Mathematica Slovaca

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Mathematica Slovaca, Vol. 32 (1982), No. 1, 35--48

Persistent URL: http://dml.cz/dmlcz/132289

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TOPOLOGY ON REGULATORS OF LATTICE ORDERED GROUPS II. COMPLETELY REGULAR REGULATORS

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In the present paper we continue investigating mutual relations between the properties of an l-group G and the induced topological space $(\mathfrak{R}, G)$ [6]. The main attention is paid to completely regular regulators $(\mathfrak{R}, \bigcup)$. These regulators are characterized as regulators formed by minimal prime subgroups of G (1.2 and 1.4). The mappings Z and Ψ are (mutually inverse) dual isomorphisms between the lattice of clopen sets of $(\mathfrak{R}, G)$ and the lattice $\Gamma(\mathfrak{R}, G)$ of all the so-called ambiguous polars of G (2.2; a polar K is said to be ambiguous with respect to $(\mathfrak{R}, \bigcup)$ if no $\bigcup x (x \in \mathfrak{R})$ contains both K and K'). If $(\mathfrak{R}, \bigcup)$ is a standard $\mathscr{C}$-regulator ($\equiv$ every solid subgroup of G is the meet of some family of $\bigcup x$), then $\Gamma(\mathfrak{R}, G)$ is the set of all direct factors of G (3.4). There are defined the equality, similarity and equivalence of regulators and it is examined which of these relations carry the property of the complete regularity or reducibility. The connection of regulators with the most comprehensive reduced and completely regular regulator, the Π'-regulator, is investigated.

The paper is a continuation of [6] where the reader can find all less frequent notions or new ones used (as a rule without any reference) in this second part. For the terminology and denotations concerning the general theory of l-groups cf. [1] and [2].

1. Completely regular elements

1.1 Definition. Let G be an l-group and $(\mathfrak{R}, \bigcup)$ a regulator of G. An element $f \in G$ is said to be *completely regular* with respect to $(\mathfrak{R}, \bigcup)$ if to an arbitrary $x \in Z(f)$ there exists $g \in G$ such that $x \in \mathfrak{R} \setminus Z(g) \subseteq Z(f)$, [5] 2.1. If every element of G is completely regular with respect to $(\mathfrak{R}, \bigcup)$, the regulator $(\mathfrak{R}, \bigcup)$ is called *completely regular*, [4] II 3; [6] I 1.6. Note that the condition $x \in \mathfrak{R} \setminus Z(g) \subseteq Z(f)$ can be replaced by the following one: $f \delta g$ and $x \bar{\in} Z(g)$ ([6] 2.15).

In the definition of the induced topology on $\mathfrak{R}$ one supposes that the regulator is standard, [6] 1.2. In all the following assertions in which the topological space $(\mathfrak{R}, G)$ occurs and the regulator $(\mathfrak{R}, \bigcup)$ is not specified, we often assume tacitly that $(\mathfrak{R}, \bigcup)$ is standard (and so $G \neq \{0\}$).

1.2 Theorem. *Let* $(\mathfrak{R}, \bigcup)$ *be a standard regulator of an l-group G. The following conditions are equivalent.*

1. *f is completely regular with respect to* $(\mathfrak{R}, \bigcup)$.
2. $Z(f)$ *is an open set of the topological space* $(\mathfrak{R}, G)$.
3. $Z(f') = \mathfrak{R} \setminus Z(f)$.
4. *There holds:*
   (a) $Z(f') \cap Z(f'') = \emptyset$
   and one of the following equivalent conditions (b), (c) *and* (d):
   (b) $g \in G,\ f'' = g'' \Rightarrow Z(f) = Z(g)$;
   (c) $Z(f) = Z(f'')$;
   (d) $Z(f) \in \mathfrak{M}(\mathfrak{R}, G)$.

(5) $f' \nsubseteq \bigcup x$ *for every* $x \in \mathfrak{R}$ *with* $f \in \bigcup x$.

Proof. (Cf. [5] 2.1) 1 $\Rightarrow$ 2. From the difinition 1.1 it is clear that $x \in Z(f)$ is an inner point of the set $Z(f)$.

2 $\Rightarrow$ 3. By [6] 2.15 we have $g \in f' \equiv Z(g) \supseteq \mathfrak{R} \setminus Z(f)$. Hence by [6] 2.2, $Z(f') = \bigcap \{Z(g) \colon g \in f'\} = \bigcap \{Z(g) \colon Z(g) \supseteq \mathfrak{R} \setminus Z(f)\} = \overline{\mathfrak{R} \setminus Z(f)} = \mathfrak{R} \setminus Z(f)$.

3 $\Rightarrow$ 4(d). $Z(f) = \mathfrak{R} \setminus Z(f')$ is a clopen set, hence 4(d).

(d) $\Rightarrow$ (c). $Z(f) \in \mathfrak{M}(\mathfrak{R}, G) \Rightarrow K = \Psi Z(f) \in \Gamma(G)$ ([6] 2.18) $\Rightarrow f \in K$ ([6] 2.4) $\Rightarrow f'' \subseteq K \Rightarrow Z(f'') \supseteq Z(K) = Z\Psi Z(f) = Z(f)$ ([6] 2.4). The converse inclusion is clear.

(c) $\Rightarrow$ (b) is evident.

(b) $\Rightarrow$ (d). If $g \in f''$, then $(|g| \vee |f|)'' = g'' \vee f'' = f''$, hence $Z(g) \supseteq Z(|g| \vee |f|) = Z(f)$. Finally $Z(f'') = \bigcap \{Z(g) \colon g \in f''\} \supseteq Z(f)$. Evidently $Z(f) \supseteq Z(f'')$, whence $Z(f) = Z(f'') \in \mathfrak{M}(\mathfrak{R}, G)$, [6] 2.18.

3 $\Rightarrow$ 4(a). There holds $Z(f'') \cap Z(f') \subseteq Z(f) \cap Z(f') = Z(f) \cap (\mathfrak{R} \setminus Z(f)) = \emptyset$.

4 $\Rightarrow$ 5. Pick $x \in Z(f)$. Then $x \in Z(f) = \mathfrak{R} \setminus Z(f') = \mathfrak{R} \setminus \bigcap \{Z(g) \colon g \in f'\} = \bigcup \{\mathfrak{R} \setminus Z(g) \colon g \in f'\}$. It follows that there exists $g \in f'$ with $x \in \mathfrak{R} \setminus Z(g)$, i.e. $g \bar{\in} \bigcup x$, whence $f' \nsubseteq \bigcup x$.

5 $\Rightarrow$ 1. Fix $x \in Z(f)$. By assumption there exists $g \in f' \setminus \bigcup x$, thus $x \bar{\in} Z(g)$ and $f \delta g$, whence 1.

1.3 Definition. Denote by $\mathscr{P}(G)$ or $\mathrm{m}\mathscr{P}(G)$ the set of all prime or minimal prime subgroups of G, respectively.

1.4 Theorem. *Let* $(\mathfrak{R}, \bigcup)$ *be a standard regulator of an l-group G. The following conditions are equivalent.*

1. $(\mathfrak{R}, \bigsqcup)$ *is completely regular.*
2. $Z(f)$ *is an open set of the topological space* $(\mathfrak{R}, G)$ *for every* $f \in G$.
3. $Z(f') = \mathfrak{R} \setminus Z(f)$ *for every* $f \in G$.
4. $\bigsqcup x$ *is a minimal prime subgroup of* G *for every* $x \in \mathfrak{R}$.
5. *For every (minimal) prime subgroup* J *of* G *and for every* $x \in Z(J)$ *we have* $J = \bigsqcup x$.
6. $\{\bigsqcup x : x \in \mathfrak{R}\}$ *is the set of all minimal prime subgroups* J *of* G *with* $Z(J) \neq \emptyset$.
7. (a) $Z(f'') \cap Z(f') = \emptyset$ *for every* $f \in G$ *holds and simultaneously one of the following equivalent conditions* (b), (c) *and* (d):
   (b) $f, g \in G,\ f'' = g'' \Rightarrow Z(f) = Z(g)$;
   (c) $Z(f) = Z(f'')$ *for every* $f \in G$;
   (d) $Z(f) \in \mathfrak{M}(\mathfrak{R}, G)$ *for every* $f \in G$.
8. Z *maps* Π' *onto* $\mathfrak{F}' = \{\mathfrak{R} \setminus Z(f) : f \in G\}$.
9. Z *maps* Π *onto* $\mathfrak{F} = \{Z(f) : f \in G\}$ *and* $Z(f') \cap Z(f'') = \emptyset$ *for every* $f \in G$.

Proof. (Cf. [4] IV 8.10). The equivalence of 1, 2, 3 and 7 and the equivalence of b, c and d in the condition 7 follow from 1.2; $1 \equiv 4$ follows from 1.2 $(1 \equiv 5)$, too, since the condition $f \in \bigsqcup x \Rightarrow f' \not\subseteq \bigsqcup x$ for every $f \in G$ characterizes $\bigsqcup x$ as a minimal prime among all prime subgroups, [4] III 7.6; [1] 3.4.13.

$4 \Rightarrow 5$. Let J be a prime subgroup of G and $Z(J) \neq \emptyset$. Then $x \in Z(J) \Rightarrow \bigsqcup x = \Psi(x) \supseteq \Psi Z(J) \supseteq J$, [6] 2.3 and 2.4. By supposition $\bigsqcup x$ is a minimal prime subgroup of G, thus $\bigsqcup x = J$.

$5 \Rightarrow 6$. Fix $x \in \mathfrak{R}$. A minimal prime subgroup J of G exists with $\bigsqcup x \supseteq J$. Hence $x \in Z\Psi(x) = Z(\bigsqcup x) \subseteq Z(J)$ and by 5, $\bigsqcup x = J$. There is proved $\{\bigsqcup x : x \in \mathfrak{R}\} \subseteq \{J \in m\mathcal{P}(G) : Z(J) \neq \emptyset\}$. Conversely, if $J \in m\mathcal{P}(G)$, $x \in Z(J)$, then (by 5) $J = \bigsqcup x$, hence $\supseteq$.

$6 \Rightarrow 1$ holds since $4 \Rightarrow 1$ is true.

$7 \Rightarrow 8$. Pick $f \in G$. By 7(a), (c), there holds $\emptyset = Z(f') \cap Z(f'') = Z(f') \cap Z(f)$. Since $f' \delta f''$, we have $\mathfrak{R} = Z(f') \cup Z(f'') = Z(f') \cup Z(f)$ ([6] 2.15), and so $Z(f') = \mathfrak{R} \setminus Z(f)$. Consequently, Z maps Π' onto $\mathfrak{F}'$.

$8 \Rightarrow 2$. For $f \in G$ there exists $h \in G$ with $Z(h') = \mathfrak{R} \setminus Z(f)$. Therefore $Z(f) = \mathfrak{R} \setminus Z(h')$ is an open set.

$7 \Rightarrow 9$. By 7(c), Z maps Π onto $\mathfrak{F}$. The remainder of the condition 9 is 7(a).

$9 \Rightarrow 7$. For $f \in G$ there exists $h \in G$ with $Z(f) = Z(h'')$, whence $Z(f) \in \mathfrak{M}(\mathfrak{R}, G)$ for every $f \in G$. Hence 7(d) is true.

Note. The condition 9 is a transparent transcription of the condition 7. It is introduced to be shown that it is not possible "to omit the dash" (over Π and $\mathfrak{F}$) in 8.

1.5 Corollary. 1. *The* Π'*-regulator is standard, completely regular and re-*

duced. Moreover, $Z(J) \neq \emptyset$ for every $J \in \mathrm{m}\mathscr{P}(G)$. Consequently, every l-group $G \neq \{0\}$ possesses a regulator which is completely regular and reduced (and thus standard).

2. *Every completely regular regulator is standard and its simplification is reduced.*

3. *Every minimal prime subgroup is a z-subgroup.*

Proof. 1. $\bigcup x$ is a minimal prime subgroup for every $x \in \mathrm{ll}(\Pi')$, [4] III 7.2. Thus $\mathfrak{R}_{\Pi'}$ is completely regular by 1.4 (see also [4] II 4.16) and standard. $\mathfrak{R}_{\Pi}$ is reduced because different minimal prime subgroups are incomparable sets.

2. If $(\mathfrak{R}, \bigcup)$ is completely regular and $f = 0$, then $x \in Z(f)$ for every $x \in \mathfrak{R}$. Hence for every $x \in \mathfrak{R}$ there exists $g \in G$ with $g \bar{\in} \bigcup x$. The latter part of 2 is proved by a similar argument to that in 1.

3. This well-known result is an immediate consequence of 1 and 1.4. Indeed, the Π'-regulator $\mathfrak{R}_{\Pi'}$ is completely regular by 1 and $\{\bigcup x \colon x \in \mathfrak{R}_{\Pi'}\}$ is the set of all minimal prime subgroups by 1 and 1.4. Again by 1.4, $Z(f'') = Z(f)$ for every $f \in G$, which is equivalent to: $f \in \bigcup x \Rightarrow f'' \subseteq \bigcup x$ for every $x \in \mathfrak{R}$.

1.6 Now we call attention to relations between the completely regular regulators and the set $\Omega(\mathfrak{R}, G)$.

The elements of the set $\Omega(\mathfrak{R}, G)$ are called *$\mathfrak{R}$-subgroups* (of G). We define $\mathscr{P}\Omega(\mathfrak{R}, G) = \Omega(\mathfrak{R}, G)) \cap \mathscr{P}(G)$ the set of all *prime $\mathfrak{R}$-subgroups* of G. The system of all minimal elements of the set $\mathscr{P}\Omega(\mathfrak{R}, G)$ will be denoted by $\mathrm{m}\mathscr{P}\Omega(\mathfrak{R}, G)$ *(minimal prime $\mathfrak{R}$-subgroups of G)*. From the Zorn Lemma it follows that every prime $\mathfrak{R}$-subgroup of G contains a minimal prime $\mathfrak{R}$-subgroup of G because by [6] 2.11 and 2.3, there holds $G \in \mathscr{P}\Omega(\mathfrak{R}, G)$ and by [6] 2.13 the meet of an arbitrary chain in $\mathscr{P}\Omega(\mathfrak{R}, G)$ belongs to $\Omega(\mathfrak{R}, G)$ and by [4] II 2.3 this meet belongs to $\mathscr{P}(G)$.

1.7 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a completely regular regulator of an l group G $(\neq \{0\})$. Then*

$$\{\bigcup x \colon x \in \mathfrak{R}\} = \{J \in \mathrm{m}\mathscr{P}(G) \colon Z(J) \neq \emptyset\} = \mathrm{m}\mathscr{P}\Omega(\mathfrak{R}, G).$$

Consequently, minimal prime $\mathfrak{R}$-subgroups of G are minimal prime subgroups of G, i.e. $\mathrm{m}\mathscr{P}\Omega(\mathfrak{R}, G) \subseteq \mathrm{m}\mathscr{P}(G)$.

Proof. The first equality follows from 1.4 ($1 \equiv 6$). From this equality it follows that every $J \in \mathrm{m}\mathscr{P}(G)$ with $Z(J) \neq \emptyset$ is equal to a prime $\mathfrak{R}$-subgroup $\bigcup x$ $(= \Psi(x))$ for some $x \in \mathfrak{R}$ and hence $\bigcup x$ is evidently a minimal prime $\mathfrak{R}$-subgroup. Consequently, $\{J \in \mathrm{m}\mathscr{P}(G) \colon Z(J) \neq \emptyset\} \subseteq \mathrm{m}\mathscr{P}\Omega(\mathfrak{R}, G)$.

Conversely, fix $J_0 \in \mathrm{m}\mathscr{P}\Omega(\mathfrak{R}, G)$. $Z(J_0) \neq \emptyset$ holds, because of $Z(J_0) = \emptyset \Rightarrow J_0 = \Psi Z(J_0) = \Psi(\emptyset) = G \supseteq \bigcup x$ for every $x \in \mathfrak{R}$. Since $\bigcup x$ $(= \Psi(x))$ is a prime $\mathfrak{R}$-subgroup, we have $(G =)\ J_0 = \bigcup x$, in contradiction with the standardness of $(\mathfrak{R}, \bigcup)$ (1.5). The prime subgroup J_0 contains a minimal prime subgroup, say J, $J \subseteq J_0$. Thus $\emptyset \neq Z(J_0) \subseteq Z(J)$. For $x \in Z(J)$ we have by 1.4 $J = \bigcup x$ $(= \Psi(x))$; then

J is a prime $\mathfrak{R}$-subgroup, hence $J_0 = J$. It follows that $\{J \in m\mathscr{P}(G): Z(J) \neq \emptyset\} \supseteq m\mathscr{P}\Omega(\mathfrak{R}, G)$, completing the proof of the theorem.

1.8 Corollary. *For the Π'-regulator $\mathfrak{R}_{\Pi'}$ of G there holds*

$$\{\bigcup x : x \in \mathfrak{R}_{\Pi'}\} = m\mathscr{P}\Omega(\mathfrak{R}_{\Pi'}, G) = m\mathscr{P}(G).$$

Proof. The Π'-regulator is completely regular (1.5) and $\{\bigcup x: x \in \mathfrak{R}_{\Pi'}\} = m\mathscr{P}(G)$, [4] III 7.2; [1] 3.4.15. Now the assertion follows from 1.7.

Problems. 1. Does the equality $\{J \in m\mathscr{P}(G): Z(J) \neq \emptyset\} = m\mathscr{P}\Omega(\mathfrak{R}, G)$ or $m\mathscr{P}(G) = m\mathscr{P}\Omega(\mathfrak{R}, G)$ characterize the completely regular regulators or the Π'-regulator, respectively?

2. Does the equality $\{\bigcup x: x \in \mathfrak{R}\} = m\mathscr{P}\Omega(\mathfrak{R}, G)$ characterize the Π'-regulator in the class of reduced regulators?

1.9 Lemma. *Let $A \subseteq \mathfrak{R}$ be a clopen and compact set of the space $(\mathfrak{R}, G)$. Then $\mathfrak{R} \setminus A = Z(f)$ for some $f \in G$, completely regular with respect to $(\mathfrak{R}, \bigcup)$, and $\Psi(\mathfrak{R} \setminus A) = f''$, $\Psi(A) = f'$.*

Proof. Let the dash denote the complement in $\mathfrak{M}$ and Γ. The set $A' = \mathfrak{R} \setminus A$ is closed (see [6] 2.21), hence $A' = Z\Psi(A') = \bigcap\{Z(g): g \in \Psi(A')\}$, [6] 2.2 and 2.3. The compact set $A = \mathfrak{R} \setminus A' = \bigcup\{\mathfrak{R} \setminus Z(g): g \in \Psi(A')\}$ is covered by open sets $\mathfrak{R} \setminus Z(g)$ $(g \in \Psi(A'))$; thus there exists a finite number of elements $g_i \in \Psi(A')$ such that $A = \bigcup_i (\mathfrak{R} \setminus Z(g_i))$. Then $A' = \bigcap_i Z(g_i) = \bigcap_i Z(|g_i|) = Z\left(\bigvee_i |g_i|\right)$. By 1.2 the element $f = \bigvee_i |g_i|$ is completely regular with respect to $(\mathfrak{R}, \bigcup)$, because the set $Z(f) = A'$ is open. By 1.2 again there holds $\Psi(A') = \Psi(\mathfrak{R} \setminus A) = \Psi Z(f) = \Psi Z(f'') = f''$, $\Psi(A) = \Psi(\mathfrak{R} \setminus Z(f)) = \Psi Z(f') = f'$.

1.10 Definition. The set of all clopen sets of the space $(\mathfrak{R}, G)$ is denoted by $\mathscr{O}(\mathfrak{R}, G)$ (briefly $\mathscr{O}_{\mathfrak{R}}$ or $\mathscr{O}$ only). By $\Pi(\mathfrak{R}, G)$ (briefly $\Pi_{\mathfrak{R}}$) there will be denoted the set of all principal polars f'' of G, where the element f is completely regular with respect to $(\mathfrak{R}, \bigcup)$.

1.11 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a standard regulator of an l-group G and $(\mathfrak{R}, G)$ a compact space. Then Ψ and Z are (mutually inverse) dual isomorphisms between the lattices $\mathscr{O}(\mathfrak{R}, G)$ and $\Pi(\mathfrak{R}, G)$ and*

$$\Pi(\mathfrak{R}, G) \subseteq \Pi(G) \cap \Pi'(G)$$

holds.

Proof. $B \in \mathscr{O}(\mathfrak{R}, G) \Rightarrow A = \mathfrak{R} \setminus B$ is clopen and hence compact $\Rightarrow$ by 1.9 there exists $f \in G$ which is completely regular with respect to $(\mathfrak{R}, \bigcup)$ such that $\mathfrak{R} \setminus A =$

$Z(f)$ and $\Psi(B) = \Psi(\mathfrak{R}\backslash A) = f''$. It follows that $\Psi(\mathcal{O}_{\mathfrak{R}}) \subseteq \Pi_{\mathfrak{R}}$. Since B is clopen and hence compact, there exists $g \in G$ such that $\Psi(B) = g'$ (1.9). It follows that $\Psi(\mathcal{O}_{\mathfrak{R}}) \subseteq \Pi(G) \cap \Pi'(G)$. The Z-image of the polar f'', where $f'' \in \Pi((\mathfrak{R}, G)$, is clopen in $(\mathfrak{R}, G)$, because the set $Z(f'') = Z(f)$ is open by 1.2. It follows that $Z(\Pi_{\mathfrak{R}}) \subseteq \mathcal{O}_{\mathfrak{R}}$ Since Z and Ψ are mutually inverse isomorphisms between the lattices Γ and $\mathfrak{M}$ ([6] 2.18) and we have proved $\Psi(\mathcal{O}_{\mathfrak{R}}) \subseteq \Pi_{\mathfrak{R}}$ and $Z(\Pi_{\mathfrak{R}}) \subseteq \mathcal{O}_{\mathfrak{R}}$, there follows the required dual isomorphism between $\mathcal{O}_{\mathfrak{R}}$ and $\Pi_{\mathfrak{R}}$.

1.12 Corollary. *Let $(\mathfrak{R}, \bigcup)$ be a completely regular regulator of an l-group G and the space $(\mathfrak{R}, G)$ compact. Then Ψ and Z are (mutually inverse) dual isomorphisms between the lattices $\Pi(G)$ and $\mathcal{O}(\mathfrak{R}, G)$, and there holds*

$$\Pi(\mathfrak{R}, G) = \Pi(G) = \Pi'(G).$$

From the complete regularity of $(\mathfrak{R}, \bigcup)$ it follows that $\Pi(\mathfrak{R}, G) = \Pi(G)$ and from 1.11 we have $\Pi(\mathfrak{R}, G) \subseteq \Pi'(G)$. Hence $\Pi(G) = \Pi'(G)$.

Problems. 3. Which G fulfil $\bigcap\{\Pi(\mathfrak{R}, G)\colon (\mathfrak{R}, \bigcup)$ a standard regulator of $G\} = \emptyset$ or $\bigcup\{\Pi(\mathfrak{R}, G)\colon (\mathfrak{R}, \bigcup)$ a standard regulator of $G\} = \Pi(G) \cap \Pi'(G)$?

4. Which G fulfil $\Pi(\mathfrak{R}, G) = \Pi(G) \cap \Pi'(G)$? (This is fulfilled, e.g., under the assumptions of 1.12.)

2. Ambiguous polars

2.1 Definition. Let $(\mathfrak{R}, \bigcup)$ be a regulator of an l-group G and $K \in \Gamma(G)$. The polar K is called *ambiguous* with respect to $(\mathfrak{R}, \bigcup)$ if for an arbitrary $x \in \mathfrak{R}$ there holds $K \subseteq \bigcup x \Rightarrow K' \nsubseteq \bigcup x$. The set of all ambiguous polars of G with respect to $(\mathfrak{R}, \bigcup)$ will be denoted by $\Gamma(\mathfrak{R}, G)$ (briefly $\Gamma_{\mathfrak{R}}$), [5] 2.4. Evidently $K \in \Gamma(\mathfrak{R}, G) \Rightarrow K' \in \Gamma(\mathfrak{R}, G)$.

2.2 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a standard regulator of an l-group G. Then Ψ and Z are (mutually inverse) dual isomorphisms between the lattices $\mathcal{O}(\mathfrak{R}, G)$ and $\Gamma(\mathfrak{R}, G)$. Hence $\Gamma(\mathfrak{R}, G)$ is a subalgebra of the Boolean algebra $\Gamma(G)$.*

Proof. Since Ψ and Z are (mutually inverse) dual isomorphisms between the lattices $\Gamma(G)$ and $\mathfrak{M}(\mathfrak{R}, G)$ and $\Gamma(\mathfrak{R}, G) \subseteq \Gamma(G)$, $\mathcal{O}(\mathfrak{R}, G) \subseteq \mathfrak{M}(\mathfrak{R}, G)$, it suffices to prove that

(*) a polar $K \in \Gamma(G)$ is ambiguous with respect to $(\mathfrak{R}, \bigcup)$ iff $Z(K)$ is open in $(\mathfrak{R}, G)$.

Pick $x \in \mathfrak{R}$ and $g \in G$. Then by [6] 2.15, $g \in K' \equiv \mathfrak{R}\backslash Z(g) \subseteq Z(K)$, hence (*).

Recall that $(\mathfrak{R}, \bigcup)$ is called a *z-regulator* if $\bigcup x$ is a z-subgroup for every $x \in \mathfrak{R}$, [6] 2.23.

It is evident that $\bigcup x$ is a z-subgroup iff $Z(f'') = Z(f)$ for every $f \in \bigcup x$.

2.2a Lemma. *A regulator* $(\mathfrak{R}, \bigcup)$ *is a z-regulator iff* $Z(f'') = Z(f)$ *for every* $f \in \bigcup\{\bigcup x : x \in \mathfrak{R}\}$.

A standard regulator $(\mathfrak{R}, \bigcup)$ *is a z-regulator iff* $Z(f'') = Z(f)$ *for every* $f \in G$.

Indeed, sufficiency is clear. If $(\mathfrak{R}, \bigcup)$ is a z-regulator, then $Z(f'') = Z(f)$ for every $f \in \bigcup\{\bigcup x : x \in \mathfrak{R}\}$. Moreover, if $(\mathfrak{R}, \bigcup)$ is standard and $f \bar{\in} \bigcup x$ for every $x \in \mathfrak{R}$, then on the one hand $x \bar{\in} Z(f)$ for every $x \in \mathfrak{R}$, i.e. $Z(f) = \emptyset$ and on the other hand $f' \subseteq \bigcup\{\bigcap x : x \in \mathfrak{R}\} = \{0\}$, hence $f'' = G$ and thus $Z(f'') = \emptyset$, [6] 2.3. Consequently again $Z(f'') = Z(f)$.

2.3 Corollary. 1. *If* $f \in G$ *is completely regular with respect to* $(\mathfrak{R}, \bigcup)$, *then* $f'' \in \Gamma(\mathfrak{R}, G)$.

2. *Pick* $f \in G$ *and suppose* $\bigcup x$ *to be a z-subgroup for every* $x \in Z(f)$. *Then there holds the converse implication in* 1. *Hence if* $(\mathfrak{R}, \bigcup)$ *is a z-regulator, then* $\Pi(\mathfrak{R}, G) \supseteq \Pi(G) \cap \Gamma(\mathfrak{R}, G)$.

3. *A regulator* $(\mathfrak{R}, \bigcup)$ *is completely regular iff* $\Pi(G) \subseteq \Gamma(\mathfrak{R}, G)$ and $Z(f'') = Z(f)$ *for every* $f \in \bigcup\{\bigcup x : x \in \mathfrak{R}\}$ *(or every* $f \in G$*).*

4. *A z-regulator* $(\mathfrak{R}, \bigcup)$ *is completely regular iff* $\Pi(G) \subseteq \Gamma(\mathfrak{R}, G)$.

Proof. 1. Let $f \in G$ be completely regular with respect to $(\mathfrak{R}, \bigcup)$, $x \in \mathfrak{R}$ and $f'' \subseteq \bigcup x$. Then $f \in \bigcup x$ and by 1.2 $f' \not\subseteq \bigcup x$. Hence $f'' \in \Gamma(\mathfrak{R}, G)$.

2. Suppose $f'' \in \Gamma(\mathfrak{R}, G)$. If $f \in \bigcup x$ for some $x \in \mathfrak{R}$, then $f'' \subseteq \bigcup x$ and so $f' \not\subseteq \bigcup x$. Hence f is completely regular with respect to $(\mathfrak{R}, \bigcup)$ by 1.2.

3. If $(\mathfrak{R}, \bigcup)$ is completely regular, then $\Pi(G) = \Pi(\mathfrak{R}, G) \subseteq \Gamma(\mathfrak{R}, G)$ by 1 and $Z(f'') = Z(f)$ for every $f \in G$ by 1.4. Conversely, let the condition of 3 be fulfilled, $f \in \bigcup\{\bigcup x : x \in \mathfrak{R}\}$ and $x \in \mathfrak{R}$. From $Z(f'') = Z(f)$ it follows that $f \in \bigcup x \Rightarrow f'' \subseteq \bigcup x$ and from $f'' \in \Gamma(\mathfrak{R}, G)$ it follows that $f'' \subseteq \bigcup x \Rightarrow f' \not\subseteq \bigcup x$. By 1.2, $(\mathfrak{R}, \bigcup)$ is completely regular.

4. follows immediately from 3 and 2.2a.

Note. Since evidently $K \in \Gamma(\mathfrak{R}, G) \equiv K' \in \Gamma(\mathfrak{R}, G)$, then

$$\Pi(G) \subseteq \Gamma(\mathfrak{R}, G) \equiv \Pi'(G) \subseteq \Gamma(\mathfrak{R}, G).$$

2.4 Corollary. 1. *Let* $(\mathfrak{R}, \bigcup)$ *be a standard regulator of an l-group* G *and let the space* $(\mathfrak{R}, G)$ *be compact. Then* $\Gamma(\mathfrak{R}, G) = \Pi(\mathfrak{R}, G) \subseteq \Pi(G) \cap \Pi'(G)$.

2. *Let* $(\mathfrak{R}, \bigcup)$ *be a completely regular regulator of* G *and let the space* $(\mathfrak{R}, \bigcup)$ *be compact. Then* $\Gamma(\mathfrak{R}, G) = \Pi((\mathfrak{R}, G) = \Pi(G) = \Pi'(G)$.

1. follows from 1.11 and 2.2. 2. The first equality follows from 1, the others according to 1.12.

In the theorem 2.23 [6], conditions characterizing the extremal disconnectedness of the space $(\mathfrak{R}, G)$ are given. The above results on $\Gamma(\mathfrak{R}, G)$ and $\mathcal{O}(\mathfrak{R}, G)$ enable us to describe some additional conditions.

2.5 Theorem. *Let* $(\mathfrak{R}, \bigcup)$ *be a standard regulator of an* l*-group* G. *The following conditions are equivalent.*

1. *The space* $(\mathfrak{R}, G)$ *is extremally disconnected.*
2. $\mathcal{O}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$.
3. $\Gamma(\mathfrak{R}, G) = \Gamma(G)$.
4. *The lattice* $\mathfrak{M}(\mathfrak{R}, G)$ *is a sublattice of the lattice* $\mathfrak{N}(\mathfrak{R}, G)$.
5. *The lattice* $\Gamma(G)$ *is a sublattice of the lattice* $\Omega(\mathfrak{R}, G)$.
6. $\Psi[\mathcal{O}(\mathfrak{R}, G)] = \Gamma(G)$.
7. $Z[\Gamma(G)] = \mathcal{O}(\mathfrak{R}, G)$.

If a standard z*-regulator* $(\mathfrak{R}, \bigcup)$ *of* G *fulfils one of the above conditions, then* $(\mathfrak{R}, \bigcup)$ *is completely regular.*

Proof. The equivalences $1 \equiv 3 \equiv 4$ are proved in [6] 2.23, $1 \equiv 2$ is evident and $4 \equiv 5$ follows from [6] 2.11 and 2.18 (Ψ is a dual isomorphism which maps $\mathfrak{N}_{\mathfrak{R}}$ on $\Omega_{\mathfrak{R}}$ and $\mathfrak{M}_{\mathfrak{R}}$ on Γ).

$7 \Rightarrow 2$. $Z(\Gamma) = \mathfrak{M}_{\mathfrak{R}}$ by [6] 2.18.

$2 \Rightarrow 6$. $\Psi(\mathfrak{M}_{\mathfrak{R}}) = \Gamma$ using [6] 2.18.

$6 \Rightarrow 7$. Ψ and Z are mutually inverse mappings by [6] 2.18. The last assertion follows from 2.3(4) because by 3 $\Gamma(\mathfrak{R}, G) = \Gamma(G) \supseteq \Pi(G)$.

2.6 Theorem. *For the* Π'*-regulator* $\mathfrak{R}_{\Pi'}$ *of an* l*-group* G *the following conditions are equivalent.*

(i) *The space* $(\mathfrak{R}_{\Pi'}, G)$ *is compact.*

(ii) $\Pi(G) = \Pi'(G)$.

(iii) Ψ *and* Z *are (mutually inverse) dual isomorphisms between the lattice* $\Pi(G)$ *and the lattice* $\mathcal{O}_c(\mathfrak{R}_{\Pi'}, G)$ *of all compact clopen sets of the space* $(\mathfrak{R}_{\Pi'}, G)$

(iv) $\mathcal{O}(\mathfrak{R}_{\Pi'}, G) = \mathcal{O}_c(\mathfrak{R}_{\Pi'}, G)$ *the family of all compact clopen sets of the space* $(\mathfrak{R}_{\Pi'}, G)$.

Proof. i $\Rightarrow$ iii follows from 1.12, because the Π'-regulator is completely regular (1.5).

iii $\Rightarrow$ ii. For every $g \in G$ the set $A = Z(g'')$ is clopen and compact in $(\mathfrak{R}_{\Pi'}, G)$, thus $A' = \mathfrak{R}_{\Pi'} \setminus A = Z(f)$ for some $f \in G$ and $\Psi(A') = f''$ by 1.9. There holds $\Psi(A) = \Psi Z(g'') = g''$ ([6] 2.18), hence $g' = [\Psi(A)]' = \Psi(A') = f''$ by [6] 2.19. We have proved $\Pi'(G) \subseteq \Pi(G)$. The equality $\Pi'(G) = \Pi(G)$ follows immediately.

ii $\Rightarrow$ i follows from [6] 1.9.

i $\Rightarrow$ iv. Closed sets of a compact space are compact.

iv $\Rightarrow$ i is evident.

2.7 Corollary. *Let* $(\mathfrak{R}, \bigcup)$ *be a completely regular relator of an* l*-group* G *and let the space* $(\mathfrak{R}, G)$ *be compact. Then* $(\mathfrak{R}_{\Pi'}, G)$ *is compact, where* $\mathfrak{R}_{\Pi'}$ *means the* Π'*-regulator of* G.

Proof. By 2.4 $\Pi(G) = \Pi'(G)$. Then the assertion follows from 2.6.

3. $\mathscr{C}$-regulator

In accordance with [1] 2.2 and [2] 1.1 we denote the lattice of all solid subgroups of an l-group G by $\mathscr{C}(G)$. In this lattice there holds $\bigwedge_\alpha C_\alpha = \bigcap_\alpha C_\alpha$, $\bigvee_\alpha C_\alpha = [\bigcup_\alpha C_\alpha]$ for $\{C_\alpha\} \subseteq \mathscr{C}(G)$, where $\left[\bigcup_\alpha C_\alpha\right]$ is the subgroup of the group G generated by the set $\bigcup_\alpha C_\alpha$ ([1] 2.2.7; [2] 1.4).

3.1 Definition. A regulator $(\mathfrak{R}, \bigcup)$ of G is called a *$\mathscr{C}$-regulator* if for every $C \in \mathscr{C}(G)$ there exists $\emptyset \subseteq A \subseteq \mathfrak{R}$ such that $C = \bigcap\{\bigcup x\colon x \in A\}$, in other words if $\mathscr{C}(G) = \Omega(\mathfrak{R}, G)$ ([6] 2.11).

3.2 An example of a standard $\mathscr{C}$-regulator. For $0 \neq a \in G$ let H_a be the set of all values of a. Let $\mathfrak{R}$ denote the union of all $H_a (0 \neq a \in G)$ and $\bigcup$ the identical mapping of $\mathfrak{R}$. Then $(\mathfrak{R}, \bigcup)$ is a standard $\mathscr{C}$-regulator.

3.3 Definition. Denote by $\Delta(G)$ the set of all direct factors of G. $\Delta(G)$ is a subalgebra of the Boolean algebra $\Gamma(G)$ and a sublattice of the lattice $\mathscr{C}(G)$, [1] 3.5.12.

3.4 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a standard $\mathscr{C}$-regulator of G. Then $\Delta(G) = \Gamma(\mathfrak{R}, G)$ and Ψ and Z are (mutually inverse) dual isomorphisms of the lattice $\Delta(G)$ and $\mathcal{O}(\mathfrak{R}, G)$.*

Proof. The second assertion follows immediately from the first one by 2.2. Fix $K \in \Delta(G)$. If $\bigcup x \supseteq K \cup K'$ for some $x \in \mathfrak{R}$, then $\bigcup x \supseteq K + K' = G$, a contradiction. Thus $\Delta(G) \subseteq \Gamma(\mathfrak{R}, G)$. The converse inclusion: If $A \in \mathcal{O}(\mathfrak{R}, G)$, then $A' = \mathfrak{R} \setminus A \in \mathcal{O}(\mathfrak{R}, G)$. Since $A, A' \in \mathfrak{N}(\mathfrak{R}, G)$ and the operations in the lattice $\mathfrak{N}(\mathfrak{R}, G)$ are $\cap$ and $\cup$: $A \wedge_{\mathfrak{N}} A' = A \cap A' = \emptyset$, $A \vee_{\mathfrak{N}} A' = A \cup A' = \mathfrak{R}$, thus $\Psi(A) \vee_\Omega \Psi(A') = G$, $\Psi(A) \cap \Psi(A') = \Psi(A) \wedge_{\mathfrak{N}} \Psi(A') = \{0\}$. Since the subgroups $\Psi(A)$ and $\Psi(A')$ are disjoint and thus permutable and $\mathscr{C}(G) = \Omega(\mathfrak{R}, G)$, we have $G = \Psi(A) \vee_{\mathscr{C}} \Psi(A') = [\Psi(A) \cup \Psi(A')] = \Psi(A) + \Psi(A')$. Therefore $\Psi(A)$ is a direct factor of G.

3.5 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a standard $\mathscr{C}$-regulator of G. Then G is strongly projectable iff one of the conditions of Theorem 2.5 is true.*

Proof. By 3.4, $\Delta(G) = \Gamma(G)$ (the strong projectability) holds iff the condition 2.5(6) is fulfilled.

Theorem 3.5 is a generalization of [4] II 5.8.

4. Similar and equivalent regulators

4.1 Definition. Let $(\mathfrak{R}_i, \bigcup_i)$ be a regulator of an l-group G_i $(i = 1, 2)$. The regulator $(\mathfrak{R}_2, \bigcup_2)$ is said to be

a) *similar* to the regulator $(\mathfrak{R}_1, \bigsqcup_1)$ or
b) *equivalent* to the regulator $(\mathfrak{R}_1, \bigsqcup_1)$

if there exists an l-isomorphism $\alpha: G_1$ onto G_2 and a) a surjection or b) a bijection $\beta: \mathfrak{R}_2$ onto $\mathfrak{R}_1$, respectively, such that for every $f \in G_1$ and every $x \in \mathfrak{R}_2$

$$(\alpha) \qquad f \in \bigsqcup_1 \beta x \equiv \alpha f \in \bigsqcup_2 x .$$

Equivalently

$$(\beta) \qquad \alpha \bigsqcup_1 \beta x = \bigsqcup_2 x \quad \text{for every} \quad x \in \mathfrak{R}_2 .$$

If we denote by $Z_{\mathfrak{R}_i}$ the mapping Z corresponding to the regulator $(\mathfrak{R}_i, \bigsqcup_i)$ $(i = 1, 2)$, then (α) has clearly the following equivalent form

$$(\gamma) \qquad \beta^{-1}(Z_{\mathfrak{R}_1}(f)) = Z_{\mathfrak{R}_2}(\alpha f) \quad \text{for every} \quad f \in G_1 .$$

4.2 Lemma. *In case* a) β *is a continuous, open and closed mapping of the topological space* $(\mathfrak{R}_2, G_2)$ *onto* $(\mathfrak{R}_1, G_1)$, *in case* b) *a homeomorphism of these spaces.*

[4] IV 8.2.

Let H be a solid subgroup of an l-group G and G/H the (say left coset) decomposition of G modulo H. Then G/H is a distributive lattice with respect to the canonical ordering. Given $f \in G$ denote by $f(H)$ the class of the decomposition G/H containing f. If $(\mathfrak{R}, \bigsqcup)$ is a regulator of G and $H = \bigsqcup x$, we write $f(x)$ instead of $f(\bigsqcup x)$. By the symbol $f(x) \geqq 0$ we replace the more extensive one $f(x) \geqq 0(x)$.

4.3 Theorem. *Let a regulator* $(\mathfrak{R}_2, \bigsqcup_2)$ *of an* l*-group* G_2 *be similar to a regulator* $(\mathfrak{R}_1, \bigsqcup_1)$ *of an* l*-group* G_1. *Then for every* $x \in \mathfrak{R}_2$ *the binary relation*

$$R_x = \{(f(\beta x), (\alpha f)(x)) : f \in G_1\}$$

is an isomorphism of the canonically (linearly) ordered sets $G_1/\bigsqcup_1 \beta x$ and $G_2/\bigsqcup_2 x$.

If $(\mathfrak{R}_i, \bigsqcup_i)$ $(i = 1, 2)$ *are realizers, then the relation* R_x *is an* l*-isomorphism of the* l*-groups* $G_1/\bigsqcup_1 \beta x$ and $G_2/\bigsqcup_2 x$ $(x \in \mathfrak{R}_2)$.

The proof coincides essentially with the proof of Theorem 8.3 [4] IV.

4.4 Theorem. 1. *Every regulator is similar to its simplification.*

2. *Let* $(\mathfrak{R}_2, \bigsqcup_2)$ *be similar to* $(\mathfrak{R}_1, \bigsqcup_1)$. *If* $(\mathfrak{R}_2, \bigsqcup_2)$ *is* (a) *standard,* (b) *reduced,* (c) *completely regular, so is* $(\mathfrak{R}_1, \bigsqcup_1)$.

3. *Equivalent regulators are standard, reduced or completely regular simultaneously.*

Proof. 1. The defining mappings are $\alpha = \mathrm{id}_G$ and $\beta =$ the projection of $\mathfrak{R}_2$ onto $\bar{\mathfrak{R}}_2$, where $\bar{\mathfrak{R}}_2$ is the partition on $\mathfrak{R}_2$ corresponding to the mapping $\bigsqcup_2$.

2. For $x \in \mathfrak{R}_2$ there holds $\bigsqcup_1 \beta x = \alpha^{-1} \bigsqcup_2 x \neq G_1$ or $\in \mathrm{m}\mathscr{P}(G)$ (in case (a) or (c), respectively). Then the assertion concerning the complete regularity follows from 1.4. If $(\mathfrak{R}_2, \bigsqcup_2)$ is reduced, then the mapping β is one-to-one, because

$\beta x = \beta y$ for some $x, y \in \mathfrak{R}_2$ implies $\sqcup_2 x = \alpha \sqcup_1 \beta x = \alpha \sqcup_1 \beta y = \sqcup_2 y$, hence $x = y$. $(\mathfrak{R}_1, \sqcup_1)$ is evidently reduced (and the similarity is an equivalence).

3. follows from 2.

4.5 Definition. Two regulators of G, $(\mathfrak{R}_1, \sqcup_1)$ and $(\mathfrak{R}_2 \sqcup_2)$ are said to be *equal*, in symbols $(\mathfrak{R}_1, \sqcup_1) = (\mathfrak{R}_2, \sqcup_2)$, if there exists a bijection γ of $\mathfrak{R}_2$ onto $\mathfrak{R}_1$ such that $\sqcup_1 \gamma x = \sqcup_2 x$ for every $x \in \mathfrak{R}_2$.

The equality is then an equivalence of two regulators of the same l-group related to the mappings $\alpha = \mathrm{id}_G$ and $\beta = \gamma$.

4.6 Lemma. *Let $(\mathfrak{R}_2, \sqcup_2)$ be a regulator of an l-group $G \neq \{0\}$. The following conditions are equivalent.*

1. *The regulator $(\mathfrak{R}_2, \sqcup_2)$ is similar to the Π'-regulator of G.*
2. $\{\sqcup_2 x : x \in \mathfrak{R}_2\} = m\mathcal{P}(G)$.
3. *The simplification of the regulator $(\mathfrak{R}_2, \sqcup_2)$ is equal to the Π'-regulator of G.*

Proof. Denote by $(\mathfrak{R}_1, \sqcup_1)$ the Π'-regulator of G and use the notation of 4.1.

$1 \Rightarrow 2$. Suppose the similarity of $(\mathfrak{R}_2, \sqcup_2)$ to $(\mathfrak{R}_1, \sqcup_1)$. Since $\{\sqcup_1 y : y \in \mathfrak{R}_1\} = m\mathcal{P}(G)$ ([4] III 7.2; [1] 3.4.15), $\{\sqcup_2 : x \in \mathfrak{R}_2\} = \{\alpha \sqcup_1 \beta x : x \in \mathfrak{R}_2\} = \{\alpha \sqcup_1 y : y \in \mathfrak{R}_1\} = m\mathcal{P}(G)$, because the l-automorphism α carries $m\mathcal{P}(G)$ onto $m\mathcal{P}(G)$.

$2 \Rightarrow 3$. Let α be the identical mapping of G. Denote by $(\bar{\mathfrak{R}}_2, \bar{\sqcup}_2)$ the simplification of the regulator $(\mathfrak{R}_2, \sqcup_2)$ and for $\bar{x} \in \bar{\mathfrak{R}}_2$ denote by $\beta \bar{x}$ the element $y \in \mathfrak{R}_1$ for which $\sqcup_1 y = \bar{\sqcup}_2 \bar{x}$. β is a bijection of $\bar{\mathfrak{R}}_2$ onto $\mathfrak{R}_1$ and $\sqcup_1 \beta \bar{x} = \bar{\sqcup}_2 \bar{x}$ holds. Hence 3.

$3 \Rightarrow 1$. Use the notation of the definition 4.5, where $(\mathfrak{R}_1, \sqcup_1)$ is the Π'-regulator of G and instead of $(\mathfrak{R}_2, \sqcup_2)$ put the simplification $(\bar{\mathfrak{R}}_2, \bar{\sqcup}_2)$ of $(\mathfrak{R}_2, \sqcup_2)$. Then the similarity is established via the identical l-automorphism α of the l-group G and the bijection β: $\mathfrak{R}_2$ onto $\mathfrak{R}_1$ defined as follows: $\beta = \gamma\pi$, where π is the projection of $\mathfrak{R}_2$ onto $\bar{\mathfrak{R}}_2$. Then for $x \in \mathfrak{R}_2$ we have $\sqcup_1 \beta x = \sqcup_1 \gamma \pi x = \sqcup_1 \gamma \bar{x} = \bar{\sqcup}_2 \bar{x} = \bar{\sqcup}_2 \pi x = \sqcup_2 x$.

4.7 Corollary. *Let $(\mathfrak{R}_2, \sqcup_2)$ be a reduced regulator of G. Then $(\mathfrak{R}_2, \sqcup_2)$ is similar to the Π'-regulator $\mathfrak{R}_{\Pi'}$ of G iff $(\mathfrak{R}_2, \sqcup_2)$ is equal to $\mathfrak{R}_{\Pi'}$.*

4.8 Lemma. *The Π'-regulator of G is similar to a regulator of G iff the former is equal to the latter.*

Proof. Let $(\mathfrak{R}_2, \sqcup_2)$ be the Π'-regulator of G and $(\mathfrak{R}_1, \sqcup_1)$ a regulator of G. Suppose the similarity and notation as in the definition 4.1. We shall show that $\sqcup_1$ is one-to-one. Pick $x_1, y_1 \in \mathfrak{R}_1$ with $x_1 \neq y_1$ and $\sqcup_1 x_1 = \sqcup_1 y_1$. Then there exist $x_2, y_2 \in \mathfrak{R}_2$, $x_2 \neq y_2$ such that $x_1 = \beta x_2$, $y_1 = \beta y_2$ and $\sqcup_2 x_2 = \alpha \sqcup_1 \beta x_2 = \alpha \sqcup_1 x_1 = \alpha \sqcup_1 y_1 = \alpha \sqcup_1 \beta y_2 = \sqcup_2 y_2$, a contradiction, because the Π'-regulator is reduced and thus $\sqcup_2$ is one-to-one. Evidently $\{\sqcup_1 y : y \in \mathfrak{R}_1\} = \{\sqcup_2 x : x \in \mathfrak{R}_2\} = m\mathcal{P}(G)$ (because α carries $m\mathcal{P}(G)$ onto itself). Define γ (definition 4.5) as a mapping which maps $x \in \mathfrak{R}_2$ onto the element $y \in \mathfrak{R}_1$ fulfilling $\sqcup_1 y = \sqcup_2 x$. This mapping is a bijection and establishes the equality of both regulators.

From theorem 1.4 it follows that $\sqcup x$ $(x \in \mathfrak{R})$ is a minimal prime subgroup whenever a regulator $(\mathfrak{R}, \sqcup)$ is completely regular and from [4] III 7.2 or [1] 3.4.15 that the set $\{\sqcup x \colon x \in \mathfrak{R}_{\Pi'}\}$ is formed by all the minimal prime subgroups of G. Thus the Π'-regulator has a special position among the completely regular regulators. In what follows we shall study relationships of the completely regular regulators to their distinguished representative $\mathfrak{R}_{\Pi'}$.

4.9 Lemma. *Let $(\mathfrak{R}, \sqcup)$ be a regulator of $G \neq \{0\}$, $\emptyset \neq \mathfrak{R}_1 \subseteq \mathfrak{R}$ and $\sqcup_1 = \sqcup|_{\mathfrak{R}_1}$. Then $(\mathfrak{R}_1, \sqcup_1)$ is a regulator of G iff $\mathfrak{R}_1$ is a dense subset of the space $(\mathfrak{R}, G)$.*

Proof. "Only if". It suffices to prove that $\cap\{\sqcup_1 x \colon x \in \mathfrak{R}_1\} = \{0\}$. If $f \in G$ belongs to the above meet, then $x \in Z_{\mathfrak{R}_1}(f)$ for every $x \in \mathfrak{R}_1$, hence $\mathfrak{R}_1 = Z_{\mathfrak{R}_1}(f) = Z_{\mathfrak{R}}(f) \cap \mathfrak{R}_1$, whence $Z_{\mathfrak{R}}(f) \supseteq \mathfrak{R}_1$. Then $\mathfrak{R} \supseteq Z_{\mathfrak{R}}(f) = \mathrm{cl}_{(\mathfrak{R}, G)} Z_{\mathfrak{R}}(f) \supseteq \mathrm{cl}_{(\mathfrak{R}, G)} \mathfrak{R} = \mathfrak{R}$ Hence $Z_{\mathfrak{R}}(f) = \mathfrak{R}$, $f = 0$, [6] 2.3.

"If". By supposition $\cap\{\sqcup_1 x \colon x \in \mathfrak{R}_1\} = \{0\}$. Then for an arbitrary $0 \neq f \in G$ there exists $x_1 \in \mathfrak{R}_1$ such that $x_1 \bar{\in} Z_{\mathfrak{R}}(f)$. For an arbitrary $x \in \mathfrak{R}$ there exists $g \in G$ such that $x \bar{\in} Z_{\mathfrak{R}}(g)$. Since $Z(|f| \vee |g|) = Z(f) \cap Z(g)$, there holds $x, x_1 \in \mathfrak{R} \setminus Z_{\mathfrak{R}}(|f| \vee |g|)$.

4.10 Theorem. *Let $(\mathfrak{R}_1, \sqcup_1)$ be the Π'-regulator of an l-group G and $(\mathfrak{R}_2, \sqcup_2)$ a regulator of G. Then $(\mathfrak{R}_2, \sqcup_2)$ is completely regular (completely regular and reduced) iff a dense subset $\mathfrak{R}_3$ of the space $(\mathfrak{R}_1, G)$ exists such that the regulator $(\mathfrak{R}_2, \sqcup_2)$ is similar (equivalent) to the regulator $(\mathfrak{R}_3, \sqcup_1|_{\mathfrak{R}_3})$.*

Proof. Let the mapping Z, concerning the regulator $(\mathfrak{R}_i, \sqcup_i)$ $(i = 1, 2)$, be denoted by the symbol Z_i and let $\sqcup_3 = \sqcup_1|_{\mathfrak{R}_3}$. Let $(\mathfrak{R}_2, \sqcup_2)$ be similar to $(\mathfrak{R}_3, \sqcup_3)$. Let α be an l-automorphism of G and $\beta \colon \mathfrak{R}_2$ onto $\mathfrak{R}_3$ a mapping fulfilling $\alpha \sqcup_3 \beta x = \sqcup_2 x$ for every $x \in \mathfrak{R}_2$. Since $\mathfrak{R}_3 \subseteq \mathfrak{R}_1$ and for every $x \in \mathfrak{R}_3$ $\sqcup_3 x = \sqcup_1 x \in \mathrm{m}\mathscr{P}(G)$, there also holds $\sqcup_2 x \in \mathrm{m}\mathscr{P}(G)$ for every $x \in \mathfrak{R}_2$. Thus by 1.4 the regulator $(\mathfrak{R}_2, \sqcup_2)$ is completely regular.

Conversely, let $(\mathfrak{R}_2, \sqcup_2)$ be completely regular. By 1.4 $\{\sqcup_1 x \colon x \in \mathfrak{R}_1\} = \mathrm{m}\mathscr{P}(G) \supseteq \{\sqcup_2 y \colon y \in \mathfrak{R}_2\}$. For an arbitrary $y \in \mathfrak{R}_2$ there exists exactly one $x \in \mathfrak{R}_1$ such that $\sqcup_2 y = \sqcup_1 x$. We define a mapping $\beta \colon \mathfrak{R}_2$ into $\mathfrak{R}_1$ by the rule $\beta y = x$. Denote $\mathfrak{R}_3 = \beta \mathfrak{R}_2 (\subseteq \mathfrak{R}_1)$. We shall prove that $\mathfrak{R}_3$ is a dense set of the space $(\mathfrak{R}_1, G)$, i.e. $\mathrm{cl}_{(\mathfrak{R}_1, G)} \mathfrak{R}_3 = \mathfrak{R}_1$. $\mathrm{cl}_{(\mathfrak{R}_1, G)} \mathfrak{R}_3$ is the meet of all $Z_1(f)$ $(f \in G)$ which contain $\mathfrak{R}_3$. Thus let $Z_1(f) \supseteq \mathfrak{R}_3$ hold for some $f \in G$. Then $f \in \cap\{\sqcup_2 y \colon y \in \mathfrak{R}_2\} = \{0\}$, whence $f = 0$ and $\mathrm{cl}_{(\mathfrak{R}_1, G)} \mathfrak{R}_3 = \mathfrak{R}_1$. The similarity of $(\mathfrak{R}_2, \sqcup_2)$ to the regulator $(\mathfrak{R}_3, \sqcup_3)$ is given by means of the mappings $\alpha = \mathrm{id}_G$ and β defined above. In fact, for an arbitrary $y \in \mathfrak{R}_2$, $\alpha \sqcup_3 \beta y = \sqcup_3 \beta y = \sqcup_1 \beta y = \sqcup_2 y$.

Proof of the second assertion of the theorem. If $(\mathfrak{R}_2, \sqcup_2)$ is equivalent to $(\mathfrak{R}_3, \sqcup_3)$, then by 4.4 the regulator $(\mathfrak{R}_2, \sqcup_2)$ is reduced because the regulator $(\mathfrak{R}_3, \sqcup_3)$ is reduced. Conversely, if the regulator $(\mathfrak{R}_2, \sqcup_2)$ is completely regular and reduced, the mapping β defined in the previous paragraph is one-to-one, because the mapping $\bigcup_2$ is one-to-one (indeed, if $y_1, y_2 \in \mathfrak{R}_2$, $\beta y_1 = \beta y_2$, then $\bigcup_2 y_1 = \bigcup_1 \beta y_1 = \bigcup_1 \beta y_2 = \bigcup_2 y_2$, thus $y_1 = y_2$).

4.11 Corollary. *Every completely regular regulator of G is similar to a reduced completely regular regulator of G.*

Proof. Using the notation of the preceding theorem the completely regular regulator $(\mathfrak{R}_2, \bigcup_2)$ is similar to the reduced completely regular regulator $(\mathfrak{R}_3, \bigcup_3)$ (4.4(1)). In fact, $(\mathfrak{R}_3, \bigcup_3)$ is reduced because the Π'-regulator is reduced and is completely regular by 1.4.

4.12 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a completely regular regulator of G. If $(\mathfrak{R}, G)$ is compact, then $(\mathfrak{R}, \bigcup)$ is similar to the Π'-regulator. If, moreover, $(\mathfrak{R}, \bigcup)$ is reduced, then it is equal to the Π'-regulator.*

Proof. Let $(\mathfrak{R}_1, \bigcup_1)$ be the Π'-regulator of G. By 1.12, $\Pi = \Pi'$ and by 2.6 the space $(\mathfrak{R}_1, G)$ is compact. By 4.10, there exists a dense subset $\mathfrak{R}_3$ of the space $(\mathfrak{R}_1, G)$ such that the regulator $(\mathfrak{R}, \bigcup)$ is similar (in the other case equivalent) to the regulator $(\mathfrak{R}_3, \bigcup_3)$, where $\bigcup_3 = \bigcup_1 |_{\mathfrak{R}_3}$. By 4.2 the space $(\mathfrak{R}_3, G)$ is a continuous image of a compact space, hence $(\mathfrak{R}_3, G)$ is a compact subspace of a Hausdorff space $(\mathfrak{R}_1, G)$. Then $\mathfrak{R}_3$ is a closed subset of $(\mathfrak{R}_1, G)$. Since $\mathfrak{R}_3$ is dense, $\mathfrak{R}_3 = \mathfrak{R}_1$. Hence both assertions are valid.

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Received November 23, 1979

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ТОПОЛОГИИ НА РЕГУЛЯТОРАХ СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУПП II. ВПОЛНЕ РЕГУЛЯРНЫЕ РЕГУЛЯТОРЫ

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Резюме

В работе продолжается изучение отношений между свойствами l-группы G и индуцированным на $\mathfrak{R}$ топологическим пространством $(\mathfrak{R}, G)$, где $(\mathfrak{R}, \bigcup)$ — регулятор в G (см. часть I.). Главное внимание посвящено понятию вполне регулярного регулятора $(\mathfrak{R}, \bigcup)$, который определен следующим образом: $f \in G$, $x \in \mathfrak{R}$, $f \in \bigcup x \Rightarrow$ существует $g \in G$ так, что $g \bar{\in} \bigcup x$, $f \delta g$ (где $f \delta g$ обозначает дизъюнктивность элементов f и g, то есть $|f| \wedge |g| = 0$). Эти регуляторы характеризованы как регуляторы, образованные минимальными простыми подгруппами в G и $\bigcup = \mathrm{id}_{\mathfrak{R}}$ (1.2 и 1.4). Отображения Z и Ψ являются (взаимно обратными) дуальными изоморфизмами между структурой $\mathcal{O}(\mathfrak{R}, G)$ открытых и замкнутых множеств в $(\mathfrak{R}, G)$ и структурой $\Gamma(\mathfrak{R}, G)$ всех поляр K в G, обладающих следующим свойством: если $x \in \mathfrak{R}$, то $\bigcup x$ не содержит одновременно K и K' (2.2). Если $(\mathfrak{R}, \bigcup)$ — Π'-регулятор, то компактность пространства $(\mathfrak{R}, G)$ эквивалентна тому, что Z и Ψ являются дуальными изоморфизмами между структурой $\Pi(G)$ всех главных поляр в G и структурой $\mathcal{O}_c(\mathfrak{R}, G)$ всех компактных открытых и замкнутых множеств пространства $(\mathfrak{R}, G)$; другие эквивалентные условия: $\mathcal{O}(\mathfrak{R}, G) = \mathcal{O}_c(\mathfrak{R}, G)$; $\Pi(G) = \Pi'(G)$ (2.6). В части I. (2.22) приведенная серия условий, эквивалентных экстремальной несвязности пространства $(\mathfrak{R}, G)$, дополняется здес напр. следующими условиями: $\mathcal{O}(\mathfrak{R}, G) = \mathfrak{M}(\mathfrak{R}, G)$; $\Psi[\mathcal{O}(\mathfrak{R}, G)] = \Gamma(G)$ (2.5). Если $(\mathfrak{R}, \bigcup)$ — стандартный $\mathscr{C}$-регулятор ($\equiv$всякая выпуклая l-подгруппа в G является пересечением некоторой системы $\bigcup x$), тогда $\Gamma(\mathfrak{R}, G)$ — множество всех прямых факторов в G (3.4). Определены равенство, подобие и эквивалентность регуляторов и исследуется вопрос, какое из этих отношений сохраняет полную регулярность или редуцированность. Особенно изучено отношение регуляторов к самому содержательному редуцированному и вполне регулярному регулятору, к Π'-регулятору (абз. 4).