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*Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday*

# BL-ALGEBRAS AND QUANTUM STRUCTURES

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ABSTRACT. Endowing a lower-bounded partially ordered set with a total addition or with a total difference operation leads to the basic notion of a NAM, that is, a naturally ordered abelian monoid, or a BCK-algebra, respectively.

BL-algebras may be alternatively viewed as certain NAMs or certain BCK-algebras. We characterize the appropriate subclasses by making use of those properties which have been so far considered in an apparently rather different context, namely for certain quantum structures.

The three most important subclasses of BL-algebras, MV-, product, and Gödel algebras, are also taken into account.

## 1. Introduction

Basic Logic ([Háj1], [Got]), introduced by Hájek several years ago, aims at formalizing in a quite general manner statements of fuzzy nature. It is a calculus of propositions which are true principally only to a certain degree, that is, to which in general no sharp yes or no is assigned.

The Lindenbaum algebras of the theories of Basic Logic are the BL-algebras. BL-algebras are a certain type of residuated lattices, and they have been examined in several papers; see e.g. [Háj2], [Tur], [Höh].

On the other hand, algebras of various kinds have been examined in the last decades in connection with foundational questions about the formalism of quantum mechanics ([DvPu]). Among these algebras, which are in general referred to

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as quantum structures, we find for instance effect algebras ([FoBe]), MV-algebras ([CiOtMu]) and BCK-algebras ([MeJu]). We find that BL-algebras are relatively closely related to the quantum structures: BL-algebras generalize MV-algebras, which in turn are special cases of effect algebras; furthermore, BL-algebras form a subclass of the BCK-algebras.

Now, since the structure theory of most quantum structures is a rather difficult matter, several of their subclasses have been considered, which are somewhat more convenient to handle. For example, the Riesz decomposition property was introduced for effect algebras ([Rav]), the relative cancellation property was defined for BCK-algebras ([DvGr]), and based on these notions, it was in both cases possible to prove *po*-group representation theorems.

Whereas the mentioned special properties play basically no role in connection with the original motivation to study quantum structures, we see that they now naturally appear in a different context — in the context of fuzzy logic. To see that some of them are actually the characteristic properties of BL-algebras among certain very basic types of algebras, this is the aim of the present paper.

We proceed as follows. BL-algebras have a conjunction-like and an implication-like operation, and each of these two basic operations is definable from the other one. We give axiomatizations with respect to the conjunction only (Section 3) as well as with respect to the implication only (Section 4). In both cases, we start from an appropriate general type of algebra — one with a total addition, one with a total difference.

So first, we shall view BL-algebras as special bounded NAMs, where a NAM is just meant to be an abelian monoid ordered in the natural manner; compare also [Küh]. Second, we consider BL-algebras as special bounded BCK-algebras; compare also [Ior]. Now, the properties which single out BL-algebras among both types of structures, are those of the mentioned kind: the Riesz decomposition property, the property of being mutually compatible, the relative cancellation property.

We moreover see to which subclasses of NAMs and of BCK-algebras MV-, PL-, and G-algebras correspond, which are the Lindenbaum algebras of the Łukasiewicz, the product, and the Gödel logic, respectively.

Finally, we briefly discuss (in Section 5) the categorical-theoretic question connected with the transition from BL-algebras to algebras which are based on one basic operation instead of two.

## 2. BL-algebras reviewed

BL-algebras are the Lindenbaum algebras of Hájek's Basic logic. Their axioms have been chosen in accordance to the nature of this logic — usually more or less in the following manner.

**DEFINITION 2.1.** A *residuated lattice* is a structure $(L; \leq, \odot, \Rightarrow, 0, 1)$ such that the following holds.

(RL1) $(L; \leq)$ is a lattice with a smallest element $0$ and a largest element $1$.

(RL2) $(L; \odot, 1)$ is a commutative monoid, that is, $\odot$ is a associative and commutative binary operation, and $1$ is a neutral element with respect to $\odot$.

(RL3) $\odot$ is isotone, that is, for any $a, b, c \in L$, $a \leq b$ implies $a \odot c \leq b \odot c$.

(RL4) For any $a, b \in L$, $a \Rightarrow b$ is the maximal element $x$ such that $a \odot x \leq b$.

Furthermore, a residuated lattice is called a *BL-algebra* under the following conditions.

(BL1) $a \wedge b = a \odot (a \Rightarrow b)$ for $a, b \in L$.

(BL2) $(a \Rightarrow b) \vee (b \Rightarrow a) = 1$ for $a, b \in L$.

We note that also the supremum of pairs of elements of a BL-algebra $L$ is definable from $\odot$ and $\Rightarrow$,

$$a \vee b = \big((a \Rightarrow b) \Rightarrow b\big) \wedge \big((b \Rightarrow a) \Rightarrow a\big) \qquad \text{for} \quad a, b \in L,$$

and that the lattice $(L; \wedge, \vee)$ is a distributive one.

The following proposition is meant to show that BL-algebras, when not considered in connection with fuzzy logic, are not as unnatural objects as they seem to be at first sight.

**DEFINITION 2.2.** Let $(L; \leq, \odot, \Rightarrow, 0, 1)$ be a residuated lattice. We say that

(a) $\odot$ is *compatible with the lattice operations* if for any $a \in L$ the mapping $L \to L$, $x \mapsto a \odot x$, is a lattice homomorphism.

(b) $\Rightarrow$ is *compatible with the lattice operations* if for any $a \in L$

($\alpha$) the mapping $L \to L$, $x \mapsto a \Rightarrow x$, is a lattice homomorphism and

($\beta$) the mapping $L \to L$, $x \mapsto x \Rightarrow a$, is a homomorphism of the lattice $L$ to its dual.

(c) $L$ is *divisible* if for any $a, b \in L$, $a \leq b$ holds if and only if $a = b \odot x$ for some $x \in L$.

**PROPOSITION 2.3.** *A residuated poset* $(L; \leq, \odot, \Rightarrow, 0, 1)$ *is a BL-algebra if and only if* $\odot$ *and* $\Rightarrow$ *are compatible with the lattice operations and* $L$ *is divisible.*

P r o o f . By (BL2), the conditions concerning the lattice compatibilities hold e.g. according to [Höh; Propositions 2.1, 2.3]. By [Höh; Lemma 2.5], the divisibility is equivalent to (BL1). And from the compatibility of $\Rightarrow$ with $\wedge$, (BL2) follows. □

Note that Proposition 2.3 in particular implies that the operations $\odot$ and $\Rightarrow$ are, with respect to any one of its arguments, isotone or antitone, respectively.

The importance of BL has apparently much to do with the fact that it possesses three extensions well-known from fuzzy logics: the Łukasiewicz, product, and Gödel logic. The algebraic counterparts of these logics are the MV-, PL-, and G-algebras, respectively.

For the original definitions of these algebras and for further details, we refer to [CiOtMu], [Háj1], and [Göd], respectively. Here we will, following the lines of [Háj1], consider MV-, PL-, and G-algebras as subclasses of the BL-algebras.

**DEFINITION 2.4.** Let $L$ be a BL-algebra. Let

$$a^\star \stackrel{\text{def}}{=} a \Rightarrow 0 \qquad \text{for} \quad a \in L$$

be the *complement* of $a$.

(i) $L$ is called an *MV-algebra* if we have:
  (MV) The complement operation is involutive, that is, $a^{\star\star} = a$ for all $a \in L$.

(ii) $L$ is called a *PL-algebra* if we have:
  (PL1) $a^{\star\star} \leq (a \odot b \Rightarrow a \odot c) \Rightarrow (b \Rightarrow c)$ for any $a, b, c \in L$.
  (PL2) $a \wedge a^\star = 0$ for $a \in L$.

(iii) $L$ is called a *G-algebra* if we have:
  (G) $a \odot b = a \wedge b$ for any $a, b \in L$.

Now, BL-algebras are lattices endowed with operations which are modelled upon a logical conjunction and implication. In what follows, we prefer to work with algebras possessing an addition-like and a difference-like operation; namely, what we will consider throughout the text are the duals of BL-algebras rather than BL-algebras themselves. This makes it more convenient to compare BL-algebras with algebras known from other fields like e.g. quantum structures.

**DEFINITION 2.5.** Let $(L; \leq_{\text{BL}}, \odot, \Rightarrow, 0_{\text{BL}}, 1_{\text{BL}})$ be a BL-algebra. Then $(L; \leq, \oplus, \ominus, 0, 1)$ is called the *dual* of $L$, where for $a, b \in L$

$$a \leq b \stackrel{\text{def}}{\Longleftrightarrow} b \leq_{\text{BL}} a\,,$$

$$a \ominus b \stackrel{\text{def}}{=} b \Rightarrow a\,, \qquad a \oplus b \stackrel{\text{def}}{=} a \odot b\,,$$

$$0 \stackrel{\text{def}}{=} 1_{\text{BL}}\,, \qquad 1 \stackrel{\text{def}}{=} 0_{\text{BL}}\,.$$

We see that the transition from a BL-algebra to its dual may be considered just as a change of notation. In particular, all statements about BL-algebras are easily reformulated for duals of BL-algebras. This is to be kept in mind in the sequel, where some results are formulated for duals of BL-algebras, but are actually to be understood as statements about BL-algebras.

**Remark 2.6.** The duals of BL-algebras are exactly the bounded DRl-monoids with the property $(a \ominus b) \wedge (b \ominus a) = 0$ for any pair $a, b$; see e.g. [Küh]. DRl-monoids have been introduced by Swamy; their basic properties may be found in [Swa].

## 3. BL-algebras as naturally ordered abelian monoids

Since the two operations $\odot$ and $\Rightarrow$ of a BL-algebra $(L; \leq, \odot, \Rightarrow, 0, 1)$ are definable from each other, one may wonder if it is not possible to axiomatize in a reasonable manner the reducts $(L; \leq, \odot, 0, 1)$ and $(L; \leq, \Rightarrow, 0, 1)$ such that there is a unique expansion to a BL-algebra. We propose here such an axiomatization, using mainly those properties which are known from related types of algebras and in particular from quantum structures.

We consider in this section the conjunction-like operation $\odot$. So with respect to the dual algebra $(L; \leq, \oplus, \ominus, 0, 1)$, given according to Definition 2.5, we have to characterize the structure $(L; \leq, \oplus, 0, 1)$.

We are given an abelian monoid which is endowed in the natural way with a partial order. Since this type of algebra arises frequently, we shall use an own term for it. The abbreviation "NAM" was chosen in analogy to the term "PAM", which stands for "partial ordered monoid" ([GuPu]).

**Definition 3.1.** A *naturally ordered abelian monoid*, or *NAM* for short, is a structure $(L; \leq, \oplus, 0)$ with the following properties:

(NAM1) $(L; \leq, 0)$ is a poset with a smallest element $0$.

(NAM2) $\oplus$ is a binary operation such that for any $a, b, c \in L$
- (a) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
- (b) $a \oplus 0 = a$;
- (c) $a \oplus b = b \oplus a$.

(NAM3) We have $a \leq b$ for $a, b \in L$ if and only if $a \oplus x = b$ for some $x \in L$.

Furthermore, a structure $(L; \leq, \oplus, 0, 1)$ is called a *bounded NAM* if $(L; \leq, \oplus, 0)$ is a NAM with a largest element $1$.

Now, duals of BL-algebras will prove to be special bounded NAMs; for their exact characterization the following properties are needed.

**DEFINITION 3.2.** Let $(L; \leq, \oplus, 0, 1)$ be a bounded NAM.

(i) We say that $L$ has the *difference property* if for any $a, b \in L$ such that $a \leq b$ there is a smallest element $x \in L$ such that $a \oplus x = b$.

(ii) We say that $L$ has the *Riesz Decomposition Property*, or (RDP) for short, if for any $a, b, c \in L$ such that $c \leq a \oplus b$ there are $a_1 \leq a$ and $b_1 \leq b$ such that
  ($\alpha$) $c = a_1 \oplus b_1$
  and
  ($\beta$) $a_1 = a$ in case $c \geq a$.

(iii) We say that $L$ is *mutually compatible* if for any $a, b \in L$ there are $a_1, b_1, c \in L$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$, and $a_1 \wedge b_1 = 0$.

Moreover, we say that a bounded NAM is *of type BL* if it has the difference property, if it fulfils (RDP), and if it is mutually compatible.

Let us briefly comment these definitions. Roughly speaking, condition (i) states that there is the shortest distance between any pair of comparable elements. Note that the difference property alone does not imply a residuation property analogous to (RL4). Furthermore, (RDP) corresponds to the equally denoted property of effect algebras ([Rav], [DvVe]), although in the case of the latter algebras, the requirement ($\beta$) is superfluous. Finally, two compatible elements of an effect algebra formally fulfil the requirements of (iii).

**THEOREM 3.3.** *Let* $(L; \leq, \oplus, \ominus, 0, 1)$ *be the dual of a BL-algebra. Then* $(L; \leq, \oplus, 0, 1)$ *is a bounded NAM of type BL.*

*Conversely, let* $(L; \leq, \oplus, 0, 1)$ *be a bounded NAM of type BL. Then L may be expanded uniquely to the dual of a BL-algebra* $(L; \leq, \oplus, \ominus, 0, 1)$.

Proof. Let $(L; \leq, \oplus, \ominus, 0, 1)$ be the dual of a BL-algebra. By (RL1), (NAM1) holds and 1 is the largest element. By (RL2), (NAM2) holds. By (RL3), we have $a = a \oplus 0 \leq a \oplus x$ for any $a, x \in L$, which is one half of (NAM3). If $a \leq b$ for some $a, b \in L$, then by (RL4) $b \ominus a$ is the smallest element $x$ such that $a \oplus x \geq b$, and by (BL1), we have $a \oplus (b \ominus a) = a \vee b = b$; so also the second half of (NAM3) as well as the difference property follows.

Assume now $c \leq a \oplus b$ for some $a, b, c \in L$. Set $a_1 = a \wedge c$ and $b_1 = c \ominus a_1$. Then $a_1 \leq a$ and, in view of Proposition 2.3 and (RL4), $b_1 = c \ominus (a \wedge c) = c \ominus a \leq (a \oplus b) \ominus a \leq b$; moreover, $a_1 \oplus b_1 = (c \ominus a_1) \oplus a_1 = c$ by (BL1). In case $c \geq a$, we have $a_1 = a$. So (RDP) is proved.

To see that $E$ is mutually compatible, let $a, b \in L$, and set $a_1 = a \ominus b$, $b_1 = b \ominus a$ and $c = a \wedge b$. We see by (BL1) and Proposition 2.3 that $a = (a \ominus (a \wedge b)) \oplus (a \wedge b) = (a \ominus b) \oplus (a \wedge b) = a_1 \oplus c$, and similarly $b = b_1 \oplus c$. Furthermore, $a_1 \wedge b_1 = 0$ holds by (BL2). So the proof that $L$ is a bounded NAM of type BL is complete.

Conversely, let $(L; \leq, \oplus, 0, 1)$ be a bounded NAM of type BL. By (NAM1) and the boundedness of $L$, $(L; \leq, 0, 1)$ is a bounded poset. To see that $L$ is a lattice, let $a, b \in L$, and let, according the mutual compatibility of $L$, $a_1, b_1, c \in L$ be such that $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \wedge b_1 = 0$. We claim that $c = a \wedge b$ and $d = a_1 \oplus b_1 \oplus c = a \vee b$.

By (NAM3), $c \leq a, b$. Let $x \leq a, b$; we shall show $x \leq c$; then it will be clear that $c = a \wedge b$. From $x \leq a = a_1 \oplus c$ we conclude by (RDP) and the difference property that $x = x_{a_1} \oplus x_c$ such that $x_{a_1} \leq a_1$, $x_c \leq c$ and $x_{a_1}$ is the minimal element summing up with $x_c$ to $x$. Choose $r \in L$ such that $c = x_c \oplus r$; then $x_c \leq x \leq b = x_c \oplus b_1 \oplus r$ implies by (RDP) that $x = x_c \oplus x'_{a_1}$ for some $x'_{a_1} \leq b_1 \oplus r$. By the minimality of $x_{a_1}$ we have $x_{a_1} \leq x'_{a_1}$, and from $x_{a_1} \leq b_1 \oplus r$ we conclude by (RDP) and $a_1 \wedge b_1 = 0$ that $x_{a_1} \leq r$. Thus $x = x_{a_1} \oplus x_c \leq r \oplus x_c = c$.

Furthermore, we have $d \geq a, b$. To see that $d$ is actually the supremum of $a$ and $b$, let $y \geq a, b$; we will show $y \geq d$. We have $c \leq b \leq y = a_1 \oplus c \oplus s$ for some $s$. From (RDP), we again conclude $b = c \oplus b'_1$ for some $b'_1 \leq a_1 \oplus s$, and since we may assume that $b_1$ was chosen as the minimal element summing up with $c$ to $b$, we have $b_1 \leq b'_1$, and we conclude by (RDP) $b_1 \leq s$. It follows that $y \geq d$.

This completes the proof of (RL1). (RL2) holds by (NAM2). (NAM3) implies that $\oplus$ is isotone; so also (RL3) holds.

Now, we see from (RL4) that there is maximally one function $\ominus$ with the property that $(L; \leq, \oplus, \ominus, 0, 1)$ is the dual of a BCK-algebra. In view of the difference property, we may for any $a, b \in L$ define $b \ominus a$ to be the smallest element $x$ such that $a \oplus x = a \vee b$.

By our definition of $\ominus$, we get $a \oplus (b \ominus a) = a \vee b$ for $a, b \in L$, which gives (BL1).

To see (RL4), we have to show that for any $a, b \in L$, $b \ominus a = \min\{x : a \oplus x \geq b\}$. We have $a \oplus (b \ominus a) = a \vee b \geq b$. Let now $x$ be such that $a \oplus x \geq b$; then we have $a \leq a \vee b \leq a \oplus x$, whence, by (RDP), $a \vee b = a \oplus x'$ for some $x' \leq x$. So by the definition of $\ominus$, $b \ominus a \leq x' \leq x$.

It remains to show (BL2), according to which for $a, b \in L$ we have $(a \ominus b) \wedge (b \ominus a) = 0$. So let again $a = a_1 \oplus c$, $b = b_1 \oplus c$ for some $a_1, b_1, c \in L$ such that $a_1 \wedge b_1 = 0$. We have seen above that then $a \vee b = a_1 \oplus b$, so $a_1 \geq a \ominus b$, and similarly, $b_1 \geq b \ominus a$. The claim follows. □

So given some bounded NAM of type BL $L$, it is the function $\ominus$ defined by

$$b \ominus a \stackrel{\text{def}}{=} \min\{x : a \oplus x = a \vee b\} \qquad \text{for any} \quad a, b \in L, \tag{1}$$

which makes $L$ the dual of a BL-algebra. We will as usual refer to $\ominus$ as the *residuum* of $L$. Note that in particular

$$(b \ominus a) \oplus a = a \vee b \qquad \text{for any} \quad a, b \in L. \tag{2}$$

Let us now turn to those well-known algebras which are frequently discussed in connection with BL-algebras: the MV-, PL-, and G-algebras. According to Definition 2.4, these three types of algebras form subclasses of the BL-algebras. Now, because, by Theorem 3.3, BL-algebras are in a one-to-one correspondence with bounded NAMs of type BL, we see that MV-, PL-, and G-algebras might be equivalently viewed as subclasses of the bounded NAMs. The respective characteristic properties of bounded NAMs are given in Proposition 3.4.

In what follows, we will denote the complement function on the dual of a BL-algebra $L$ by $^{\circledast}$, that is,

$$a^{\circledast} \stackrel{\text{def}}{=} 1 \ominus a \qquad \text{for} \quad a \in L .$$

**PROPOSITION 3.4.** *Let* $(L; \leq_{\mathrm{BL}}, \odot, \Rightarrow, 0_{\mathrm{BL}}, 1_{\mathrm{BL}})$ *be a BL-algebra, and let* $(L; \leq, \oplus, 0, 1)$ *be the corresponding bounded NAM according to Definition* 2.5 *and Theorem* 3.3.

(i) *$L$ as a BL-algebra is an MV-algebra if and only if $L$ as a bounded NAM has the following property:*

($\mathrm{MV}_\oplus$) *If $a \oplus b = c$ for some $a, b, c \in L$ such that $a$ is the smallest element $x$ fulfilling $x \oplus b = c$, then also $b$ is the smallest element $y$ fulfilling $a \oplus y = c$.*

(ii) *$L$ as a BL-algebra is a PL-algebra if and only if for $L$ as a bounded NAM the following holds:*

($\mathrm{PL}_\oplus 1$) *Let $a \in L$ be such that $a \oplus x = 1$ holds only for $x = 1$. Then, for any $b, c \in L$, $a \oplus b = a \oplus c$ implies $b = c$.*

($\mathrm{PL}_\oplus 2$) *If $a \oplus a \oplus b = 1$ for some $a, b \in L$, then $a \oplus b = 1$.*

(iii) *$L$ as a BL-algebra is a G-algebra if and only if for $L$ as a bounded NAM we have:*

($\mathrm{G}_\oplus$) *$a \oplus b = a \vee b$ for $a, b \in L$.*

P r o o f. Let us treat $L$ as a bounded NAM throughout the proof. Let $\ominus$ be the residuum of $L$.

(i) Let condition (MV) hold, that is, assume $a^{\circledast\circledast} = a$ for all $a \in L$. To prove ($\mathrm{MV}_\oplus$), assume $a \oplus b = c$ for $a, b, c \in L$, where $a$ is chosen minimal, that is, $a = c \ominus b$. We have to show that $b = c \ominus a$. Clearly, $b \geq c \ominus a$. From (RL4) we see that $c \ominus a \geq (c \oplus c^{\circledast}) \ominus (a \oplus c^{\circledast}) = 1 \ominus (a \oplus c^{\circledast})$; and since $b \leq c \leq 1$, we have $a \oplus c^{\circledast} = (1 \ominus c) \oplus (c \ominus b) = 1 \ominus b = b^{\circledast}$ by [Swa; Lemma 15]. So $c \ominus a \geq 1 \ominus b^{\circledast} = b^{\circledast\circledast} = b$, and it follows that $b = c \ominus a$.

Conversely, let ($\mathrm{MV}_\oplus$) hold, and let $a \in L$. Then $a \oplus (1 \ominus a) = 1$ implies that $a$ is minimal, that is, $a = 1 \ominus (1 \ominus a) = a^{\circledast\circledast}$, which is (MV).

(ii) Assume that (PL1) and (PL2) hold. To prove ($\mathrm{PL}_\oplus 1$), let $a, b, c \in L$ be such that $a \oplus x = 1$ for some $x \in L$ implies $x = 1$, and such that $a \oplus b = a \oplus c$.

Then $1 = \min\{x :\ a \oplus x = 1\} = a^{\circledast}$. According to (PL1) we have

$$(c \ominus b) \ominus \big((a \oplus c) \ominus (a \oplus b)\big) \leq a^{\circledast\circledast}\,. \tag{3}$$

It follows that $c \ominus b = 0$, whence, by (1), $c \leq b$. Similarly, we conclude $b \leq c$. Thus $b = c$, and ($\mathrm{PL}_{\oplus}1$) is proved.

Furthermore, by (PL2), (2) and [Swa; Lemma 6], $1 = a \vee a^{\circledast} = \big[(1 \ominus a) \ominus a\big] \oplus a = \big[1 \ominus (a \oplus a)\big] \oplus a = (a \oplus a)^{\circledast} \oplus a$ for any $a \in L$. Consequently, $a^{\circledast} \leq (a \oplus a)^{\circledast}$ and, since $^{\circledast}$ is antitone, even $a^{\circledast} = (a \oplus a)^{\circledast}$. Now, if $a \oplus a \oplus b = 1$ for some $b \in L$, then $b \geq (a \oplus a)^{\circledast} = a^{\circledast}$, so $a \oplus b = 1$. This proves ($\mathrm{PL}_{\oplus}2$).

Assume now ($\mathrm{PL}_{\oplus}1$) and ($\mathrm{PL}_{\oplus}2$) to hold. For any $a \in L$, we conclude from $a \oplus a \oplus (a \oplus a)^{\circledast} = 1$ by ($\mathrm{PL}_{\oplus}2$) that already $a \oplus (a \oplus a)^{\circledast} = 1$. According to the previous paragraph, $a \oplus (a \oplus a)^{\circledast} = a \vee a^{\circledast}$; so (PL2) holds.

Let now $a, b, c \in L$. We have to show (3). By (PL2) and the distributivity of the lattice order of $L$, we have $a = a_0 \vee a_1$, where $a_0 = a \wedge a^{\circledast}$ and $a_1 = a \wedge a^{\circledast\circledast} = a^{\circledast\circledast}$. Then, according to Proposition 2.3, we may write the expression on the left side of (3) as

$$\bigvee_{i=0,1} \bigwedge_{j=0,1} (c \ominus b) \ominus \big[(a_j \oplus c) \ominus (a_i \oplus b)\big]\,. \tag{4}$$

From (PL2) and [Tur; (12)] we get ${a_0}^{\circledast} = \big(a \wedge a^{\circledast}\big)^{\circledast} = a^{\circledast} \vee a^{\circledast\circledast} = 1$; so by ($\mathrm{PL}_{\oplus}1$), $a_0 \oplus \big((a_0 \oplus c) \ominus a_0\big) = a_0 \oplus c$ implies $(a_0 \oplus c) \ominus a_0 = c$. We conclude, by [Swa; Lemma 6], $(c \ominus b) \ominus \big[(a_0 \oplus c) \ominus (a_0 \oplus b)\big] = (c \ominus b) \ominus \big([(a_0 \oplus c) \ominus a_0] \ominus b\big) = 0$. Thus, in (4), the term $i = 0$ may be deleted.

Furthermore, we have $(c \ominus b) \ominus \big[(a_1 \oplus c) \ominus (a_1 \oplus b)\big] \leq a_1$, because $\big[(a_1 \oplus c) \ominus (a_1 \oplus b)\big] \oplus a_1 = \big[\big((a_1 \oplus c) \ominus b\big) \ominus a_1\big] \oplus a_1 \geq (a_1 \oplus c) \ominus b \geq c \ominus b$. Thus, in (4), the term $i = 1$ is smaller than $a_1$.

All in all, the term (4) is below $a_1$, and $a_1 = a^{\circledast\circledast}$, so (PL1) is proved.

(iii) This is evident. □

## 4. BL-algebras as BCK-algebras

We now consider the implication-like operation $\Rightarrow$: We will axiomatize BL-algebras on the base of this operation only. We will further work with the duals of BL-algebras $(L; \leq, \oplus, \ominus, 0, 1)$; so we shall characterize their reduct $(L; \leq, \ominus, 0, 1)$. Again, we will concentrate on those properties which have already been defined in other contexts.

In the previous case, when we restricted ourselves to the $\oplus$ operation, we had to do with a bounded poset endowed with a total addition determining the order. Now here, we are given a poset which is endowed with a total difference operation being connected to the order in a natural way.

A genuine difference operation – on some lower-bounded poset $(L; \leq, 0)$ should be defined for comparable elements only. If we do so, we are led to the notion of a *poset with a difference* ([KoCh]), the basic properties of which are $a - (a - b) = b$ for $b \leq a$, $(a - b) - (a - c) = c - b$ for $b \leq c \leq a$, and $a - 0 = a$. Now, the analogous notion for the case of a total difference are the BCK-algebras, as might be nicely seen from the axioms below.

BCK-algebras have been originally defined by Imai and Iséki; the basic reference is [MeJu]. The connection between BL- and BCK-algebras has been already pointed out in [Ior]; our characterization of the relevant subclass is, however, an alternative one.

In accordance to the nature of this article, the basic BCK-operation is denoted by $\ominus$ rather than the usual $\star$; and our operation $\oplus$ replaces the $\circ$ used elsewhere. Furthermore, the BCK-ordering $\leq$ is added as an own relation.

**DEFINITION 4.1.** A *BCK-algebra* is a structure $(L; \leq, \ominus, 0)$ such that

(BCK1) $(L; \leq, 0)$ is a poset with a smallest element $0$.

(BCK2) $\ominus$ is a binary operation such that for any $a, b, c \in L$
- (a) $a \ominus (a \ominus b) \leq b$;
- (b) $(a \ominus b) \ominus (a \ominus c) \leq c \ominus b$;
- (c) $a \ominus 0 = a$.

(BCK3) For any $a, b \in L$, $a \leq b$ if and only if $a \ominus b = 0$.

Furthermore, a structure $(L; \leq, \ominus, 0, 1)$ is called a *bounded BCK-algebra* if $(L; \leq, \ominus, 0)$ is a BCK-algebra with a largest element $1$.

BL-algebras may be understood as special BCK-algebras; for an exact characterization, we chose the following properties.

**DEFINITION 4.2.** Let $(L; \leq, \ominus, 0, 1)$ be a bounded BCK-algebra.

(i) $L$ is said to have the *addition property* if for any $a, b \in L$ there is a $c \in L$ such that for all $d \in L$ we have $(d \ominus a) \ominus b = d \ominus c$.

(ii) $L$ is said to be *strongly cancellative* if for any $a, b, c \in L$ such that $c \leq a, b$ we have

$$a \leq b \iff a \ominus c \leq b \ominus c .$$

(iii) $L$ is said to be *mutually compatible* if for any $a, b \in L$

$$(a \ominus b) \wedge (b \ominus a) = 0 .$$

Moreover, we say that a bounded BCK-algebra is *of type BL* if it has the addition property, if it is strongly cancellative, and if it is mutually compatible.

Here, the addition property is meant to ensure that an addition may be defined in a natural manner, in analogy to the difference property for NAMs. As outlined in [Ise], it is actually equivalent to the *condition* (S), which was introduced by Iséki:

(S) for any $a$ and $b$ there is a largest element $y$ fulfilling $y \ominus a \leq b$.

Furthermore, the strong cancellativity is a strengthened version of the *relative cancellation property*, which has been defined in [DvGr]; according to the latter, elements $a$ and $b$ such that $a \ominus c = b \ominus c$ for some $c \leq a, b$, are equal. Finally, the mutual compatibility is analogous to the equally denoted property for NAMs.

**THEOREM 4.3.** *Let* $(L; \leq, \oplus, \ominus, 0, 1)$ *be the dual of a BL-algebra. Then* $(L; \leq, \ominus, 0, 1)$ *is a bounded BCK-algebra of type BL.*

*Conversely, let* $(L; \leq, \ominus, 0, 1)$ *be a bounded BCK-algebra of type BL. Then* $L$ *may be expanded uniquely to the dual of a BL-algebra* $(L; \leq, \oplus, \ominus, 0, 1)$.

Proof. Let $(L; \leq, \oplus, \ominus, 0, 1)$ be the dual of a BL-algebra. Then $(L; \leq, \ominus, 0, 1)$ is a bounded BCK-algebra. This may be verified directly with the help of (RL4); or see [Ior].

Furthermore, given $a, b \in L$, we have, by [Swa; Lemma 6], $(d \ominus a) \ominus b = d \ominus (a \oplus b)$ for any $d \in L$, which proves the addition property.

Let now $a, b, c \in L$ such that $c \leq a, b$. By the isotonicity properties of $\ominus$ and $\oplus$ and (2), $a \leq b$ implies $a \ominus c \leq b \ominus c$, which in turn implies $a = (a \ominus c) \oplus c \leq (b \ominus c) \oplus c = b$. So $L$ is strongly cancellative.

Finally, we have by (BL2) that $E$ is mutually compatible. So $L$ is a bounded BCK-algebra of type BL.

Conversely, let $(L; \leq, \ominus, 0, 1)$ be a bounded BCK-algebra of type BL. It follows from (RL4) that there is maximally one function $\oplus$ making $L$ the dual of a BL-algebra. Given some $a, b \in L$, let us, in accordance with the addition property, define $a \oplus b$ to be the element $c$ such that $(d \ominus a) \ominus b = d \ominus c$ for all $d$; there is, by (BCK3), for every pair maximally one such element. Then, for any $a, b \in L$, $a \oplus b$ is the largest element $y$ such that $y \ominus a \leq b$. Indeed, we have $0 = (a \oplus b) \ominus (a \oplus b) = \big((a \oplus b) \ominus a\big) \ominus b$, whence $(a \oplus b) \ominus a \leq b$; and if $y \ominus a \leq b$ for some $y \in L$, then $0 = (y \ominus a) \ominus b = y \ominus (a \oplus b)$, whence $y \leq a \oplus b$.

(RL2), (RL3), and (RL4) hold by [MeJu; Theorems I.7.7, I.7.10].

We next show that $L$ is a lattice; then (RL1) follows. Let $a, b \in L$; we claim that $a \vee b = a \oplus (b \ominus a)$, which is also the content of (BL1). We have $a \oplus (b \ominus a) = \max\{y : y \ominus a \leq b \ominus a\} \geq a, b$. If $z \geq a, b$ for some $z \in L$ and furthermore $y \ominus a \leq b \ominus a$ for some $y$, then $z \ominus a \geq b \ominus a \geq y \ominus a$; under the assumption $y \geq a$ we conclude by the strong cancellativity $z \geq y$; so $z \geq a \oplus (b \ominus a)$.

Furthermore, we claim that $c = \big(b \ominus (b \ominus a)\big) \vee \big(a \ominus (a \ominus b)\big) = a \wedge b$. By (BCK2)(a), $c$ is a lower bound of $a$ and $b$. Assume $x \leq a, b$. It follows $x \leq a = a \vee (a \ominus b) = \big(a \ominus (a \ominus b)\big) \oplus (a \ominus b) \leq c \oplus (a \ominus b)$, which means $x \ominus c \leq a \ominus b$. Similarly, we see $x \ominus c \leq b \ominus a$, and since $L$ is mutually compatible, we get $x \ominus c = 0$, that is, $x \leq c$ by (BCK3).

(BL2) follows from the fact that $L$ is mutually compatible. This completes the proof that $L$ is the dual of a BL-algebra. □

So given some bounded BCK-algebra of type BL $L$, the function $\oplus$ is definable by

$$a \oplus b \overset{\text{def}}{=} \max\{y : \ y \ominus b \leq a\} \qquad \text{for any} \quad a, b \in L, \tag{5}$$

which makes $L$ the dual of a BL-algebra. We will refer to $\oplus$ as the *S-function* of $L$, its existence being the subject of condition (S).

**Remark 4.4.** We may also characterize the bounded BCK-algebras of type BL in the following way, which makes use (practically) only of known terms. Namely, BL-algebras are in a one-to-one correspondence to BCK-algebras which are

(i) bounded,
(ii) lattice-ordered,
(iii) such that $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$,
(iv) with condition (S),
(v) which fulfil the relative cancellation property.

In analogy to Proposition 3.4 in Section 3, we shall now see how MV-, PL-, and G-algebras may be understood as subclasses of BCK-algebras.

MV-, PL-, and G-algebras are, by Definition 2.4, special BL-algebras; so these three algebras may also be viewed, by Theorem 4.3, as special BCK-algebras. In the case of MV-algebras, this is a well-known fact ([Mun]). We further note that, in order to characterize MV- or G-algebras among the BCK-algebras, we may refer to standard terminology, recalled in the following definition ([MeJu]). This is apparently not the case for PL-algebras.

**Definition 4.5.** Let $(L; \leq, \ominus, 0)$ be a BCK-algebra.

(i) $L$ is called *commutative* if $a \ominus (a \ominus b) = b \ominus (b \ominus a)$ holds for any $a, b \in L$.

(ii) $L$ is called *positive implicative* if $(a \ominus c) \ominus (b \ominus c) = (a \ominus b) \ominus c$ holds for any $a, b, c \in L$.

We will again set $a^{\circledast} \overset{\text{def}}{=} 1 \ominus a$ for an element $a$ of a bounded BCK-algebra.

**PROPOSITION 4.6.** *Let* $(L; \leq_{\mathrm{BL}}, \odot, \Rightarrow, 0_{\mathrm{BL}}, 1_{\mathrm{BL}})$ *be a BL-algebra, and let* $(L; \leq, \ominus, 0, 1)$ *be the corresponding bounded BCK-algebra according to Theorem* 4.3.

(i) *$L$ as a BL-algebra is an MV-algebra if and only if $L$ as a bounded BCK-algebra is commutative if and only if for $L$ as a bounded BCK-algebra we have:*

($\mathrm{MV}_\ominus$) *For any $a, b \in L$, $a \leq b$ if and only if $a = b \ominus x$ for some $x \in L$.*

(ii) *$L$ as a BL-algebra is a PL-algebra if and only if for $L$ as a bounded BCK-algebra the following conditions hold:*

($\mathrm{PL}_\ominus 1$) *Let $a, b, c \in L$ such that $a^{\circledast} = 1$ and such that, for all $x \in L$, $x \ominus a \leq b$ if and only if $x \ominus a \leq c$. Then $b = c$.*

($\mathrm{PL}_\ominus 2$) $a \vee a^{\circledast} = 1$ *for $a \in L$.*

(iii) *$L$ as a BL-algebra is a G-algebra if and only if $L$ as a bounded BCK-algebra is positive implicative.*

P r o o f . Throughout this proof, we will treat $L$ as a bounded BCK-algebra. Let $\oplus$ be the S-function of $L$.

(i) Assume (MV), that is, let the complement $^{\circledast}$ be involutive. Then, for $a, b, c \in L$, $a, b \leq c$ and $c \ominus b \leq c \ominus a$ imply $a \leq b$. Indeed, from [MeJu; Theorem I.3.4] and $^{\circledast\circledast} = \mathrm{id}$ we have $b^{\circledast} \ominus c^{\circledast} = c \ominus b \leq c \ominus a = a^{\circledast} \ominus c^{\circledast}$. From $c^{\circledast} \leq a^{\circledast}, b^{\circledast}$ and the strong cancellativity, we conclude $b^{\circledast} \leq a^{\circledast}$ and thus $a \leq b$. Now, by [MeJu; Theorem I.5.6], this property implies the commutativity of $B$.

If the BCK-algebra $L$ is commutative, then again by [MeJu; Theorem I.5.6], we have $a = b \ominus (b \ominus a)$ whenever $a \leq b$, which proves ($\mathrm{MV}_\ominus$).

Assume now ($\mathrm{MV}_\ominus$). We know that $^{\circledast}$ is order-reversing and that, for $a \in L$, $a^{\circledast\circledast} \leq a$; thus $a^{\circledast\circledast\circledast} = a^{\circledast}$. Now, ($\mathrm{MV}_\ominus$) implies that $^{\circledast}$ is surjective; so $a = b^{\circledast}$ for some $b$. It follows $a^{\circledast\circledast} = b^{\circledast\circledast\circledast} = b^{\circledast} = a$, that is, (MV) holds.

(ii) (PL2) evidently coincides with ($\mathrm{PL}_\ominus 2$). Moreover, note that ($\mathrm{PL}_\ominus 1$) is equivalent to saying that for $a, b, c \in L$, $a^{\circledast} = 1$ and $a \oplus b = a \oplus c$ imply $b = c$, which is ($\mathrm{PL}_\oplus 1$). From Proposition 3.4(ii) and its proof it follows that, when assuming (PL2), ($\mathrm{PL}_\oplus 1$) is equivalent to (PL1).

(iii) The BL-algebra $L$ is a G-algebra if and only if $a \oplus b = a \vee b$ for any $a, b \in L$. By [MeJu; Theorem I.7.12], this condition holds if and only if $L$ is positive implicative. □

## 5. Categorical-theoretical aspects

As seen in this paper, BL-algebras, respectively duals of BL-algebras, still possess reasonable axiomatizations when we restrict to one of their two binary operations. Namely, we proved that BL-algebras and, for instance, bounded NAMs of type BL, are in a one-to-one correspondence, the latter being a reduct of the dual of the former structure.

What we certainly cannot expect is the equivalence of the appropriate categories; the respective congruences do not correspond to each other. Let us conclude this paper with a simple example illustrating that homomorphisms of bounded NAMs of type BL do not necessarily preserve the residuum.

EXAMPLE 5.1. Let $(\{0, \frac{1}{2}, 1\}; \leq, \oplus, 0, 1)$ and $(\{0, 1\}; \leq, \oplus, 0, 1)$ be the three- and two-element MV-chain, respectively, understood as bounded NAMs of type BL. Define $\varphi\colon \{0, \frac{1}{2}, 1\} \to \{0, 1\}$, $0 \mapsto 0$, $\frac{1}{2} \mapsto 1$, $1 \mapsto 1$. Then $\varphi$ preserves the order, $\oplus$, and the constants. But we have, with respect to the three-element algebra, $1 \ominus \frac{1}{2} = \frac{1}{2}$, whereas $\varphi(1) \ominus \varphi(\frac{1}{2}) = 0 \neq 1 = \varphi(\frac{1}{2})$. So $\varphi$ is homomorphism of bounded NAMs of type BL, but not of the corresponding duals of BL-algebras.

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