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ON FACE–VECTORS AND VERTEX–VECTORS OF POLYHEDRAL MAPS ON ORIENTABLE 2–MANIFOLDS

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ABSTRACT. Let $p_k(M)$ and $v_k(M)$ denote the number of k-gonal faces and k-valent vertices, respectively, of a polyhedral map M on closed connected orientable 2-manifold T_g of genus g, $g \geq 0$. A pair of sequences $\big(p_k(M) \mid k \geq 3\big)$ and $\big(v_k(M) \mid k \geq 3\big)$ associated in a natural way with M is called the face-vector and the vertex-vector of M, respectively. Let $p = (p_k \mid 3 \leq k \neq 6)$ and $v = (v_k \mid k \geq 4)$ be a pair of sequences of non-negative integers satisfying a necessary combinatorial condition $\sum\limits_{k\geq 3}(6-k)p_k + 2\sum\limits_{k\geq 3}(3-k)v_k = 12(1-g)$. Denote by $P_6(p,v,g)$ the set of all non-negative integers p_6 such that if p_6 is added to p and $v_3 = \frac{1}{3}\Big(\sum\limits_{k\geq 3} kp_k - \sum\limits_{k\geq 4} kv_k\Big)$ is added to v, the face-vector and the vertex-vector of a polyhedral map M on T_g for given integer g, $g \geq 0$, is obtained. In the present paper we determine, for each triple (p,v,g) up to two ones, the set $P_6(p,v,g)$ except for a finite number of its members.

1. Introduction and main results

Let T_g be a closed connected orientable 2-manifold of genus g. A map M is said to be a *polyhedral map* on T_g provided that M is a 2-dimensional topological cell-complex decomposing T_g or, equivalently, M is a cellular embedding of a graph G on T_g having properties analogous to the ones of convex polyhedra. (That is each face of M is a 2-cell and no two faces have a multiply connected union. See [2], [8], [21], [22].)

2-cells, 1-cells and 0-cells of M are called *faces*, *edges* and *vertices*, respectively. A face (vertex) is *i-gonal* (*i-valent*) if it is incident with i edges. By $p_i(M)$ or $v_i(M)$ we denote the cardinality of the set of i-gonal faces or i-valent vertices, respectively. Clearly $p_1(M) = p_2(M) = v_1(M) = v_2(M) = 0$.

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Much effort has been devoted to study of vectors $\big(p_i(M) \mid i \geq 3\big)$ and $\big(v_i(M) \mid i \geq 3\big)$ associated in a natural way with a map M on T_g and called the *face-vector* and the *vertex-vector of* M, respectively. For a survey see e. g. [9], [11], [21], [22], [24].

The problem of determining which pair of sequences of non-negative integers (A) can appear as the face-vector

$$(p_i \mid i \geq 3) \quad \text{and} \quad (v_i \mid i \geq 3) \tag{A}$$

and the vertex-vector of a polyhedral map M on T_g for a given non-negative integer g seems to be difficult. On the one hand the famous Euler formula $p(M)+e(M)-v(M)=2(1-g)$ (where $p(M)$ or $e(M)$ or $v(M)$ is the number of faces or edges or vertices of M on T_g, respectively) as applied to the elements of (A) provides the following necessary condition

$$\sum_{i\geq 3}(6-i)p_i + 2\sum_{i\geq 3}(3-i)v_i = 12(1-g)\,. \tag{1}$$

An interesting property of (1) is that it gives no information about the values p_6 and v_3. However, the next evident necessary condition

$$\sum_{i\geq 3} i v_i = \sum_{i\geq 3} i p_i \equiv 0 \pmod 2 \tag{2}$$

yields a relationship among v_3, p_6 and other elements of the sequences (A).

On the other hand there are the pairs of sequences (A) which satisfy the conditions (1) and (2) for some g and which are not the face-vectors and the vertex-vectors of maps on T_g. See e. g. [1], [8], [9], [12], [15], [18], [19], [20], [22].

The equality (2) allows the following reformulation of the problem:

Consider a pair of sequences of non-negative integers

$$p = (\,p_i \mid 3 \leq i \neq 6)\,, \qquad v = (v_i \mid i \geq 4) \tag{B}$$

and a non-negative integer g satisfying (1). The triple (p,v,g) determines the set $P_6(p,v,g)$ of all non-negative integers such that the sequence p with any element of $P_6(p,v,g)$ added as p_6 and the sequence v supplemented by $v_3 = \frac{1}{3}\Big(\sum_{i\geq 3} i p_i - \sum_{i\geq 4} i v_i\Big)$ is the face-vector and the vertex-vector of a polyhedral map M on T_g, respectively. The problem consists in characterizing the set $P_6(p,v,g)$ for all triples (p,v,g).

Eberhard [3] was the first to consider questions of the above type. He proved that the set $P_6(p, v^*, 0)$ is non-empty for any p. (Here and in the sequel $v^* = (v_i \mid v_i = 0$ for all $i \geq 4)$.) Eberhard's result served as a starting point for many different investigations mainly thanks to Grünbaum who renewed the interest in Eberhard's Theorem, gave a clear proof [8] and some ramifications and analogues of it, see [9], [10], [11], [13], [14]. Some interesting properties of the set $P_6(p, v^*, 0)$ were found by Grünbaum and Motzkin [12], Fisher [6], Kraeft [23] and Jendroľ [15]. Jendroľ and Jucovič in [18], [19], generalized Eberhard's result by determining all the triples (p, v, g) for the set $P_6(p, v, g)$ to be non-empty.

The next result is a generalization of some mentioned above.

THEOREM 1. ([17]) *Let* $p = (p_i \mid 3 \leq i \neq 6)$ *and* $v = (v_i \mid i \geq 4)$ *be a pair of sequences of non-negative integers satisfying* (1) *and* (2).

(i) *If* $\sum_{k \geq 3} p_k = 0$ *for* $k \equiv 1 \pmod 2$ *and* $\sum_{k \geq 3} v_k = 1$ *for* $k \not\equiv 0 \pmod 3$, *then the set* $P_6(p, v, 0)$ *is empty.*

(ii) *If the condition of* (i) *are not satisfied,*

$$\sum_{k \geq 3} (p_k + v_k) \leq 2 \quad \text{for} \quad k \not\equiv 0 \pmod 3 \quad \text{and}$$

$$\sum_{3 \leq m \neq 6} p_m + \sum_{n \geq 4} v_n \equiv 0 \pmod 2,$$

then there exists a constant $d = d(p, v)$ *depending on the elements of* p *and* v *such that* $P_6(p, v, 0)$ *contains all even integers* $\geq d$ *and no odd integers.*

(iii) *If the conditions of* (i) *are not satisfied,*

$$\sum_{k \geq 3} (p_k + v_k) \leq 2 \quad \text{for} \quad k \not\equiv 0 \pmod 3 \quad \text{and}$$

$$\sum_{3 \leq m \neq 6} p_m + \sum_{n \geq 4} v_n \equiv 1 \pmod 2,$$

then there exists a constant $d = d(p, v)$ *depending on the elements of* p *and* v *such that* $P_6(p, v, 0)$ *contains all odd integers* $\geq d$ *and no even integers.*

(iv) *If the conditions of* (i) *are not satisfied and*

$$\sum_{k \geq 3} (p_k + v_k) \geq 3 \quad \text{for} \quad k \not\equiv 0 \pmod 3,$$

then there exists a constant $d = d(p, v)$ *depending on the elements of* p *and* v *such that the set* $P_6(p, v, 0)$ *contains all integers* $\geq d$.

The main results of the paper generalize and extend previous results of [3], [6], [12], [15], [16], [17], [18], [19], [23]. We show that the phenomena like those of (i), (ii) and (iii) of Theorem 1 for $g = 0$ does not occur for $g \geq 2$. We have

THEOREM 2. *For any triple (p, v, g) satisfying (1) and (2) with $g \geq 2$ there is a constant d depending on the triple (p, v, g) such that the set $P_6(p, v, g)$ contains all integers $\geq d$.*

For $g = 1$ i.e. for toroidal polyhedral maps the situation is as follows:

THEOREM 3. *Suppose the triple $(p, v, 1)$ satisfies the conditions (1) and (2).*

(i) *If $\sum_{3 \leq k \neq 6} p_k \neq 2$ or $\sum_{k \geq 4} v_k \neq 0$, then there exists a constant d depending on the triple $(p, v, 1)$ such that $P_6(p, v, 1)$ contains all integers $\geq d$.*

(ii) *If $p_5 = p_7 = 1$, $p_k = 0$ for $k \neq 5, 7$ and $v_k = 0$ for all $k \geq 4$, then the set $P(p, v, 1)$ is empty.*

(iii) *If $p_4 = p_8 = 1$, $p_i = 0$ for $i \neq 4, 8$ or $p_3 = p_9 = 1$, $p_i = 0$ for $i \neq 3, 9$, and $v_i = 0$ for all $i \geq 4$, then there is a constant d depending on the triple $(p, v, 1)$ that $P_6(p, v, 1)$ contains every even number $\geq d$.*

The rest of the paper is organized as follows:

In Section 2 we give the necessary definition and the elementary constructions. In Section 3 there are formulated some existence lemmas. In Sections 4 and 5 we bring the proofs of our results. Section 6 contains some discussion on some relatives of our results and of a few open problems.

2. Basic construction elements

Basic face construction elements (see [15], [16]):

The face-aggregate of a map M as in Fig. 1a or 2a or 3a (or their mirror images) called an A_m configuration, or a B_m configuration or a C_m configuration consists of an x-valent vertex, $x \geq 3$ (denoted by small black circles in the said Figures) trivalent vertices and an m-gon, $m \geq 6$, two hexagons and one quadrangle, or of an m-gon, $m \geq 6$, two hexagons and two quadrangles, or of an m-gon, $m \geq 6$, two hexagons and three quadrangles, respectively; the m-gon mentioned will be called a basic face of the configuration. (We note that in the sequel $g, h, i, j, k, l, m, n, t, x$, mean non-negative integers. We shall denote in the figures the size of every non-hexagonal face excluding faces of the X configurations, $X \in \{A_m, B_m, C_m, D, E, F, G, U_m, V_m, W_m\}$, bounded by heavy lines, hexagons are to be denoted only in more important cases. Non-trivalent vertices will be denoted by small black circles.)

Basic face construction steps: A basic construction step transforms a starting map M into a map M'; it uses the presence of the X_m configuration, $X \in \{A, B, C\}$ in M (see Figs. 1, 2, 3). For the map M' we have $p_4(M') = p_4(M) + 1$, $p_{m+2}(M') = p_{m+2}(M) + 1$, $p_j(M') = p_j(M)$, $j \neq 4, 6, m, m+2$ and $p_6(M') = p_6(M) + z$ ($z = 2, 3$ or 7 for $X = A, B$ or C, respectively), $p_m(M') = p_m(M) - 1$ (if $m \neq 6$) or $p_6(M') = p_6(M) + z - 1$ (if $m = 6$), $v_m(M') = v_m(M)$ for $m \neq 3$, $v_3(M') = \frac{1}{3}\Big(\sum\limits_{k \geq 3} kp_k - \sum\limits_{k \geq 4} kv_k\Big)$. For continuing the construction it is important that transforming an A_m configuration (a B_m or a C_m configuration) we get a B_{m+2} configuration (a C_{m+2} or an A_{m+2} configuration) and a B_6 configuration (a C_6 or an A_6 configuration, respectively) (differing only in their basic faces). If an $(m+2)$-gon is needed, we use the basic construction step to the X_6 configuration; if not, use the X_{m+2} configuration producing an $(m+4)$-gon. Note that the transformation of a C_m configuration yields a new C_6 configuration face-disjoint from A_{m+2} and A_6 configurations (see Fig. 3b); this C_6 configuration is not used in basic construction steps.

Basic vertex construction elements ([17]) are the face-aggregates in Fig. 4a or 5a or 6a (or their mirror images) called a U_m configuration or V_m configuration or W_m configuration, respectively. The U_m configuration, V_m configuration or W_m configuration consist of an m-valent vertex, $m \geq 3$ (denoted in Figures by small black circles), at most one other non-trivalent vertex and a quadrangle and a hexagon or two adjacent quadrangles or a triple of quadrangles and a hexagon, respectively.

The basic vertex construction step transforms a map M into a map M'; it uses the presence of the Y_m configuration $Y \in \{U, V, W\}$ in M and changes it as in Figures 4b, 5b and 6b. In the map M' we have $p_4(M') = p_4(M) + 3$, $p_6(M') = p_6(M) + z$ ($z = 10, 9$ or 11 for $Y = U, V$ or W, respectively). $p_i(M') = p_i(M)$ for all $i \geq 3$, $i \neq 4, 6$; $v_i(M') = v_i(M)$ for $i \geq 3$, $i \neq 3, m, m+3$; $v_m(M') = v_m(M) - 1$, $v_{m+3}(M') = v_{m+3}(M) + 1$, $v_3(M') = v_3(M) + t$ ($t = 21$, 21 or 25 for $Y = U, V$ or W respectively). For continuing the construction it is important that transforming a Y_m configuration we get a Y_{m+3} configuration and a C_6 configuration. If a $(m+3)$-valent vertex is needed, for continuing the construction the X_6 configuration is used, where $X = A, B$ or C if $Y = U, V$ or W respectively. (Note that the Y_{m+3} configuration is a part of the X_6 configuration (see Figs. 4b, 5b, 6b).) If not, we continue in the construction by using the Y_{m+3} configuration.

Let $M = M(q, w, g, a, b, c)$ be a polyhedral map on an orientable surface Γ_g of genus g with the following properties:

(i) The sequences $q = (q_i \mid i \geq 3)$ and $w = (w_i \mid i \geq 3)$ are the face-vector and the vertex-vector of M respectively.

(ii) M contains as submaps at least a A_6 configurations, $a \geq 0$, b B_6 configurations, $b \geq 0$, and c C_6 configurations, $c \geq 0$, such that all configurations mentioned are pairwise face-disjoint.

Auxiliary construction elements: The configurations shown in Fig. 7 will play an important role together with the basic construction elements. The configuration shown in Fig. 7a will be designated as a D configuration (and its mirror image as a D' configuration). Figs. 7b, 7c and 7e show configurations which will henceforth be designated as E, F and G configurations, respectively. All the vertices of the configuration E and F are trivalent. All vertices of the configurations G and D but one are trivalent.

3. Existence lemmas

In this chapter some lemmas are stated which will be useful to the proofs of an existence of polyhedral maps on the orientable surface of genus g for any $g \geq 0$.

Agreements:

1. An assumption in some lemmas in the sequel that an X configuration, $X \in \{D, E, F, G\}$, is in the map $M = M(q, w, g, a, b, c)$ will also mean that the X configuration is face-disjoint with any of a A_6 configurations, b B_6 configurations and c C_6 configurations of the map.

2. As a simplification we will not write down the value w_3 in the records of vertex-vectors of maps in lemmas below. As shown by (2) the value w_3 is uniquely determined by the other members of the vertex-vector and all the members of the face-vectors of the map.

LEMMA 1.α ($\alpha \in \{1, 2, \ldots, 9\}$). (cf. [17]) *Let $u = (u_1 \mid i \geq 4)$ be a sequence of non-negative integers with a finite number of non-zero elements with $\sum u_k \equiv 0 \pmod 2$ for $4 \leq k \not\equiv 0 \pmod 3$ and let*

$$j = 3 + \sum_{i \geq 4} (i - 3) u_i .$$

If there is a map $M = M(q, w, g, a, b, c)$ with $a + b + c \neq 0$, then there is a map $M' = M(q', w', g, a', b', c')$ with $q' = (q'_i \mid q'_i = q_i$ for all $i \geq 3$, $i \neq 4, 6$, $q'_4 = q_4 + r_4$, $q'_6 = q_6 + r_6)$ and $w' = (w'_i \mid w'_i = w_i + u_i$ for all $i \geq 4$, $w'_3)$. For the values α, r_4, a', b', c' see Table 1.α $\alpha \in \{1, 2, 3\}$ if $a \neq 0$, $\alpha \in \{4, 5, 6\}$ if $b \neq 0$ and $\alpha \in \{7, 8, 9\}$ if $c \neq 0$. The value r_6 is a constant depending on the sequence u.

Table 1.

α	j	r_4	a'	b'	c'
1	$3k$	$3k$	a	b	$c+k$
2	$3k+1$	$3k+1$	$a-1$	$b+1$	$c+k$
3	$3k+2$	$3k+2$	$a-1$	b	$c+k+1$
4	$3k$	$3k$	a	b	$c+k$
5	$3k+1$	$3k+1$	a	$b-1$	$c+k+1$
6	$3k+2$	$3k+2$	$a+1$	$b-1$	$c+k+1$
7	$3k$	$3k$	a	b	$c+k$
8	$3k+1$	$3k+1$	$a+1$	b	$c+k$
9	$3k+2$	$3k+2$	a	$b+1$	$c+k$

LEMMA 2.α ($\alpha \in \{1,2,\dots,27\}$). (cf. [15, p. 172, Lemma 3.α]) *Let $f = (f_i \mid i \ge 7)$ be a sequence of non-negative integers with a finite number of non-zero elements and let*

$$\ell = 6 + \sum_{i \ge 7} (i-6) f_i .$$

If there is a map $M = M(q,w,g,a,b,c)$ with $a+b+c \ne 0$, then there is a map $M' = M'(q',w',g,a',b',c')$ with $q' = (q'_i \mid q'_r = q_r + s_r$ for $3 \le r \le 6$, $q'_i = q_i + f_i$ for all $i \ge 7)$ and $w' = (w'_i \mid w'_i = w_i$ for all $i \ge 4$, $w'_3)$; for the values $s_3, s_4, s_5, a', b', c'$ see Table 2.α $\alpha \in \{1,2,\dots,9\}$ if $a \ne 0$; $\alpha \in \{10,\dots,18\}$ if $b \ne 0$, $\alpha \in \{19,\dots,27\}$ if $c \ne 0$. The value s_6 is a constant depending on the sequence f.

LEMMA 3. (cf. [16]) *If there is a map $M = M(q,v,g,a,b,c)$ with $c \ge 2$, then there is a map $M' = M(q',v',g+1,a,b,c-2)$ such that $q' = (q'_i \mid q'_i = q_i$ for all $i \ne 4$, $q'_4 = q_4 - 6)$ and $v' = (v'_i \mid v'_i = v_i$ for all $i \ge 4$, $v'_3 = v_3 - 8)$.*

LEMMA 4. ([15, p. 174]) *Let $M = M(q,w,g,a,b,c)$ be a map and let f_3, f_4, f_5 be non-negative integers satisfying following conditions*

(i) $3f_3 + 2f_4 + f_5 = 3q_3 + 2q_4 + q_5$;
(ii) $f_3 \ge q_3$, $q_5 \le f_5 \le q_5 + 1$;
(iii) $f_3 \le 2c + q_3$ *or* $f_3 = 2c + q_3 + 1$ *and* $b \ne 0$.

Then there is a map $M' = M(q',w',g,a',b',c')$ with

$$q' = (q'_i \mid q'_r = f_r,\ 3 \le r \le 5;\ q'_6 = q_6 - (f_5 - q_5),\ q'_i = q_i \text{ for all } i \ge 7),$$
$$w' = (w'_i \mid w'_i = w_i \text{ for all } i \ge 4,\ w'_3) \quad \text{and} \quad a' \ge 0,\ b' \ge 0,\ c' \ge 0 .$$

Table 2.

α	ℓ	s_3	s_4	s_5	a'	b'	c'
1	$6k$	0	$3k-3$	0	a	b	$c+k-1$
2	$6k+1$	1	$3k-4$	0	$a-1$	b	$c+k-1$
3	$6k+1$	0	$3k-3$	1	$a-1$	b	$c+k-1$
4	$6k+2$	0	$3k-2$	0	$a-1$	$b+1$	$c+k-1$
5	$6k+3$	1	$3k-3$	0	a	b	$c+k-1$
6	$6k+3$	0	$3k-2$	1	$a-1$	$b+1$	$c+k-1$
7	$6k+4$	0	$3k-1$	0	$a-1$	b	$c+k$
8	$6k+5$	1	$3k-2$	0	$a-1$	$b+1$	$c+k-1$
9	$6k+5$	0	$3k-1$	1	$a-1$	b	$c+k-1$
10	$6k$	0	$3k-3$	0	a	b	$c+k-1$
11	$6k+1$	1	$3k-4$	0	$a+1$	$b-1$	$c+k-1$
12	$6k+1$	0	$3k-3$	1	a	$b-1$	$c+k-1$
13	$6k+2$	0	$3k-2$	0	a	$b-1$	$c+k$
14	$6k+3$	1	$3k-2$	0	a	b	$c+k-1$
15	$6k+3$	0	$3k-2$	1	a	$b-1$	$c+k-1$
16	$6k+4$	0	$3k-1$	0	$a+1$	$b-1$	$c+k$
17	$6k+5$	1	$3k-2$	0	a	$b-1$	$c+k$
18	$6k+5$	0	$3k-1$	1	a	$b-1$	$c+k$
19	$6k$	0	$3k-3$	0	a	b	$c+k-1$
20	$6k+1$	1	$3k-4$	0	a	$b+1$	$c+k-2$
21	$6k+1$	0	$3k-3$	1	a	b	$c+k-2$
22	$6k+2$	0	$3k-2$	0	$a+1$	b	$c+k-1$
23	$6k+3$	1	$3k-3$	0	a	b	$c+k-1$
24	$6k+3$	0	$3k-2$	1	a	b	$c+k-1$
25	$6k+4$	0	$3k-1$	0	a	$b+1$	$c+k-1$
26	$6k+5$	1	$3k-2$	0	$a+1$	b	$c+k-1$
27	$6k+5$	0	$3k-1$	1	a	b	$c+k-1$

LEMMA 5. (cf. [16]) *If there is a map $M = M(q, w, g, a, b, c)$ with at least one G configuration, then there is a map $M' = M(q', w', g, a, b, c)$ with one less G configuration such that*

$$q' = (q'_i \mid q'_i = q_i \text{ for all } i \neq 4,5,6\,,\ q'_4 = q_4 - 1\,,\ q'_5 = q_5 + 2\,,\ q'_6 = q_6 - 2)\,,$$

$$w' = (w'_i \mid w'_i = w_i \text{ for all } i \geq 4\,,\ w'_3 = w_3 - 2)\,.$$

LEMMA 6.α ($\alpha \in \{1,2,3,4,5\}$). *If there is a map $M = M(q,w,g,a,b,c)$ with e, $e \geq 1$, mutually face-disjoint E configurations, then there is a map $M' = M(q',w',g,a,b,c)$ with e' E configurations,*

$$q' = (q_i' \mid g_j' = q_j + r_j\,,\ 3 \leq j \leq 6\,,\ q_i = q_i \text{ for all } i \geq 7) \quad \text{and}$$
$$w' = (w_i' \mid w_i' = w_i \text{ for all } i \geq 4\,,\ w_3')\,,$$

where for:

$\alpha = 1$ $\quad e' = e\,,\ r_3 = r_4 = r_5 = 0\,,\ r_6 = 4t$ *for any* $t \geq 1\,,\ g' = g\,;$

$\alpha = 2$ $\quad e' = e - 1\,,\ r_3 = 2\,,\ r_4 = -3\,,\ r_5 = 0\,,\ r_6 = 5\,,\ g' = g\,;$

$\alpha = 3$ $\quad e' = e - 1\,,\ r_3 = 1\,,\ r_4 = -2\,,\ r_5 = 1\,,\ r_6 = 2\,,\ g' = g\,;$

$\alpha = 4$ $\quad e' = e - 1\,,\ r_3 = 1\,,\ r_4 = -3\,,\ r_5 = 3\,,\ r_6 = 1,\ g' = g\,;$

$\alpha = 5$ $\quad e' = e - 2\,,\ e \geq 2\,,\ r_3 = r_5 = 0\,,\ r_4 = -6\,,\ r_6 = -4\,,\ g' = g + 1\,.$

P r o o f. For $\alpha = 1$ or 5 see [16]. The necessary changes in the interior of the E configuration for the remaining cases are left to the reader. □

LEMMA 7. (cf. [16]) *If there is a map $M = M(q,v,g,a,b,c)$ with at least one F configuration, then there is a map $M' = M(q',v',g,a,b,c)$ with one less F configuration and such that*

$$q' = (q_i' \mid q_i' = q_i \text{ for all } i \neq 4,5,6\,,\ q_4' = q_4 - 2\,,\ q_5' = q_5 + 4\,,\ q_6' = q_6 - z)\,,$$
$$w' = (w_i' \mid w_i' = w_i \text{ for all } i \geq 4\,,\ w_3')\,, \quad \text{where} \quad z = 3,4,5 \text{ or } 6\,.$$

LEMMA 8. *Let $M = M(q,v,g,a,b,c)$ be a map with $c \neq 0$, $a + b \leq 1$ and a pair of adjacent quadrangles face-disjoint with a A_6, b B_6 and c C_6 configurations of M and let f_3, f_4, f_5, h are non-negative integers such that*

(i) $f_3 \geq q_3,\ f_5 \geq q_5 + 4\,,$
(ii) $3q_3 + 2q_4 + q_5 = 3f_3 + 2f_4 + f_5 + 12h\,,$
(iii) $3(f_3 - q_3) + (f_5 - q_5) \leq 6c + 4b + 2a\,,$
(iv) $2h \leq c,$

then there is a map $M' = M(q',v',g+h,a',b',c')$ such that

$$q' = (q_i' \mid q_j' = f_j \text{ for } 3 \leq j \leq 5\,,\ q_i' = q_i\,,\ i \geq 7\,,\ q_6' = t \text{ for all } t \geq r)\,,$$
$$w' = (w_i' \mid w_i' = w_i \text{ for } i \geq 4\,,\ w_3') \quad \text{and} \quad a' \geq 0\,,\ b' \geq 0\,,\ c' \geq 0\,.$$

r is constant depending on the sequence $(q'_i \mid i \geq 3,\ i \neq 6)$ *and* $(w'_i \mid i \geq 4)$.

Proof. A very useful transformation of a map M into a map M_1 called the replacing of edges by hexagons (or $\mathcal{E}$-transformation) will be used first (cf. [8], [18], [19], [22]). In this transformation every edge of M is replaced by a hexagon in such a way that a pair of neighbouring faces in M consisting of a k-gon K and an ℓ-gon L is replaced by a k-gon K^* and an ℓ-gon L^* in M_1 which are separated by a hexagon. The vertices of K^* and L^* are trivalent and at the same time to every r-valent vertex of M there corresponds in M_1 an r-valent vertex in the same position which is incident with r hexagons. If two edges have a common vertex, then the hexagons corresponding to these edges are adjacent in M_1. The $\mathcal{E}$-transformation changes configurations A_m, B_m, C_m, $m \geq 6$ into configurations which will be designated as $\mathcal{E}(A_m)$, $\mathcal{E}(B_m)$ and $\mathcal{E}(C_m)$ respectively. The map M_1 obtained contains c $\mathcal{E}(C_6)$, one F (as the result of $\mathcal{E}$-transformation to the pair of quadrangles), at most one $\mathcal{E}(B_6)$ or $\mathcal{E}(A_6)$ configurations, $w_i(M_1) = w_i$ i-valent vertices for all $i \geq 4$, $q_i(M_1) = q_i$ i-gons, $i \geq 3$, $i \neq 6$, and $q_6(M_1) = e(M) + q_6$ hexagons. All the configurations of M_1 mentioned are pairwise disjoint. Note that every $\mathcal{E}(C_6)$ configuration contains an E configuration or three G configurations as submaps. By using Lemma 7 and Lemma 6.1 to the map M_1 a map M_2 with $q_i(M_2) = q_i$ for all $i \geq 3$, $i \neq 4,5,6$; $q_4(M_2) = q_4 - 2$, $q_5(M_2) = q_5 + 4$, $q_6(M_2) = q_6(M_1) + t$, $t \geq 0$, $w_i(M_2) = w_i$ for all $i \geq 4$ and c $\mathcal{E}(C_6)$ configurations is obtained. Then, starting with the map M_2, Lemma 6.5 is step by step applied h times. The result is a map $M_3 = M(\overline{q}, \overline{w}, g+h, \overline{a}, \overline{b}, \overline{c})$ with $c-2h$ $\mathcal{E}(C_6)$ configurations (and therefore with $c - 2h$ E configurations), with the same number of other configurations as in the map M_2, and with $\overline{q}_i = q_i(M_2)$ for all $i \geq 3$, $i \neq 4,6$, $\overline{q}_4 = q_4(M_2) - 6h$, $\overline{q}_6 = q_6(M_2) - 4h$, $w_i(M_3) = w_i(M_2)$ for all $i \geq 4$. To obtain the additional number of $f_3 - q_3$ triangles and $f_5 - q_5 - 4$ pentagons of the map M' required, starting with the map M_3. Lemma 6.2 is applied $\left\lfloor \frac{f_3 - q_3}{2} \right\rfloor$-times, then Lemma 5 $\left\lfloor \frac{f_5 - q_5 - 4}{2} \right\rfloor$-times and, if $f_3 - q_3$ and $f_5 - q_5$ are odd, Lemma 6.3 or 6.4 or $\mathcal{E}(B_6)$ configuration is changed into one triangle, one pentagon and two more hexagons. (The last in the case $f_3 - q_3 = 2c + 1$ and $b = 1$). For the resulting map

$$M' = M(q', w', g + h, a', b', c'), \qquad a' \geq 0,\ b' \geq 0,\ c' \geq 0,$$

$$q' = \Big(q'_i \mid q'_i = f_i,\ 3 \leq i \leq 5,\ q'_j = q_j \text{ for all } j \geq 7,$$

$$q'_6 = q_6(M_1) + 5\left\lfloor \frac{f_3 - q_3}{2} \right\rfloor - 2\left\lfloor \frac{f_5 - q_5 - 4}{2} \right\rfloor - 4h + t,\ t \geq 0\Big),$$

$$w' = (w'_i \mid w'_i = w_i \text{ for all } i \geq 4,\ w'_3).$$

□

LEMMA 9.α ($\alpha \in \{1,2\}$). (see [16])
If there is a polyhedral map $M = M(q,w,g,a,b,c)$ containing e pairwise face-disjoint D configurations, $e \geq 2$, face-disjoint from a A_6, b B_6 and c C_6 configurations, then there is a map $M' = M(q',w',g',a,b,c)$ containing e' D-configurations such that:

1. $q' = (q'_i \mid q'_i = q_i$ *for all* $i \geq 3,\ i \neq 6,\ q'_6 = q_6 + t,$ *for all* $t \geq 0),$
   $w' = (w'_i \mid w'_i = w_i$ *for all* $i \geq 4,\ v'_3 = v_3 + 2t),\ g' = g$ *and* $e' = e.$
2. $q' = (q'_i \mid q'_i = q_i$ *for all* $i \geq 6,\ q'_3 = q_3 - 2,\ q'_4 = q_4 - 2,\ q'_5 = q_5 - 2),$
   $w' = (w'_i \mid v'_i = v_i$ *for all* $i \geq 4,\ v'_3 = v_3 - 8),\ g' = g + 1$ *and* $e' = e - 2.$

4. Basic polyhedral maps

LEMMA 10.α ($\alpha \in \{1,2,\ldots,10\}$). *Let $k \geq 1,\ m \geq 1,\ n \geq 1$. There exist polyhedral maps:*

1. $N_1 = M(q,w,1,1,0,k)$ *with*
   $q = (q_i \mid q_i = 0$ *for all* $i \geq 3,\ i \neq 4,6;$
   $q_4 = 3k+1,\ q_6 = t$ *for all* $t \geq d_0)$ *and*
   $w = (w_i \mid w_i = 0,$ *for all* $i \geq 4,\ i \neq 3k+1,\ w_{3k+1} = 1,\ w_3).$
2. $N_2 = M(q,w,1,0,1,k)$ *with*
   $q = (q_i \mid q_i = 0$ *for all* $i \geq 3,\ i \neq 4,6;$
   $q_4 = 3k+2,\ q_6 = t$ *for all* $t \geq d_0)$ *and*
   $w = (w_i \mid w_i = 0$ *for all* $i \geq 4,\ i \neq 3k+2,\ w_{3k+2} = 1,\ w_3).$
3. $N_3 = M(q,w,1,0,0,0)$ *with*
   $q = (q_i \mid q_i = 0$ *for all* $i \geq 3,\ i \neq 6;\ q_6 = t$ *for all* $t \geq 9).$
4. $N_4 = M(q,w,1,0,0,m+n-1)$ *with*
   $q = (q_i \mid q_i = 0$ *for all* $i \geq 3,\ i \neq 4,6;$
   $q_4 = 3(m+n-1),\ q_6 = t$ *for all* $t \geq d_0)$ *and*
   $w = (w_i \mid w_i = 0$ *for all* $i \geq 4,\ i \neq 3m+1,\ 3n+2,$
   $w_{3m+1} = w_{3n+2} = 1,\ w_3).$

5. $N_5 = M(q, w, 1, 0, 1, m+n-2)$ *with*

$$q = (q_i \mid q_i = 0 \text{ for } i \geq 3,\ i \neq 4, 6;$$
$$q_4 = 3(m+n) - 4,\ q_6 = t \text{ for all } t \geq d_0) \quad \text{and}$$
$$w = (w_i \mid w_i = 0 \text{ for all } i \geq 4,\ i \neq 3m+1,\ 3n+1;$$
$$w_{3m+1} = w_{3n+1} = 1,\ m, \neq n;\ \text{or } w_{3m+1} = 2 \text{ if } m = n).$$

6. $N_6 = M(q, w, 1, 1, 0, m+n-1)$ *with*

$$q = (q_i \mid q_i = 0 \text{ for all } i \geq 3,\ i \neq 4, 6;$$
$$q_4 = 3(m+n) - 2,\ q_6 = t \text{ for all } t \geq d_0) \quad \text{and}$$
$$w = (w_i \mid w_i = 0 \text{ for all } i \geq 4,\ i \neq 3m+2,\ 3n+2;$$
$$w_{3m+2} = w_{3n+2} = 1 \text{ for } m \neq n \text{ or } w_{3m+2} = 2 \text{ for } m = n).$$

7. $N_7 = M(q, w, 1, 0, 0, m+n)$ *with* $m \geq n \geq 2$,

$$q = (q_i \mid q_i = 0 \text{ for all } i \geq 3,\ i \neq 4, 6;$$
$$q_4 = 3m + 3n,\ q_6 = t \text{ for all } t \geq d_0) \quad \text{and}$$
$$w = (w_i \mid w_i = 0 \text{ for all } i \geq 4,\ i \neq 3m,\ 3n;$$
$$w_{3m} = v_{3n} = 1 \text{ for } m \neq n \text{ or } w_{3m} = 2 \text{ for } m = n).$$

8. $N_8 = M(q, w, 0, 0, 0, k+1)$ *with*

$$q = (q_i \mid q_i = 0 \text{ for all } i \neq 4, 5, 6;$$
$$q_4 = 3k + 3,\ q_5 = 2,\ q_6 = t \text{ for all } t \geq 11k + 3) \quad \text{and}$$
$$w = (w_i \mid w_i = 0 \text{ for all } i \neq 3,\ 3k+1;\ w_{3k+1} = 1,\ w_3).$$

9. $N_9 = M(q, w, 0, 1, 0, k+1)$ *with*

$$q = (q_i \mid q_i = 0 \text{ for all } i \neq 4, 5, 6;$$
$$q_4 = 3k + 4,\ q_5 = 2,\ q_6 = t \text{ for all } t \geq 11k + 12) \quad \text{and}$$
$$w = (w_i \mid w_i = 0 \text{ for all } i \neq 3,\ 3k+2,\ w_{3k+2} = 1,\ w_3).$$

10. $N_{10} = M(q, w^*, 0, 0, 0, 2)$ *with*

$$q = (q_i \mid q_i = 0 \text{ for all } i \geq 3,\ i \neq 4, 6;\ q_4 = 6,\ q_6 = 12).$$

d_0 *is a constant depending on* k *or* m *and* n*, respectively.*

Proof. For $\alpha = 1$ we start with the planar polyhedral map P_1 in Fig. 8. It contains a U_4 configuration, eight hexagons and two face disjoint D configurations. The basic vertex construction steps are used k times to the U_4 configuration of the map P_1. A map P_1^* with the $(3k+1)$-valent vertex, k

C_6 configurations, two D configurations (all mutually face-disjoint) and $9k+9$ hexagons is obtained. Then Lemma 9.1 is used t times followed by Lemma 9.2. A map N_1 with $d_0 = 9k+9$ is obtained. Analogously we proceed in the cases $\alpha = 2$, 5 and 6. The proof starts with the planar polyhedral map in Fig. 9, in Fig. 10 or in Fig. 11 respectively. In the case $\alpha = 5$, $m = 1$ and $n \geq 1$ we start with a V_4 configuration of the map in Fig. 10a. For the case $m \geq 2$ and $n \geq 2$ we insert into the pair of the quadrangles of the map in Fig. 10a the configuration of Fig. 10b. A map P_5 with V_7 and W_7 configurations is the result. For $\alpha = 7$ we start with the trivalent polyhedral map N_3. Two adjacent hexagons of N_3 are divided by new edges as in Fig. 12. A toroidal polyhedral map P_3 with two W_6 configurations and $t+17$ hexagons, $t \geq 0$ is obtained. Then basic vertex construction steps are used gradually $(m-1)$-times starting with one W_6 configuration and $(n-1)$-times starting with the second W_6 configuration. The result is a map N_7 required with $11(m-1)+(n-1)+t+17 = d_0+t$ hexagons, $t \geq 0$. For $\alpha = 4$ we proceed analogously as in the case $\alpha = 7$. The change of a pair of adjacent hexagons of the map N_3 for $m = 1$ (without dashed lines) or $n = 1$ (with them) is in the Fig. 13. If $m \geq 2$ and $n \geq 2$ we insert new edges into "upper" two hexagons of Fig. 13 in the same way as into two hexagons in Fig. 12. A U_7 configuration and a V_8 configuration are obtained and used for creating the $(3m+1)$-valent and $(3m+2)$ valent vertices required. For $\alpha = 3$ see [16] and for $\alpha = 8, 9$ and 10 see [17]. □

5. Proofs of Theorems 2 and 3

Consider a pair of sequences of non-negative integers (B) satisfying (1) with an integer g, $g \geq 1$. We show that there is a map M on T_g with $p_i(M) = p_i$ for all $i \geq 3$, $i \neq 6$, $v_i(M) = v_i$ for all $i \geq 4$, and with $p_6(M) = p_6$ for every $p_6 \geq d$, $v_3(M) = \frac{1}{3}\Big(\sum\limits_{i\geq 3} ip_i - \sum\limits_{i\geq 4} iv_i\Big)$ where d is a constant depending on the triple (p, v, g). Dependence d on the triple (p, v, g) is given by the construction presented (implicitly contained in lemmas used).

We will only consider the case $\sum\limits_{i\geq 4} v_i \geq 1$ because of Jendrol' [16], where the proof for the case $\sum\limits_{i\geq 4} v_i = 0$ is made. Let us denote $\sigma = \sum\limits_{k\geq 1} v_{3k+1}$, $\varrho = \sum\limits_{k\geq 1} v_{3k+2}$ and $\tau = \sum\limits_{k\geq 2} v_{3k}$.

Three basic cases will be considered.

1. $\sigma = \varrho = 0$ and $\tau = 1$,
2. $3p_3 + 2p_4 + p_5 \neq 1$ and $\sigma + \varrho \neq 0$ or $\tau \geq 2$,
3. $3p_3 + 2p_4 + p_5 = 1$ and $\sigma + \varrho \neq 0$ or $\tau \geq 2$.

5.1. Instead of the pair of sequences $p = (p_i \mid i \geq 3,\ i \neq 6)$, $v = (v_i \mid\ v_i = 0$

for all $i \geq 4$, $i \neq 3k$, $k \geq 2$, $v_{3k} = 1$) and $g \geq 1$ let us consider the pair $p' = (p'_i \mid p'_i = p_i$ for all $i \geq 3$, $i \neq 6$, $6k+1$ $p'_{6k} = p_{6k} + 1)$ and $v' = (v'_i \mid v'_i = 0$ for all $i \geq 4)$. By [16] there is a constant d such that the set $P_6(p', v', g)$ contains all $p_6 \geq d$. Now it is sufficient to transform a polyhedral map M' on T_g realizing the triple (p', v', g) to the polyhedral map M on T_g realizing the triple (p, v, g). Therefore let us transform a $6k$-gon of M' into a $3k$-valent vertex required in the following way:

Let $x_1, x_2, \ldots, x_{6k}$ be vertices of the $6k$-gon and let $y_1, y_2, \ldots, y_{6k}$ be neighbours of these vertices (some of them can be identical). Insert a new vertex x into the $6k$-gon, delete the vertices x_{6i} and join the vertices x_{6i-1}, x_{6i+1} and y_{6i} with the vertex x for every $i = 1, 2, \ldots, k$ (indices are taken modulo $6k$). A map M required with $p_i(M) = p_i(M')$ for all $i \geq 3$, $i \neq 6, 6k$, $p_6(M) = p_6(M') + k$, $p_{6k}(M) = p_{6k}(M') - 1$, $v_i(M) = v_i(M')$, for all $i \geq 4$, $i \neq 3k$, $v_{3k}(M) = 1$, $v_3(M) = v_3(M') - k$ is the result.

5.2. We will distinguish 19 cases listed below. These 19 cases cover all pairs of the sequences (B) which have to be considered in the basic case 2.

For $\sigma \equiv 1 \pmod 2$, $\varrho \equiv 0 \pmod 2$ we consider cases:

1. $p_5 \leq 1$	N_1	A_6
2. $2 \leq p_5 \leq 3$	N_8	C_6
3. $4 \leq p_5 \leq 5$	N_8	$B_6 \subset C_6$
4. $p_6 \geq 6$	N_8	$A_6 \subset C_6$

For $\sigma \equiv 0 \pmod 2$, $\varrho \equiv 1 \pmod 2$ the cases considered are:

5. $p_5 \leq 1$	N_2	B_6
6. $2 \leq p_5 \leq 3$	N_9	A_6
7. $4 \leq p_5 \leq 5$	N_9	C_6
8. $6 \leq p_5 \leq 7$	N_9	$B_6 \subset C_6$
9. $p_5 \geq 8$	N_9	$A_6 \subset C_6$

For $\sigma \equiv \varrho \equiv 1 \pmod 2$ we consider cases:

10. $p_5 \leq 1$	N_4	C_6
11. $2 \leq p_5 \leq 3$	N_4	$B_6 \subset C_6$
12. $p_5 \geq 4$	N_4	$A_6 \subset C_6$

For $\sigma \equiv \varrho \equiv 0 \pmod 2$ we consider cases:

13. $p_5 \leq 1$, $\sigma \geq 2$	N_5	B_6
14. $p_5 \leq 1$, $\sigma = 0$, $\varrho \geq 2$	N_6	A_6
15. $p_5 \leq 1$, $\sigma = \varrho = 0$	N_7	C_6
16. $2 \leq p_5 \leq 3$, $\sigma \geq 2$	N_5	$A_6 \subset B_6$
17. $2 \leq p_5 \leq 3$, $\sigma = 0$, $\varrho \geq 2$	N_6	$B_6 \subset C_6$
18. $2 \leq p_5 \leq 3$, $\sigma = \varrho = 0$	N_7	$B_6 \subset C_6$
19. $p_5 \geq 4$	N_{10}	$A_6 \subset C_6$

The proof of the existence of a required polyhedral map begins with a suitable planar or toroidal polyhedral map which contains none, one or two of the vertices of the valencies ≥ 4 required, respectively. The second column of the list above indicates the map M_S which suits to be a starting map in the corresponding case. To obtain all other vertices of valencies ≥ 4 required Lemma 1.α is applied to an X configuration of the map M_S. The choice of the suitable α depends on the X configuration of M_S and on the value j defined in Lemma 1.α, where we consider $u_i = v_i - w_i(M_s)$ for all $i \geq 4$. The third column of the list denotes the X configuration of M_S. A record $X \subset Y$ in the list means that the X configuration is used in the sequel while the rest of Y configuration of M_S is considered to be a special one (a G configuration in the cases 3, 8, 11, 16, 17, 18 or a pair of adjacent quadrangles in the cases 4, 9, 12 and 19 respectively). A polyhedral map M_V is the result of an application of Lemma 1.α.

The map M_V contains at most one of A_6 or B_6 configuration. Let Z denote this configuration. If none of A_6 and B_6 configuration appears in the map M_V, a C_6 configuration is considered to be a Z configuration.

To obtain all faces of the sizes ≥ 7 Lemma 2.β is applied to the Z configuration of the map M_V. The choice of β depends on the Z configuration, the value ℓ defined in Lemma 2.β, where $f_i = p_i$ for all $i \geq 7$, and, if ℓ odd, on p_3 ($= 1$ if $p_3 \neq 0$ and $= 0$ if not).

A map M_{VF} obtained contains all, up to several pentagons and may be a triangle, its "small" faces in c C_6 configurations a A_6 configurations and b B_6 configurations with $a + b \leq 1$, one (cases 3, 7, 11, 16, 17, 18) or two (a case 8) G-configurations or a pair of adjacent quadrangles (cases 4, 9, 12, 19), respectively, face disjoint with the quadrangles of the above mentioned a A_6, b B_6 and c C_6 configurations. In the cases 4, 9, 12 and 19 applying Lemma 8 to the map M_{VF} h times ($h = g$ for the cases 2, 3, 4, 6, 7, 8, 9, 12 or $h = g - 1$ in the rest of cases) we obtain a map M with $v_i(M) = v_i$ for all $i \geq 4$, $p_i(M) = p_i$ for all $i \geq 3$, $i \neq 6$ and with any $P_6 \geq d$ for a constant d.

In the rest of cases we proceed as follows. First the quadrangles of the D configurations of M_{VF} are changed into pentagons required (Lemma 5 is used in the cases 3, 7, 8, 11, 16, 17 and 18). The Lemma 3 is employed g times in the cases 2, 3, 4, 6, 7, 8, 9, 19 and $g - 1$ times in the rest of cases. A polyhedral map M_g on T_g is obtained. The proof ends by applying Lemma 4 to the map M_g.

5.3. The conditions of this case imply $p_4 = p_3 = 0$, $p_5 = 1$.

If $\sigma \geq 1$, then there is $k \geq 1$ such that $v_{3k+1} \neq 0$. Instead of the triple

(p, v, g) consider the triple $(p', v', 1)$ with

$$p' = \Big(p'_i \mid p'_i = p_i\,, \text{ for all } i \ge 7\,,\ p'_3 = p_3 = 0\,,\ p'_5 = p_5 = 1\,,$$
$$p'_4 = \frac{1}{2}\Big(\sum_{i\ge 7}(i-6)p'_i + 2\sum_{i\ge 4}(i-3)v'_i\Big) - 1\Big)\,,$$
$$v' = (v'_i \mid v'_i = v_i \text{ for all } i \ge 4\,,\ i \ne 3k+1\,,\ v_{3k+1} = v_{3k+1} - 1)\,.$$

We proceed as in the case 2 (subcases 1, 5, 10 or 13 in dependence on the properties of v', respectively). After using Lemmas 1.α and 2.β for a suitable α and β a map M_1 realizing the triple $(p', v', 1)$ with any $p_6(M_1) \ge d_0$ (d_0 is a constant) is obtained. All quadrangles of M_1 are only in C_6 configurations and in the configuration as in Fig. 14a (see J e n d r o ľ [15], [19]). Changing this configuration in the way as in Fig. 14b a map M_2 with a W_4 configuration and a pentagon required is obtained. The W_4 configuration is used to create, using basic vertex construction steps, the last required $(3k+1)$-valent vertex. The toroidal map M_3 having all faces of the valencies ≥ 7 and all vertices of the valencies ≥ 4 required is obtained. A $(g-1)$-multiple using of Lemma 3 provides the map M on T_g required.

If $\sigma = 0$ and $\varrho \ge 1$, $v_{3k+2} \ne 0$ for some $k \ge 1$.
Instead of the triple (p, v, g) we first consider the triple $(p', v', 1)$ with

$$p' = \Big(p'_i \mid p'_i = p_i \text{ for all } i \ge 7\,,\ p'_3 = 1\,,\ p'_5 = 0\,,$$
$$p'_4 = \frac{1}{2}\Big(\sum_{i\ge 7}(i-6)p'_i + 2\sum_{i\ge 4}(i-3)v'_i - 3\Big)\Big) \quad \text{and}$$
$$v' = (v'_i \mid v'_i = v_i \text{ for all } i \ge 4\,,\ i \ne 3k+2\,,\ v'_{3k+2} = v_{3k+2} - 1)\,.$$

Analogously as above we obtain toroidal polyhedral map M_1 realizing the triple $(p', v', 1)$ with $p'_6(M_1) = p_6$ for any $p_6 \ge d_0$ (d_0 is a constant). All quadrangles of M_1 are contained in C_6 configurations. A triangle of M_1 is adjacent to a hexagon with all vertices trivalent. By inserting new edges into the hexagon as in Fig. 15 we obtain a map M_2 with a W_5 configuration and a pentagon required but without a triangle. The W_5 configuration is used for creating the last $(3k+2)$-valent vertex required. The proof ends by using Lemma 3 $g-1$ times.

If $\sigma = \varrho = 0$ then there is $k \ge 2$ such that $v_{3k} \ne 0$. The conditions of the case require a $(6m+r)$-gon, $m \ge 1$, $r = 1, 3$ or 5 respectively. There is a toroidal polyhedral map M_1 containing a $6k$-gon, a $(6m+r)$-gon, a pentagon,

p_6 hexagons for any $p_6 \geq d_0$, d_0 is a constant, $3(k+m)$ C_6 configurations and for $r \neq 1$ an A_6 configuration (if $r=3$) or a B_6 configuration (if $r=5$), respectively. The existence of such a map is guaranteed by Jendroľ [16]. Then Lemma 1.α with $u_i = v_i$ for all $i \geq 4$, $i \neq 3k$, $u_{3k} = v_{3k} - 1$ and suitable α is employed to the map M_1. After that Lemma 2.β with $f_i = p_i$ for all $i \geq 7$, $i \neq 6m+r$, $f_{6m+r} = p_{6m+r} - 1$ follows. $\beta = 1, 4$ or 7 if $r = 1, 3$ or 5 respectively. β depends on α and the value ℓ defined in Lemma 2.β. A toroidal polyhedral map M_2 is obtained. To the map M_2 Lemma 3 is applied $(g-1)$ times. The proof of the existence finishes by a transformation of a $6k$-gon of the latter map to a new $3k$-valent vertex of the map on T_g in the same way as in the case 5.1 above.

6. Remarks

6.1. Euler's formula provides also the following condition for the pair of sequences (A) to be a face-vector and a vertex-vector of a polyhedral map on T_g for a given non-negative integer g

$$\sum_{i\geq 4}(4-i)(p_i+v_i) = 8(1-g)\,. \tag{3}$$

Considering the pair of sequences of non-negative integers

$$\hat{p} = (p_i \mid 3 \leq i \neq 4)\,, \qquad \hat{v} = (v_i \mid 3 \leq i \neq 4) \tag{C}$$

and a non-negative integer g satisfying the conditions (2) and (3), the problem of a characterization of the set $P_4(\hat{p}, \hat{v}, g)$ of suitable values of p_4 (and therefore v_4) can be posed. Many papers are devoted to the study of the set $P_4(\hat{p}, \hat{v}, g)$ especially for the case of 4-valent planar polyhedral maps, see e.g. Enns [4], Grünbaum [10], Jucovič [22], Trenkler [25]. The most general result concerning the set $P_4(p, v, g)$ is the following one due to Jucovič [21], [22].

THEOREM 4. *To every pair of sequences* (C) *and a non-negative integer* g*, not excluded below, satisfying* (2) *and* (3) *there exists a non-negative integer* d *such that the set* $P_4(\hat{p}, \hat{v}, g)$ *contains all integers* $\geq d$. *The set* $P_4(\hat{p}, \hat{v}, 1)$ *is empty for the following two pairs* $(\hat{p}, \hat{v})$

(i) $\hat{p} = (p_i \mid p_i = 0$ *for all* $i \geq 6$, $p_3 = p_5 = 1)$ *and* $\hat{v} = (v_i \mid v_i = 0$ *for all* $i \geq 3$, $i \neq 4)$.

(ii) $\hat{p} = (p_i \mid p_i = 0$ *for all* $i \geq 3$, $i \neq 4)$ *and* $v = (v_i \mid v_3 = v_5 = 1$, $v_i = 0$ *for all* $i \geq 6)$.

6.2. Barnette [1] and Jucovič [20] have found two different lower bounds for $\min\{p_6 \mid p_6 \in P_6(p, v^*, g)\}$. What is the minimum of the set $P_6(p, v, g)$?

6.3. Theorem 3 can also be interpreted as an theorem of Eberhard's type for periodic tilings. Compare with Grünbaum and Shephard [13].

6.4. The problems can be investigated not requiring the maps to be polyhedral and assuming $\Sigma(p_i + v_i) \neq 0$ for $i \leq 3$. However, greater complications are expected in this case (cf. Grünbaum and Zaks [14], Enns [5]).

6.5. An interesting and probably very difficult problem is the next one (see Barnette [2] or Gritzmann [8]): Which pairs of sequences (A) are realizable as face-vectors and vertex-vectors of polyhedra of genus g?

6.6. We do not know if there exists a polyhedral toroidal map $M = M(p, v^*, 1)$ with $p = (p_i \mid p_i = 0$ for $i \geq 3$, $i \neq 4, 6, 8$, $p_4 = p_8 = 1$, p_6 odd) or with $p = (p_i \mid p_i = 0$ for $i \geq 3$, $i \neq 3, 6, 9$, $p_3 = p_9 = 1$ and p_6 odd).

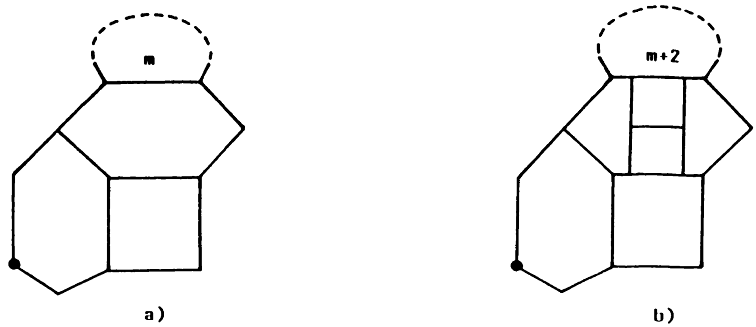


Figure 1.

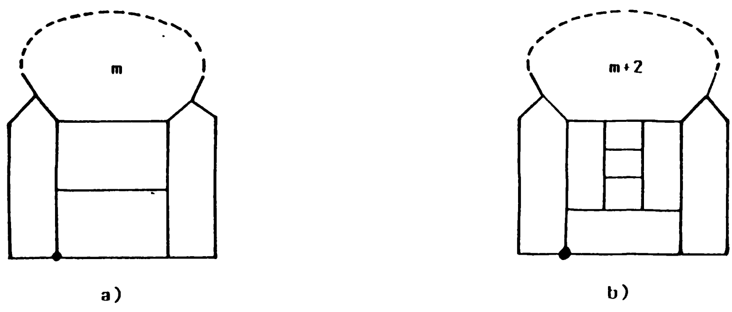
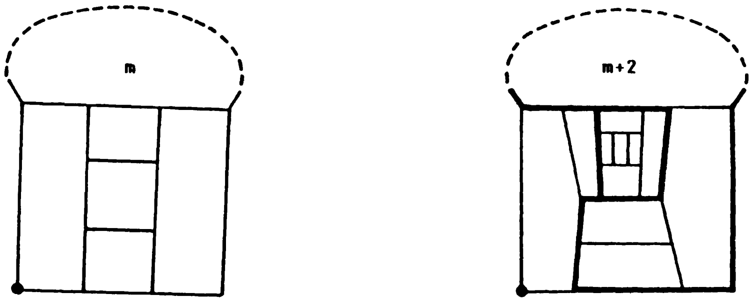
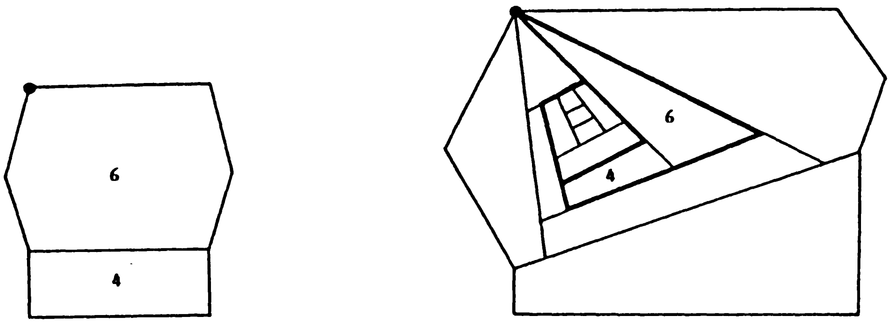


Figure 2.

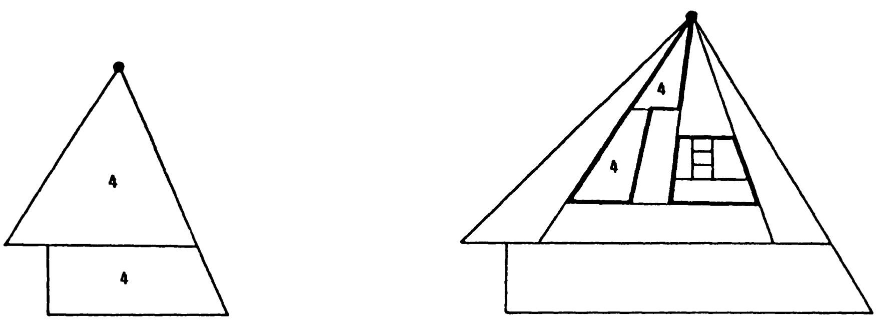


a) Figure 3. b)



a) b)

Figure 4.



a) Figure 5. b)

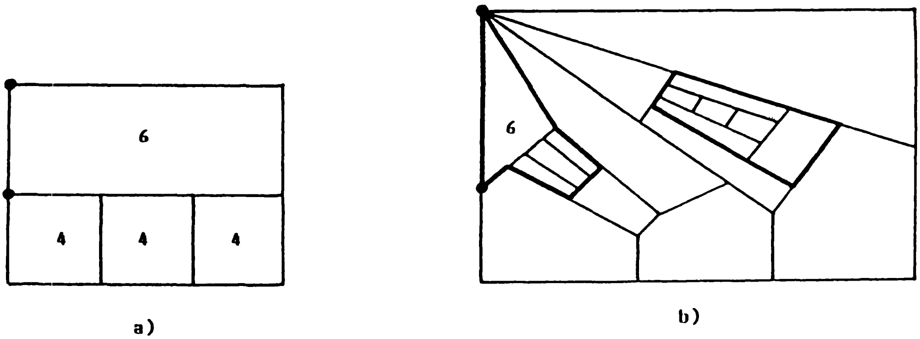
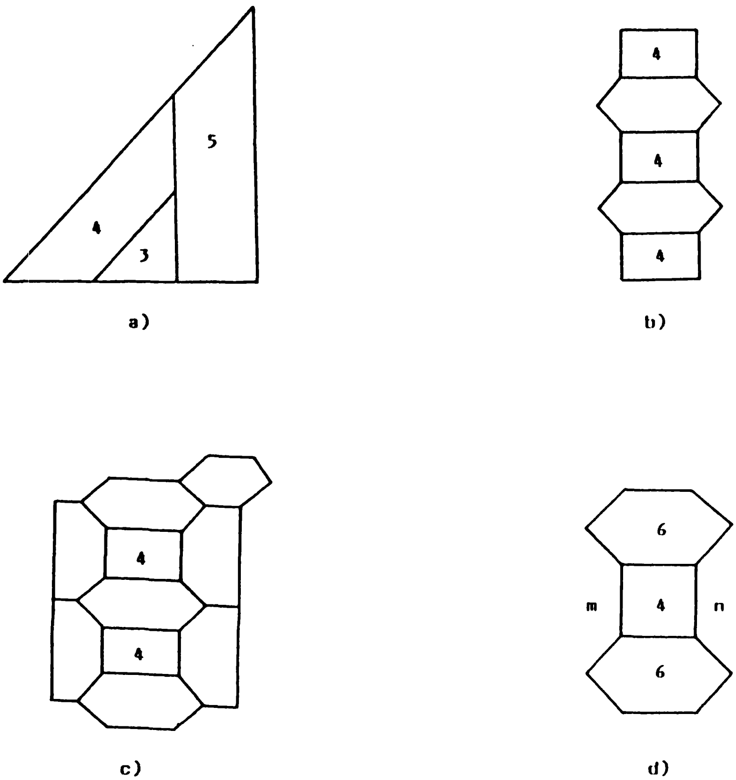


Figure 6.



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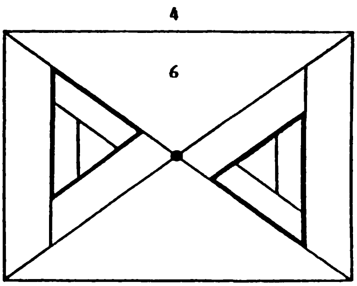

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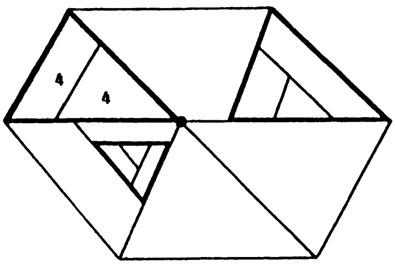

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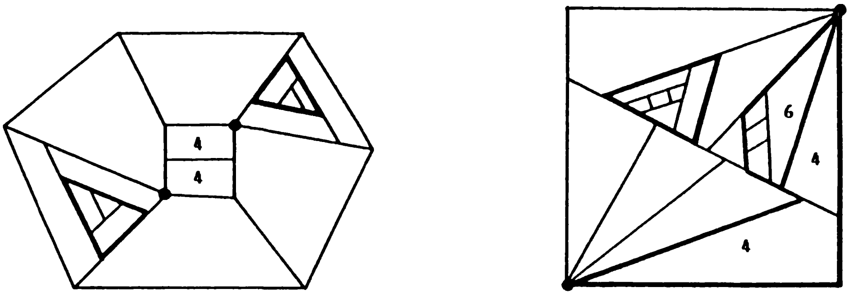

a) b)

Figure 10.

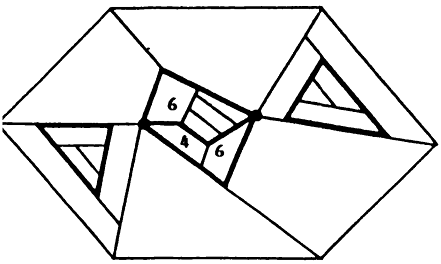

Figure 11.

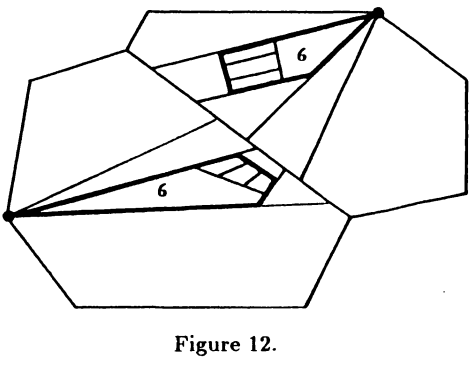

Figure 12.

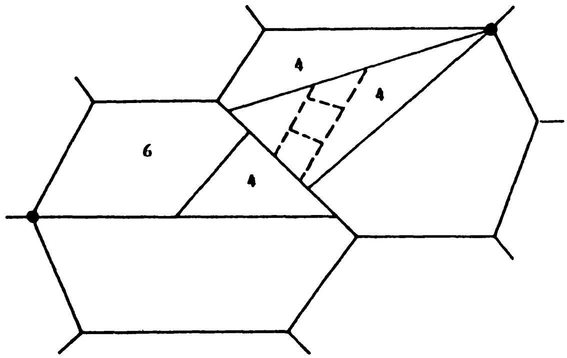

Figure 13.

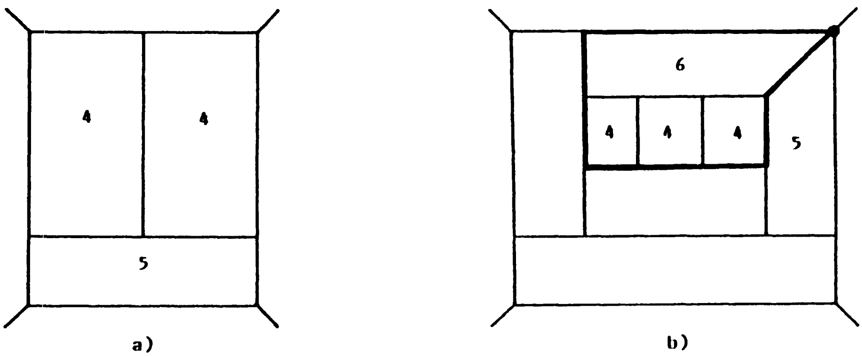

Figure 14.

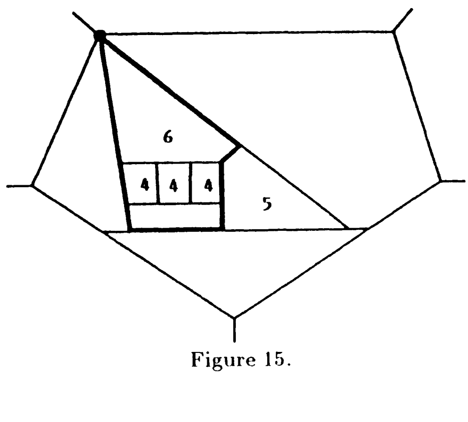

Figure 15.

REFERENCES

[1] BARNETTE, D.: *On p-vectors of 3-polytopes*, J. Combin. Theory Ser.? **7** (1969), 89–103.

[2] BARNETTE, D.: *Polyhedral maps on 2-manifolds.* In: Convexity and Related Combinatorial Geometry (D. C. Kay and M. Breen, eds.), Marcel Dekker Inc., New York and Basel, 1982, pp. 7–19.

[3] EBERHARD, V.: *Zur Morphologie der Polyeder*, Teubner, Leipzig, 1891.

[4] ENNS, T. C.: *Convex 4-valent polytopes*, Discrete Math. **30** (1980), 227–234.

[5] ENNS, T. C.: *4-valent graphs*, J. Graph Theory **6** (1982), 255-281.

[6] FISHER, J. C.: *An existence theorem for simple convex polyhedra*, Discrete Math. **7** (1974), 75–97.

[7] GRITZMANN, P.: *The toroidal analogue to Eberhard's theorem*, Mathematika **30** (1983), 274–290.

[8] GRÜNBAUM, B.: *Convex Polytopes*, Interscience, New York, 1967.

[9] GRÜNBAUM, B.: *Some analogues of Eberhard's theorem on convex polytopes*, Israel J. Math. **6** (1968), 398–411.

[10] GRÜNBAUM, B.: *Planar maps with prescribed types of vertices and faces*, Mathematika **16** (1969), 28–36.

[11] GRÜNBAUM, B.: *Polytopal graphs.* In: Studies in Graph Theory. MAA Stud. Math. 12 (D. R. Fulkerson, ed.), Math. Assoc. America, Washington, DC, 1975, pp. 201–224.

[12] GRÜNBAUM, B.—MOTZKIN, T. S.: *The number of hexagons and the simplicity of geodesics of certain polyhedra*, Canad. J. Math. **15** (1963), 744–751.

[13] GRÜNBAUM, B.—SHEPHARD, G. C.: *The theorems of Euler and Eberhard for tilings of the plane*, Resultate Math. **5** (1982), 19–44.

[14] GRÜNBAUM, B.—ZAKS, J.: *The existence of certain planar maps*, Discrete Math. **10** (1974), 93–115.

[15] JENDROĽ, S.: *On the face-vector of trivalent convex polyhedra*, Math. Slovaca **33** (1983), 165–180.

[16] JENDROĽ, S.: *On face vectors of trivalent maps*, Math. Slovaca **36** (1986), 367–386.

[17] JENDROĽ, S.: *On face-vectors and vertex-vectors of convex polyhedra*, Discrete Math. **118** (1993), 119–144.

[18] JENDROĽ, S.—JUCOVIČ, E.: *On a conjecture by B. Grünbaum*, Discrete Math. **2** (1972), 35–49.

[19] JENDROĽ, S.—JUCOVIČ, E.: *Generalization of a theorem by V. Eberhard*, Math. Slovaca **27** (1977), 383–407.

[20] JUCOVIČ, E.: *On the number of hexagons in a map*, J. Combin. Theory Ser. B **10** (1971), 232–236.

[21] JUCOVIČ, E.: *On face-vectors and vertex-vectors of cell-decompositions of orientable 2-manifolds*, Math. Nachr. **73** (1976), 285–295.

[22] JUCOVIČ, E.: *Convex 3-polytopes.* (Slovak), Veda, Bratislava, 1981.

[23] KRAEFT, J.: *Über 3-realisierbare Folgen mit beliebigen Sechseckzahlen*, J. Geom. **10** (1977), 32–44.

[24] MALKEVITCH, J.: *Polytopal graphs.* In: Selected Topics in Graph Theory 3, Academic Press, London, 1988, pp. 169–188.

[25] TRENKLER, M.: *Convex 4-valent polytopes with prescribed types of faces*, Comment. Math. Univ. Carolin. **25** (1984), 171–179.

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