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Introduction

M. Kolibiar [8] has studied properties of lines in a lattice and he has shown that the Jordan-Hölder theorem for lines is true in modular lattices. E. Gedeonová [4] has investigated lines in nonmodular lattices and she has proved that the Jordan-Hölder theorem for lines is valid in a p-modular and semimodular lattice. The aim of this paper is to investigate properties of lines in directed distributive multilattices. The main result of the paper is Jordan-Hölder theorem for lines in a directed distributive multilattice. The method of this paper is a modification of the methods used in [8].

A multilattice [2] is a poset M in which the condition (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $v \geqq a$, $v \geqq b$, and (b) $z \in M$, $z \leqq v$, $z \geqq a$, $z \geqq b$ implies $z = v$.

Analogously as in [2] denote by $(a \vee b)_h$ the set of all elements $v \in M$ of (i). Let $(a \wedge b)_d$ have the dual meaning. Set

$$a \vee b = \bigcup_{\substack{a \leqq h \\ b \leqq h}} (a \vee b)_h , \qquad a \wedge b = \bigcup_{\substack{d \leqq a \\ d \leqq b}} (a \wedge b)_d .$$

Let A and B be nonempty subsets of M, then we define

$$A \vee B = \bigcup \{a \vee b \mid a \in A, b \in B\} , \qquad A \wedge B = \bigcup \{a \wedge b \mid a \in A, b \in B\} .$$

In the whole paper we write $[(a \vee x) \wedge (b \vee x)]_x = x$ $([(a \wedge x) \vee (b \wedge x)]_x = x)$ instead of $[(a \vee x) \wedge (b \vee x)]_x = \{x\}$ $([(a \wedge x) \vee (b \wedge x)]_x = \{x\})$.

A poset A is called directed if for each pair of elements $a, b \in A$ there exist elements $d, h \in A$ such that $d \leqq a$, $d \leqq b$ and $a \leqq h$, $b \leqq h$.

A multilattice M is distributive [2] iff for every a, b, b', d, $h \in M$ satisfying the conditions $d \leqq a \leqq h$, $d \leqq b \leqq h$, $d \leqq b' \leqq h$, $(a \vee b)_h = (a \vee b')_h = h$, $(a \wedge b)_d = (a \wedge b')_d = d$ we have $b = b'$.

Let M be a multilattice and N a nonempty subset of M. N is called a submultilattice [2] of M iff $N \cap (a \vee b)_h \neq \emptyset$ and $N \cap (a \wedge b)_d \neq \emptyset$ for every a, b, d, $h \in N$ satisfying $a \leqq h$, $b \leqq h$, $d \leqq a$, $d \leqq b$.

Recall the following definition and results from [7]:

Multilattices M and M' and said to be isomorphic (denoted as $M \sim M'$) if there exists a bijection f of M onto M' satisfying: $x \leqq y$ iff $f(x) \leqq f(y)$ $(x, y \in M)$.

Let M be the Cartesian product of two posets M_1, M_2. M is directed iff M_1 and M_2 are. M is a multilattice iff M_1 and M_2 are. For each element $x \in M$ denote by x_1, x_2 $(x_i \in M_i)$ its Cartesian coordinates. Then for all a, b, h, $v \in M$, $v \in (a \vee b)_h$ $(v \in (a \wedge b)_h)$ iff $v_i \in (a_i \vee b_i)_{h_i}$ $(v_i \in (a_i \wedge b_i)_{h_i})$ for $i = 1, 2$.

Properties of lines in directed distributive multilattices

Throughout the paper M and M' denote directed distributive multilattices. Let a, b, $x \in M$. We say that x is between a and b and write axb if

(r) $$[(a \wedge x) \vee (b \wedge x)]_x = x = [(a \vee x) \wedge (b \vee x)]_x .$$

Theorem A ([6, Theorem 1]). *Let M be a directed distributive multilattice, a, b, $x \in M$. Then (r) is equivalent with*

(s) $$(a \wedge x) \wedge (b \wedge x) \subset a \wedge b , \qquad (a \vee x) \vee (b \vee x) \subset a \vee b .$$

Analogously as in [6] denote by $B(a, b)$ the set of all elements $x \in M$ for which axb holds.

Theorem B ([6, Lemma 12]). *Let M be a directed distributive multilattice, $a, b \in M$. Then*

$$B(a, b) = \bigcup_{\substack{u \in a \wedge b \\ v \in a \vee b}} \langle u, v \rangle .$$

Lemma 1. *The relation (r) in M has the properties:*

(1) *xyz implies zyx,*
(2) *xyz and xzy iff $y = z$,*
(3) *xyz and xzu imply yzu,*
(4) *xyx implies $x = y$,*
(5) *xyz and xzu imply xyu.*

Proof. The assertions (1), (2), (3) follow from [6, Lemma 6, Lemma 14, Lemma 15]. We prove (4). By Theorem A, from xyx we get $(x \wedge y) \wedge (x \wedge y) \subset x \wedge x = x$, hence $x \in (x \wedge y) \wedge (x \wedge y)$ and $x \leqq y$. By duality we get $y \leqq x$, hence $x = y$. Proof of (5): By Theorem A, from xyz we get $(x \wedge y) \wedge (y \wedge z) \subset x \wedge z$ and $(x \vee y) \vee (y \vee z) \subset x \vee z$. Let us choose $u_1 \in (x \wedge y) \wedge (y \wedge z)$ and $v_1 \in (x \vee y) \vee (y \vee z)$. Obviously $u_1 \leqq y \leqq v_1$. Using Theorem A, from xzu we get

$$u_1 \wedge (z \wedge u) \subset (x \wedge z) \wedge (z \wedge u) \subset x \wedge u ,$$

$$v_1 \vee (z \vee u) \subset (x \vee z) \vee (z \vee u) \subset x \vee u .$$

Choose $u_2 \in u_1 \wedge (z \wedge u)$, $v_2 \in v_1 \vee (z \vee u)$. Then

$$u_2 \leqq u_1 \leqq y \leqq v_1 \leqq v_2 ,$$

hence $y \in \langle u_2, v_2 \rangle \subset B(x, u)$ (Theorem B).

Four different elements $a, b, c, d \in M$ form a pseudolinear quadruple [8] when they satisfy abc, bcd, cda, dab.

If A, B are subsets of some multilattices and a bijection φ from A onto B is given so that abc iff $\varphi(a)\varphi(b)\varphi(c)$, we say that A, B are b-equivalent. A subset A of M is called a line if there exists a chain that is b-equivalent to A. An element a is an end element of a line A, if $a \in A$ and for each two elements x, y of the line A, ayx or axy. Evidently, a chain in M is a line in M.

Lemma 2 (see [1]). *Let C be a chain in M. The relation (r) in C has the* property:

(6) xyz, yzu, $y \neq z$ imply xyu.

Remark. If A is a line in M, then A is b-equivalent with a chain, hence (6) holds in A.

Let A be a line in M. Two elements $a, b \in A$, $a \neq b$, are called neighbouring if $\{x \mid x \in A, axb\} = \{a, b\}$.

A lenght of a finite line is defined to be $n-1$, if n is a number of the elements of A.

A line $A \subset M$ is called connected when it has the following property: If $x \in M$ and if there exist elements $a, b \in A$ such that axb and $A \cup \{x\}$ is a line in M, then $x \in A$.

An interval $\langle a, b \rangle$ $(a<b)$ in M is called a prime interval, if $\langle a, b \rangle = \{a, b\}$.

Lemma 3. *Let $\varphi: M \to M'$ be a b-equivalence of M onto M'. Then the image of each prime interval in M is a prime interval in M'.*

Proof. Let $\langle a, b \rangle$ be a prime interval in M. Denote $\varphi(a) = a'$, $\varphi(b) = b'$. Since $B(a, b) = \{a, b\}$, we get $B(a', b') = \{a', b'\}$. If a', b' are incomparable, then there exists $v' \in a' \vee b'$, $v' \neq a'$, $v' \neq b'$ and $v' \in B(a', b')$, which is impossible. Hence the elements a', b' form a prime interval.

We shall use the following result from [1]:

Theorem C. *Let A be a set with a ternary relation axb satisfying the conditions* (1), (3), (4) *of Lemma 1,* (6) *of Lemma 2 and*

(7) *for each three elements $x, y, z \in A$ at least one of the relations xyz, yzx, zxy holds.*

Then there exists a partial order $\leqq$ on A in which xyz iff $x \leqq y \leqq z$ or $z \leqq y \leqq x$.

Theorem 1. *A subset A of a directed distributive multilattice is a line iff it satisfies the following conditions:*

(7) *from Theorem C and*

(8) *A does not contain any pseudolinear quadruple.*

Remark. The proof of this Theorem is a modification of the proof of [8, Theorem 2.1].

Proof. Let A be a line. Then A is b-equivalent to a chain. In the chain the condition (7) is valid, hence it is valid in A, too. Assume that A contains a pseudolinear quadruple a, b, c, d. From abc, bcd, $b \neq c$ we get adb (Lemma 2). Using (2), from dba, dab we get $a = b$, which is contradictory. Hence, if A is a line, then the conditions (7) and (8) are valid. Conversely, let A satisfy (7) and (8). We have to verify that the relation (r) in A satisfies the conditions (1), (3), (4), (6), (7) of Theorem C. Obviously, (1), (3), (4), (7) are valid in A. We shall prove the validity of (6). Let xyz, yzu, $y \neq z$. From (7) it follows that we get the following cases: 1. yux, 2. uxy, 3. xyu. In the first case from xyz and xuy we get uyz (use (3)). This and yzu implies $y = z$, which is contradictory. In the second case for x, z, u at least one of the relations zux, uxz, xzu holds by (7). If zux holds, then the elements x, y, u, z are not pairwise different by (8). Let $x = y$. Then from yzu and zuy we get $z = u$, and from this xyu, hence (6) is valid. Let $x = u$. Then from yzx and xyz we get $y = z$, which is contradictory. Let $x = z$. Then from zux we get xux and this implies $x = u$, which is contradictory. Let $z = u$, then from xyz we have xyu, hence (6) is valid. Let $y = u$, then from uxy we get uxu and this implies $x = u$, which is contradictory. If uxz holds, then from uxz, uzy we get xzy by (3) from Lemma 1. From xzy, xyz we have $y = z$, which is contradictory. If xzu holds, then from xzu and xyz we get xyu by (5) from Lemma 1.Hence (6) is valid. This completes the proof.

Corollary. *Let A be a line in M, $a, b \in A$, and let a, b be incomparable. Then there exists at most one element $v \in a \vee b$ and at most one element $u \in a \wedge b$, such that $u, v \in A$.*

Proof. Let $v_1, v_2 \in A$, $v_1, v_2 \in a \vee b$, $v_1 \neq v_2$. Then it holds av_1b, av_2b. From (7) it follows that for the elements a, v_1, v_2, we get the following cases: 1. av_1v_2, 2. av_2v_1, 3. v_1av_2. Analogously for b, v_1, v_2 we have one of the possibilities: 1'. bv_1v_2, 2'. bv_2v_1, 3'. v_1bv_2. In the first case from $a \leqq v_2$ we get $v_1 \leqq v_2$, which is impossible. The second case is analogous. Similarly we verify that neither bv_1v_2 nor bv_2v_1 can hold. Hence the elements a, b, v_1, v_2 satisfy av_1b, v_1bv_2, bv_2a, v_2av_1. We get a pseudolinear quadruple, which contradicts (8). The second part of the assertion is dual.

Theorem 2. *Each two neighbouring elements of a connected line A in a directed distributive multilattice M form a prime interval in M.*

Remark. The proof of this Theorem is a modification of the proof [8, Theorem 2.3].

Proof. Let a, b be neighbouring elements of a connected line A in M. Let $t \in M$ and let atb hold. We shall prove that the set $A \cup \{t\}$ is a line in M. Hence we have to verify:

(a) for each $x, y \in A$ at least one of the relations txy, xty, xyt is valid;

(b) if x, y, $z \in A$, then the elements x, y, z, t do not form a pseudolinear quadruple.

In view of the symmetry, for the elements a, b, x, $y \in A$ it sufficies to consider the following cases: 1. xab, yab; 2. xab, aby. In the first case it holds xya or yxa. If xya holds, then we get xyb by (6) of Lemma 2. From bay and bta we have bty by (5) of Lemma 1. The relations bty and byx imply tyx by (3) of Lemma 1. If yxa holds, then the proof is analogous. In the second case we get xay by (6) of Lemma 2. From aby and atb we have aty by (5) of Lemma 1. The relations yax and yta imply ytx by (5) of Lemma 1. Consequently (a) is proved.

Now we shall prove (b). Let the elements x, y, z, $t \in A$ form a pseudolinear quadruple. In view of the symmetry it suffices to consider the following cases for the elements $a, b, x, y, z \in A$: 1. xab, yab, zab; 2. xab, yab, zba; 3. xab, yba, zab.

In the first case we have either xza or zxa. Assume that xza holds. From xza and xyz we get xya by (5) of Lemma 1. The relations xya and xab imply xyb. From bay and bta we have bty. This and byx imply tyx. From tyx and txy it follows that $x = y$, which is contradictory. The case zxa is analogous.

In the second case we get bty from bta and bay. The relations ytb and yzt imply ztb. This and zba imply tba. From tba and bta we get $t = b$, which is contradictory.

In the third case we get aty from aby and atb. The relations bta, baz imply taz. This and tzy imply tay. From this and aty we get $a = t$, which is contradictory.

We have proved that $A \cup \{t\}$ is a line in M. Since A is connected, then $t \in A$. Because a, b are neighbouring in A, we get $a = t$ or $b = t$. By Theorem B we get that the elements a, b are comparable, hence they form a prime interval. Consequently Theorem 2 is proved.

Theorem 3. *Each line A in M is a submultilattice of M.*

Proof. Let A be a line in M, a, $b \in A$ and assume that there exists $h \in A$ such that $a \leqq h$, $b \leqq h$. Obviously $(a \vee b)_h \cap A \neq \emptyset$, if a, b are comparable. Let a, b be incomparable. At least one of the relations ahb, abh, bah holds. From ahb it follows that $h \in \langle u, v \rangle$, where $u \in a \wedge b$, $v \in a \vee b$ (Theorem B), hence $h \leqq v$, and $h = v$. Consequently $(a \vee b)_h \cap A \neq \emptyset$. In the case abh, from $a \leqq h$ we get $a \leqq b \leqq h$, which is impossible. Analogously bah is impossible. Hence $(a \vee b)_h \cap A \neq \emptyset$, if $a, b, h \in A$ $a \leqq h$, $b \leqq h$. The dual assertion can be proved analogously.

Lemma 4. *Let A be a line in M with end elements a, b. If $a < b$, then A is a chain in M.*

Proof. Let $x, y \in A$. From axb and ayb we get $a \leqq x \leqq b$, $a \leqq y \leqq b$. For the elements a, x, y one of the relations axy, ayx holds. From axy and ayb we get xyb, hence $x \leqq y \leqq b$. Analogously from ayx we get $y \leqq x \leqq b$. Hence x and y are comparable.

Lemma 5. *Let A be a finite connected line in M with end elements a, b. Then there exists an interval $\langle u, v\rangle$, $u \in a \wedge b$, $v \in a \vee b$ such that $A \subset \langle u, v\rangle$.*

Proof. The assertion is evident, if a and b are comparable. Let a, b be incomparable. We prove the assertion by induction with respect to the length of A. Since A is connected and its end elements are incomparable, then A has at least three elements. If A has three elements a, x, b, then we get the following cases: 1. $a \leqq x$, $x \geqq b$; 2. $a \geqq x$, $x \leqq b$. In the first case obviously $x \in a \vee b$ and $A \subset \langle u, x\rangle$ for an element $u \in a \wedge b$. The second case is dual. Now we assume that the assertion is true for lines having length $n-1$ $(n \geqq 3)$ and prove it for n. Let A have the length n $(n \geqq 3)$ and denote its elements $a = a_0, a_1, \ldots, a_{n-1}, a_n = b$, where a_i, a_{i+1} are neighbouring elements $(i = 1, \ldots, n-1)$. The elements $a_0, \ldots, a_{n-1}$ form a line with lenght $n-1$, hence there exist $u_1 \in a_0 \wedge a_{n-1}$ and $v_1 \in a_0 \vee a_{n-1}$ such that $A \subset \langle u_1, v_1\rangle$. Let $u_2 \in a_{n-1} \wedge a_n$, $v_2 \in a_{n-1} \vee a_n$. From $a_0 a_{n-1} a_n$ by Theorem A it follows that there exist $u \in u_1 \wedge u_2$, $u \in a_0 \wedge a_n$ and $v \in v_1 \vee v_2$, $v \in a_0 \vee a_n$ such that $A \subset \langle u, v\rangle$.

Jordan-Hölder Theorem for Lines

A subset $\{a, b, u, v\}$ of M is called an elementary quadruple if $u \in a \wedge b$, $v \in a \vee b$ and the intervals $\langle u, a\rangle$, $\langle u, b\rangle$, $\langle a, v\rangle$, $\langle b, v\rangle$ are prime intervals.

Lemma 6. *Let $\varphi: M \to M'$ be a b-equivalence. Then the image of an elementary quadruple in M is an elementary quadruple in M'.*

Proof. Let $a, b, u, v \in M$, $u \in a \wedge b$, $v \in a \vee b$ and $\{a, b, u, v\}$ be an elementary quadruple. Denote $x' = \varphi(x)$ for each $x \in M$. By Lemma 3 the images of $\langle u, a\rangle$, $\langle u, b\rangle$, $\langle a, v\rangle$ and $\langle b, v\rangle$ are prime intervals in M'. Now the assertion of the lemma follows immediately from [5, Lemma 5 and Lemma 6].

Let $\langle u, a\rangle$ and $\langle b, v\rangle$ be intervals of a directed distributive multilattice. The intervals $\langle u, a\rangle$ and $\langle b, v\rangle$ are called transposes if $u \in a \wedge b$ and $v \in a \vee b$. The intervals $\langle u, a\rangle$ and $\langle b, v\rangle$ are called projective if there exists a finite sequence of intervals $\langle u, a\rangle = \langle x_0, y_0\rangle$, $\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle = \langle b, v\rangle$ such that $\langle x_{i-1}, y_{i-1}\rangle$ and $\langle x_i, y_i\rangle$ are transposes for $i = 1, 2, \ldots, n$. The intervals $\langle x_i, y_i\rangle$ are called middle for $i = 1, 2, \ldots, n-1$.

From the paper [2] it follows that the following theorem is true.

Theorem D. *Let A, B be finite connected chains with end elements a, b in a directed distributive multilattice. Then the chains A, B have the same length and* *there exists a one-to-one mapping of the set of all prime intervals of the chain A onto the set of all prime intervals of the chain B such that the corresponding prime intervals are projective and the middle intervals* $\langle x_i, y_i \rangle$ *satisfy* $\langle x_i, y_i \rangle \subset \langle a, b \rangle$.

Lemma 7. *Let* $a_0, a_1, \ldots, a_{n+k} \in M$ *and* $a_0 < a_1 < \ldots < a_n$, $a_n > a_{n+1} > \ldots a_{n+k}$. *The elements* $a_0, a_1, \ldots, a_{n+k}$ *form a finite line with end elements* a_0, a_{n+k} *in M if and only if* $a_n \in a_0 \vee a_{n+k}$.

Proof. Let $a_n \in a_0 \vee a_{n+k}$. We prove that the elements $a_0, a_1, \ldots, a_{n+k}$ form a finite line in M. According to Theorem 1 we have to verify that the conditions (7) and (8) hold. First we prove the condition (7). Since the elements $a_0, a_1, \ldots, a_n$ form a chain, we get $a_i a_j a_m$ for $i<j<m$, $i=0, 1, \ldots, n-2$, $m=2, 3, \ldots, n$. Analogously we get $a_r a_s a_t$ for $r<s<t$, $r=n, n+1, \ldots, n+k-2$, $t=n+2, n+3, \ldots, n+k$. Let $i=0, 1, \ldots, n$, $j=n, n+1, \ldots, n+k$: Since $a_n \in a_0 \vee a_{n+k}$, then $a_n \in a_i \vee a_j$ and we have $a_i a_n a_j$. From this and $a_i a_p a_n$ $(i<p<n)$ we get $a_i a_p a_j$. The relations $a_j a_n a_i$, $a_j a_q a_n$ $(n<q<j)$ imply $a_j a_q a_i$. Hence we have proved that for each three elements $a_x, a_y, a_z \in A$, $x<y<z$, $x=0, 1, \ldots, n+k-2$, $z=2, 3, \ldots, n+k$ it holds $a_x a_y a_z$. Consequently the condition (7) is true. Now we prove the condition (8). Let the elements $a_x, a_y, a_z, a_w \in A$ form a pseudolinear quadruple $(x, y, z, w=0, 1, \ldots, n+k)$, hence we have $a_x a_y a_z$, $a_y a_z a_w$, $a_z a_w a_x$, $a_w a_x a_y$. In view of the symmetry it suffices to consider the following case: $x<y<z<w$. Then $a_x a_z a_w$. This and $a_x a_w a_z$ imply $a_z = a_w$, which is contradictory. We have proved that the elements $a_0, a_1, \ldots, a_{n+k}$ form a line in M. Evidently the elements a_0, a_{n+k} are end elements of this line. Conversely, we prove that $a_n \in a_0 \vee a_{n+k}$. Since $a_n > a_0$, $a_n > a_{n+k}$, $a_0 a_n a_{n+k}$ we get $a_n \in a_0 \vee a_{n+k}$ by [5, Lemma 2].

Lemma 8. *Let* $a_0, a_1, \ldots, a_{n+k} \in M$, $a_0 < a_1 < \ldots < a_n$, $a_n > a_{n+1} \ldots > a_{b+k}$ *and let these elements form a finite connected line A with end elements* a_0, a_{n+k} *in M. Let* $b_k \in M$, $b_k \in a_0 \wedge a_{n+k}$. *Then there exists a finite connected line B with end elements* a_0, a_{n+k}, *which has elements* $b_0 > b_1 > \ldots > b_k$, $b_k < b_{k+1} < \ldots < b_{k+n}$, $b_0 = a_0$, $b_{k+n} = a_{n+k}$, *such that the intervals* $\langle a_i, a_{i+1} \rangle$, $\langle b_{k+i}, b_{k+i+1} \rangle$ *are transposes and* $\langle a_{j+n}, a_{j+n+1} \rangle$, $\langle b_j, b_{j+1} \rangle$ *are transposes for* $i=0, 1, \ldots, n-1$, $j=0, 1, \ldots, k-1$.

Proof. Denote $a_0 = b_0$, $a_{n+k} = b_{k+n}$. By Lemma 7 $a_n \in a_0 \vee b_{k+n}$. By [5, Lemma 10] the intervals $\langle a_0, a_n \rangle$, $\langle b_k, b_{k+n} \rangle$ are isomorphic and there exist elements $b_{k+i} \in \langle b_k, b_{k+n} \rangle$ such that $b_{k+i} = (a_i \wedge b_{k+n})_{b_k}$ and $(a_0 \vee b_{k+i})_{a_n} = a_i$ for $i = 1, 2, \ldots, n-1$. Next it holds $b_{k+i} \in (a_i \wedge b_{k+i+1})_{b_k}$ and $a_{i+1} \in (a_i \vee b_{k+i+1})_{a_n}$. Consequently the intervals $\langle a_i, a_{i+1} \rangle$, $\langle b_{k+i}, b_{k+i+1} \rangle$ are transposes for $i = 1, 2, \ldots, n-1$. Analogously we get the elements $b_j \in \langle b_k, b_0 \rangle$ and the validity of the assertion "the intervals $\langle a_{j+n}, a_{j+n+1} \rangle$, $\langle b_j, b_{j+1} \rangle$ are transposes", for $j = 1, 2, \ldots, k-1$. From this and from Lemma 7 it follows that the elements $b_0, b_1, \ldots, b_{k+n}$ form the finite connected line B.

Remark. Evidently, the dual of Lemma 7 and the dual of Lemma 8 are valid too.

Lemma 9. *Let a, b be incomparable elements in M. Let A be a finite connected line with end elements a, b in M. Further we assume that $u, v \in M$, $u \in a \wedge b$, $v \in a \vee b$ such that $A \subset \langle u, v \rangle$. Then there exists a finite connected line B with end elements a, b, which has the elements $a = b_0 > b_1 > \ldots > b_k = u$, $u < b_{k+1} < b_{k+n} = b$ ($a = b_0 < b_1 < \ldots < b_k = v$, $v > b_{k+1} > \ldots > b_{k+n} = b$), and a one-to-one mapping of the set of all prime intervals of line A onto the set of all prime intervals of the line B such that the corresponding prime intervals are projective and the middle intervals* $\langle x_r, y_r \rangle$ satisfy ax_rb, ay_rb.

Proof. We can restrict our consideration to the following line A: $a = a_0 < a_1 < \ldots < a_{r_1}$, $a_{r_1} > a_{r_1+1} > \ldots > a_{r_2}$, $a_{r_2} < a_{r_2+1} < \ldots < a_{r_3}$, $\ldots$, $a_{r_{s-1}} < a_{r_{s-1}+1} < \ldots < a_{r_s}$, $a_{r_s} > a_{r_s+1} > \ldots > a_{r_{s+1}} = b$. (The proof is analogous, if the line A has another form.) The elements $a_{r_1}, a_{r_2}, \ldots, a_{r_s}$ are called edges. We prove the assertion of the lemma by induction with respect to the number of the edges in A. Let $s = 1$, then the assertion follows by Lemma 8. Now we assume that the assertion holds for $s = m - 1$ and we prove it for $s = m$. The elements $a = a_0 < a_1 < \ldots < a_{r_1}$, $a_{r_1} > a_{r_1+1} > \ldots > a_{r_2}$, $\ldots$, $a_{r_{m-1}} < a_{r_{m-1}+1} < \ldots < a_{r_m}$ form a connected line C, which has $m - 1$ edges $a_{r_1}, a_{r_2}, \ldots, a_{r_{m-1}}$. Evidently, $C \subset \langle u, v \rangle$. From this it follows that there exist $w \in (a \vee a_{r_m})_v$, $z \in (a \wedge a_{r_m})_u$ such that $C \subset \langle z, w \rangle$. Since C has $m - 1$ edges, there exists a finite connected line D: $a = d_0 > d_1 > \ldots > d_p = z$, $z < d_{p+1} < \ldots < d_{p+n} = a_{r_m}$ and a one-to-one mapping of the set of all prime intervals of the line C onto the set of all prime intervals of the line D such that the corresponding prime intervals are projective and the middle intervals $\langle x_t, y_t \rangle$ satisfy $ax_ta_{r_m}$, $ay_ta_{r_m}$ ($\langle x_t, y_t \rangle$ denotes an arbitrary middle interval of the projective corresponding intervals). Next it holds $a_{r_m} > b$, $a_{r_m} < v$, hence $a_{r_m} \in \langle b, v \rangle$. By [5, Lemma 10] we have that the intervals $\langle b, v \rangle$, $\langle u, a \rangle$ are isomorphic. The isomorphism described in this lemma and $z \in (a \wedge a_{r_m})_u$ imply $(z \vee b)_v = a_{r_m}$. By Lemma 7 we get that the elements $z < d_{p+1} < \ldots < a_{r_m}$, $a_{r_m} > a_{r_m+1} > \ldots a_{r_{m+1}} = b$ form a line E. Evidently, E is connected. By Lemma 8 we get that there exists a line F: $z > e_1 > \ldots > e_h = u$, $u < e_{h+1} < e_{h+2} < \ldots < e_{h+n} = b$ such that the intervals $\langle d_{p+i}, d_{p+i+1} \rangle$, $\langle e_{h+i}, e_{h+i+1} \rangle$ are transposes and $\langle a_{r_m+j}, a_{r_m+j+1} \rangle$, $\langle e_j, e_{j+1} \rangle$ are transposes for $i = 1, 2, \ldots, n - 1$, $j = 1, 2, \ldots, h - 1$. Denote $d_i = b_i$ for $i = 0, 1, \ldots, p$ (evidently $b_p = z$), $e_j = b_{j+p}$, $j = 1, 2, \ldots, h + n$ (evidently $b_{h+p} = u$). Obviously, the elements $a = b_0 > b_1 > \ldots > b_{h+p} = u$, $u < b_{h+p+1} < \ldots < b_{h+p+n} = b$ form a finite connected line B. From the construction of B it follows that there exists a one-to-one mapping of the set of all prime intervals of the line A onto the set of all prime intervals of the line B such that the corresponding prime intervals are projective. The middle intervals of the corresponding projective intervals are the intervals $\langle x_t, y_t \rangle$ and the intervals $\langle d_{p+i}, d_{p+i+1} \rangle$ for $i = 0, 1, \ldots, n - 1$. Since $ax_ta_{r_m}$, $ay_ta_{r_m}$, $ad_{p+i}a_{r_m}$, $aa_{r_m}b$ we get ax_tb, ay_tb, $ad_{p+i}b$ for $i = 0, 1, \ldots, n$. Hence the middle intervals have the demanded property. The assertion of the lemma in the brackets can be proved analogously.

Theorem 4. *Let A, B be finite connected lines with end elements a, b in a directed distributive multilattice. Then the lines A, B have the same length and there exists a one-to-one mapping of the set of all prime intervals of the line A onto the set of all prime intervals of the line B such that the corresponding prime intervals are projective and the middle intervals $\langle x_i, y_i\rangle$ satisfy ax_ib, ay_ib.*

Proof. If a, b are comparable, then the assertion is true by Theorem D. Let a, b be incomparable. Let A, B be finite connected lines with end elements a, b. By Lemma 5 there exist $u, u' \in a \wedge b$, $v, v' \in a \vee b$ such that $A \subset \langle u, v\rangle$ and $B \subset \langle u', v'\rangle$. By Lemma 9 there exists a finite connected line C with end elements a, b, which has elements $a = c_0 > c_1 > \ldots > c_k = u$, $u < c_{k+1} < \ldots < c_{k+n} = b$, and a one-to-one mapping φ_1 of the set of all prime intervals of the line A onto the set of all prime intervals of the line C such that the corresponding intervals are projective and the middle intervals $\langle x_c, y_c\rangle$ satisfy ax_cb, ay_cb. Analogously there exist a finite connected line D with end elements a, b, which has elements $a = d_0 < d_1 < \ldots < d_r = v'$, $v' > d_{r+1} > \ldots > d_{r+m} = b$, and a one-to-one mapping φ_2 of the set of all prime intervals of the line B onto the set of all prime intervals of the line D such that the corresponding intervals are projective and the middle intervals $\langle x_d, y_d\rangle$ satisfy ax_db, ay_db. By Lemma 8 there exists a finite connected line E, which has elements $a = e_0 > e_1 > \ldots > e_m = u$, $u < e_{m+1} < \ldots < e_{m+r} = b$, such that the intervals $\langle d_p, d_{p+1}\rangle$ and $\langle e_{m+p}, e_{m+p+1}\rangle$ are transposes, $\langle d_{r+q}, d_{r+q+1}\rangle$ and $\langle e_q, e_{q+1}\rangle$ are transposes for $p = 0, 1, \ldots, r-1$, $q = 0, 1, \ldots, m-1$. Hence there exists a one-to-one mapping φ_3 of the set of all prime intervals of the line D onto the set of all prime intervals of the line E such that the corresponding intervals are transposes. The elements $a = e_0 > e_1 > \ldots > e_m = u$ form a finite connected chain E_1. The elements $a = c_0 > c_1 > \ldots > c_k = u$ form a finite connected chain C_1. The chains C_1, E_1 have the same end elements a, u. By Theorem D $k = m$ and there exists a one-to-one mapping of the set of all prime intervals of the chain C_1 onto the set of all prime intervals of the chain E_1 such that the corresponding prime intervals are projective and the middle intervals $\langle x_{e_1}, y_{e_1}\rangle$ satisfy $\langle x_{e_1}, y_{e_1}\rangle \subset \langle u, a\rangle$. Analogously we get the chain C_2 with the elements $u < c_{k+1} < \ldots < c_{k+n} = b$ and the chain E_2 with the elements $u < e_{k+1} < \ldots < e_{k+r} = b$. By Theorem D $r = n$ and there exists a one-to-one mapping of the set of all prime intervals of the chain C_2 onto the set of all prime intervals of the chain E_2 such that the corresponding prime intervals are projective and the middle intervals $\langle x_{e_2}, y_{e_2}\rangle$ satisfy $\langle x_{e_2}, y_{e_2}\rangle \subset \langle u, b\rangle$. Consequently, the line C and the line E have the same length and there exists a one-to-one mapping φ_4 of the set of all prime intervals of the line C onto the set of all prime intervals of the line E such that the corresponding prime intervals are projective. The middle intervals of the corresponding prime intervals under φ_4 are the intervals $\langle x_{e_1}, y_{e_1}\rangle$ and the intervals $\langle x_{e_2}, y_{e_2}\rangle$. Since $\langle x_{e_1}, y_{e_1}\rangle \subset \langle u, a\rangle$ and $\langle x_{e_2}, y_{e_2}\rangle \subset \langle u, b\rangle$ we get $ax_{e_1}b$, $ay_{e_1}b$, $ax_{e_2}b$, $ay_{e_2}b$ by Theorem B. From these considerations it follows that the line A and the line B have the same length and the mapping $\varphi = \varphi_2^{-1}\varphi_3^{-1}\varphi_4\varphi_1$ is a one-to-one mapping of the set of all prime intervals of the line A onto the set of all prime intervals of the line B such that the corresponding intervals are projective. The middle intervals $\langle x_i, y_i \rangle$ of the corresponding prime intervals under φ are the intervals $\langle x_c, y_c \rangle$, $\langle c_j, c_{j+1} \rangle$, $\langle x_{e_1}, y_{e_1} \rangle$, $\langle x_{e_2}, y_{e_2} \rangle$, $\langle e_j, e_{j+1} \rangle$, $\langle d_j, d_{j+1} \rangle$, $\langle x_d, y_d \rangle$, which have the demanded property. This completes the proof.

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ЛИНИИ В НАПРАВЛЕННЫХ ДИСТРИБУТИВНЫХ МУЛЬТИСТРУКТУРАХ

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Резюме

Понятие дистрибутивной мультиструктуры, которым мы пользуемся в этой работе, совпадает с понятием, введенным М. Бенадо [2]. В работе определяется понятие линий в направленной дистрибутивной мультиструктуре при помощи отношения «между». Исследуются некоторые свойства линий (теорема 1, теорема 2, теорема 3). Показывается далее, что для линий в направленной дистрибутивной мультиструктуре справедлива теорема Жордана-Гёльдера.