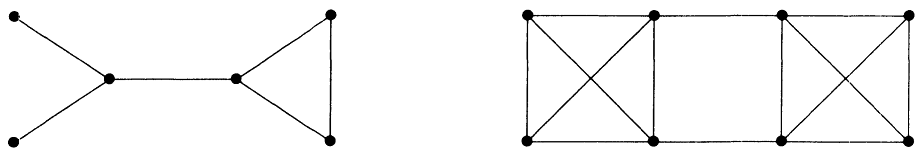
(1) *for any bridge e of G, if $G-e = G_1 \cup G_2$, then $\beta(G_1)\beta(G_2)$ is even; and*

(2) *$b(G-A)+c(G-A)-1 \leq |A|$ for every subset A of $E(G)$ except A consists of bridges or is an empty set.*

P r o o f . Assume that G is strongly upper embeddable and that G contains bridges. Then (1) is obvious. Now, if $b(G - A_0) + c(G - A_0) - 1 > |A_0|$ for some nonempty subset A_0 of $E(G)$ where A_0 does not only contain bridges, then by letting e be an edge in A_0 which is not a bridge we obtain that $G - e$ is not upper embeddable. Hence G is not strongly upper embeddable.

Conversely, let (1) and (2) hold. Then Theorem 2 readily implies that G is upper embeddable. Now, if G is not strongly upper embeddable, then there exists an edge $e \in E(G)$ (not a bridge) such that $G - e$ is not upper embeddable. This implies that there exists an subset $A_1 \subseteq E(G-e)$ such that $b\big((G-e)-A_1\big) + c\big((G-e)-A_1\big) - 2 > |A_1|$. Then $b\big(G - (A_1 \cup \{e\})\big) + c\big(G - (A_1 \cup \{e\})\big) - 1 > |A_1 \cup \{e\}|$ for the nonempty subset $A_1 \cup \{e\}$ of $E(G)$. This is a contradiction. □

We remark that there are strongly upper embeddable graphs which are not absolutely upper embeddable and vice versa. In Figure 1.4, (a) is strongly upper embeddable but not absolutely upper embeddable, and (b) is absolutely upper embeddable but not strongly upper embeddable.



(a) (b)

Figure 1.4.

2. Vertex operations

First we deal with the addition of only one vertex. For convenience, we will let the resulting graph be G and the original graph be $G - u$.

For a spanning tree T of a graph G, let $\xi(G,T)$ denote the number of components of $G - E(T)$ which have odd size. Then it is well known that $\xi(G)$, the *Betti deficiency*, is equal to $\min\{\xi(G,T) \mid T$ is a spanning tree of $G\}$, see [10]. Note that G is upper embeddable if and only if $\xi(G) \le 1$.

LEMMA 2.1. *Let G be a connected graph. If there exists a vertex of odd degree u such that $G - u$ is connected, then $\xi(G) \le \xi(G-u)$.*

P r o o f . Let T_0 be a spanning tree of $G - u$ such that $\xi(G - u, T_0) = \xi(G - u)$. Now let v be an arbitrary vertex in $V(G-u)$ such that $uv \in E(G)$. It is clear that $T_0 \cup \{uv\} = T$ is a spanning tree of G and $\xi(G,T) \le \xi(G-u, T_0)$ since $\deg_G(u)$ is odd. Hence, $\xi(G) \le \xi(G-u)$. □

For another proof, see Nedela and Škoviera [8].

With the above lemma, it is easy to see that, if we start with an upper embeddable graph, then we can always add an odd vertex to it to obtain a new upper embeddable graph. This is not true in general for an even vertex; see Figure 2.1, for example.

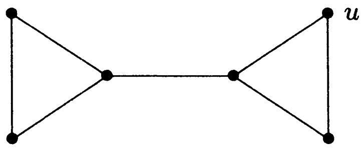


Figure 2.1.

For the addition of an even vertex we need one more condition.

LEMMA 2.2. *Let G be a connected multigraph with an even vertex u such that $G-u$ is connected. Then G is upper embeddable whenever $G-u$ is an absolutely upper embeddable multigraph and no multi-edge is incident with u.*

Proof. Let G' be the graph obtained by deleting $\deg_G u-2$ edges in $E(G)$ which are incident to u. Then G' is homeomorphic to a supergraph of $G-u$, and hence, G' is upper embeddable due to the absolute upper embeddability of $G-u$. Futhermore, $\xi(G) \le \xi(G') \le 1$, whence G is upper embeddable. $\square$

Here, no multi-edge is allowed between the vertex u and the vertex in $G-u$, otherwise we may obtain a graph which is not upper embeddable; see Figure 2.2, for example.

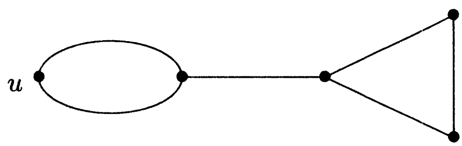


Figure 2.2.

By a similar idea, we can add more than just one even vertex as long as these vertices are not adjacent. Hence the following corollary is obtained.

COROLLARY 2.3. *Let G be a connected multigraph with an independent set S such that $G-S$ is an absolutely upper embeddable multigraph and no multi-edge is incident with a vertex in S. Then G is upper embeddable.*

If we want to add several even vertices to an upper embeddable graph in order to obtain a new upper embeddable graph, we can add two even vertices at one time.

LEMMA 2.4. *Let G be a connected multigraph. If there exist two nonadjacent even degree vertices u and v such that $N(u) \cap N(v) \neq \emptyset$ and $G - u - v$ is connected, then $\xi(G) \leq \xi(G - u - v)$.*

Proof. Let T_0 be a spanning tree of $G - u - v$ such that $\xi(G{-}u{-}v, T_0) = \xi(G{-}u{-}v)$, and $w \in N(u) \cap N(v)$. Since $\deg_G u$ and $\deg_G v$ are even, there exist u' and v' such that $u' \neq w$, $v' \neq w$ and uu', vv' are edges of G. By letting $T = T_0 \cup \{uu', vv'\}$, we obtain a spanning tree T of G, and $\xi(G) \leq \xi(G,T) \leq \xi(G{-}u{-}v, T_0) = \xi(G{-}u{-}v)$ can be checked directly. □

COROLLARY 2.5. *A connected graph G is upper embeddable if there exist two nonadjacent even degree vertices u and v such that $N(u) \cap N(v) \neq \emptyset$, $G - u - v$ is connected and $G - u - v$ is upper embeddable.*

The above lemmas enable us to construct an infinite class of upper embeddable multigraphs starting from a given one by adding either a vertex of odd degree or two nonadjacent even degree vertices which have at least one common neighbour. If we start with an absolutely upper embeddable multigraph, then an independent set of vertices can be added with no multi-edge incident with any vertex of the set. Of course, this can be done with the complete graphs; the resulting class of upper embeddable graphs is that of split graphs. Recall that a simple graph G is a *split graph* if its vertex set can be partitioned into two subsets V_1 and V_2 such that $\langle V_1 \rangle_G$ is a clique and V_2 is independent in G.

PROPOSITION 2.6. *A split graph is upper embeddable.*

Proof. The result follows immediately from Corollary 2.3 and the fact that the complete graph is absolutely upper embeddable. □

PROPOSITION 2.7. *A split graph is strongly upper embeddable.*

Proof. Let G be a split graph, and let $V(G) = V_1 \cup V_2$, where $\langle V_1 \rangle_G$ is complete, and V_2 is independent in G. Let $e = uv \in E(G)$, not a bridge. Clearly, $u \in V_1$ or $v \in V_1$, without loss of generality, let $v \in V_1$. We consider the following two cases.

(i) $u \in V_1$. Then $G - e - V_2$ has diameter 2. Recall that a multigraph of diameter 2 is absolutely upper embeddable, see Škoviera [9] and the remarks preceding Theorem 1.2. Hence $G - e$ is upper embeddable.

(ii) $u \in V_2$. Then $G - e$ is also a split graph, which is upper embeddable by Proposition 2.6. □

Since 4-edge-connected graphs and graphs of diameter 2 are all absolutely upper embeddable multigraphs, we have the following:

PROPOSITION 2.8. *Let G be a multigraph, and let $S \subseteq V(G)$ be an independent set. If $G - S$ is 4-edge-connected or has diameter 2, and no multi-edge is incident with S, then G is upper embeddable.*

Finally, we remark that some more new classes of graphs can be obtained by using the ideas mentioned in this section. The results will appear elsewhere.

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