Jarosław Morchało

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF DISCRETE VOLTERRA EQUATION

Jarosław Morchało

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ABSTRACT. The purpose of this paper is to prove asymptotic properties (for $n \to \infty$) of some discrete Volterra equations.

1. Introduction

Difference equations occur in many branches of applied mathematics: numerical analysis, physics, control theory and optimization.

In recent years, considerable attention has been paid to the development of the qualitative theory for difference equations.

Difference equation of Volterra type have been introduced in [3], [7] as the discrete analogue of Volterra integrodifferential equations. The discrete Volterra equations have been intensively investigated in recent years and new and interesting results for these equations have been proved [1], [2], [4], [5], [8].

The paper [2] is devoted to studying the linear discrete Volterra equation with control in a finite dimensional Hilbert space.

In paper [4], Kolmanovski, Myshkis and Richard investigated some Volterra discrete equations and obtained comparison theorems for resolvent and solutions.

Kolmanovski and Myshkis [5] investigated conditions under which stability (asymptotic stability) of the linear Volterra discrete equation implies stability (asymptotic stability) of the zero solution of nonlinear Volterra equation.

The purpose of this paper is to investigate asymptotic properties (as $n \to \infty$) of solutions of some discrete Volterra equations. Here, however, we shall use

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technique based on the resolvent matrix associated with the kernel in (1). Convolution equations and nonlinear equations will be presented by the author in a future papers.

We adopt the following notation in this paper:

Z is the set of all non-negative integers,
$N(n_0) = \{n_0, n_0 + 1, \dots\}$, $n_0 \in Z$,
$\mathbb{R}^k$ — the k-dimensional real Euclidean space with norm $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1, \dots, x_k)$,
M^k — the space of all $k \times k$ matrices $A = (a_{ij})$ with norm $|\cdot|$ given by $|A| = \sum_{i=1}^{k} \sum_{j=1}^{k} |a_{ij}|$.
The identity matrix is denoted by E.

Let us now consider the difference equation

$$y(n) = f(n) + \sum_{s=0}^{n-1} K(n,s)y(s) \tag{1}$$

where $K(n,s)$ is from M^k and $y, f \in \mathbb{R}^k$. Let us assume that a unique solution y of system (1) does exist for all finite $n \in Z$.

Let us find the solution y as a function f and auxiliary $k \times k$ matrix R, referred as a resolvent [4]. Let $K^{(1)}(n,s) = K(n,s)$,

$$K^{(q)}(n,s) = \sum_{r=s+1}^{n-1} K^{(q-1)}(n,r)K^{(1)}(r,s) = \sum_{r=s+1}^{n-1} K^{(1)}(n,r)K^{(q-1)}(r,s)$$

and

$$R(n,s) = \sum_{q=1}^{\infty} K^{(q)}(n,s). \tag{2}$$

The $k \times k$ matrix $R(n,s)$ is called the resolvent kernel associated with the kernel $K(n,s)$. It is easy to see that

$$R(n,s) = K(n,s) + \sum_{q=s+1}^{n-1} K(n,q)R(q,s) \tag{3}$$

and

$$R(n,s) = K(n,s) + \sum_{q=s+1}^{n-1} R(n,q)K(q,s). \tag{3'}$$

In terms of resolvent matrix $R(n,s)$ of (2) the solution of (1) can be written as

$$y(n) = f(n) + \sum_{s=0}^{n-1} R(n,s)f(s). \tag{4}$$

Multiplying both sides of the equation

$$y(j) = f(j) + \sum_{s=0}^{j-1} K(j,s)y(s)$$

by $R(n,j)$ from the left and summing with respect to j from $j=0$ to $n-1$, we obtain

$$\sum_{j=0}^{n-1} R(n,j)\big(y(j) - f(j)\big) = \sum_{j=0}^{n-1}\Bigg(\sum_{s=j+1}^{n-1} R(n,s)K(s,j)\Bigg)y(j)\,.$$

Then, by virtue (3') we have (4).

We now prove that

$$\sum_{s=j+1}^{n-1} K(n,j)R(j,s) = \sum_{s=j+1}^{n-1} R(n,j)K(j,s)\,. \qquad (*)$$

Substituting $y(n)$ from (4) into (1), we have

$$\sum_{l=0}^{n-1}\big[R(n,l) - K(n,l)\big]f(l) = \sum_{l=0}^{n-1}\sum_{s=l+1}^{n-1} K(n,s)R(s,l)f(l)\,.$$

From here and arbitrariness of f we obtain that the resolvent satisfies also the equation (3). Comparing (3) and (3'), one verifies $(*)$.

II. Asymptotic properties

Lemma. *Suppose that*

1) *the functions $f(n)$ and $K(n,s)$ are defined for $n, s \in Z$,*
2) $\overline{\lim\limits_{n\to\infty}}\, |f(n)| = M < \infty$,
3) $\overline{\lim\limits_{n\to\infty}} \sum_{s=0}^{n-1} |K(n,s)| = \mu < 1$; $\lim\limits_{n\to\infty} \sum_{s=0}^{n_0} |K(n,s)| = 0$ *for each* $n_0 \ge 0$,
4) *the equation* (1) *has a solution* $\overline{y}(n)$ *such that* $|\overline{y}(n)| \le L$ *for* $n \in Z$.

Then the following inequality holds:

$$\overline{\lim_{n\to\infty}}\, |\overline{y}(n)| \le \frac{M}{1-\mu}\,.$$

Proof. For given $\varepsilon \in (0, 1-\mu)$ we choose $n_0 \ge 0$ and then $n_1 \ge n_0$, so that $\sum_{s=0}^{n_0} |K(n,s)| \le \varepsilon$, $\sum_{s=n_0}^{n-1} |K(n,s)| \le \mu+\varepsilon$, $|f(n)| \le M+\varepsilon$ and $|\overline{y}(n)| \le L_1+\varepsilon$

$(n \geq n_1)$ where $L_1 = \overline{\lim}_{n\to\infty} |\overline{y}(n)|$. We obtain from (1)

$$|\overline{y}(n)| \leq M + \varepsilon + L \sum_{s=0}^{n_0} |K(n,s)| + (L_1 + \varepsilon) \sum_{s=n_0+1}^{n-1} |K(n,s)|$$
$$\leq M + \varepsilon + L\varepsilon + (L_1 + \varepsilon)(\mu + \varepsilon).$$

Hence

$$\overline{\lim_{n\to\infty}} |\overline{y}(n)| \leq \frac{M}{1-\mu}.$$

□

THEOREM 1. *Let $f(n)$, $F(n)$, $\psi(n)$ be defined and bounded for $n \in N(n_0)$ and let $N(n,s)$ be defined for $n \geq s \geq n_0$. Suppose that*

1° $\sum_{l=s}^{n-1} |N(n,l)||N(l,s)|^\alpha \leq \lambda |N(n,s)|^\alpha$
with some $\alpha \in \langle 0,1 \rangle$ and some $\lambda < 1$ for $n \geq s \geq n_0$,

2° $|N(n,s)| \leq F(s)$ *for $n \geq s \geq n_0$,*

3° $\overline{\lim}_{n\to\infty} \sum_{s=n_0}^{n-1} \big(F(s)\big)^{1-\alpha} |N(n,s)|^\alpha < \infty$,
$\overline{\lim}_{n\to\infty} \sum_{s=n_0}^{n-1} |N(n,s)| + \sum_{n=n_0}^{\infty} |\psi(n)| = \mu < 1$,

4°_{a} $\overline{\lim}_{n\to\infty} |f(n)| = M < \infty$,

or

4°_{b} $\overline{\lim}_{n\to\infty} f(n) = s$ $(|s| < \infty)$,
$\lim_{n\to\infty} \sum_{s=n_0}^{n-1} N(n,s) = \sigma$, $(E-\sigma)^{-1}$ *exists and*
$\overline{\lim}_{n\to\infty} \sum_{s=n_0}^{n_1} |N(n,s)| = 0$ *for fixed $n_1 \geq n_0$.*

Then the unique solution $\overline{y}(n)$ of the equation

$$y(n) = f(n) + \sum_{s=n_0}^{n-1} K^{(1)}(n,s) y(s) \tag{5}$$

where $K^{(1)}(n,s) = N(n,s) + \psi(s)$ remains bounded for $n \to \infty$. In case 4°_{b} it is convergent for $n \to \infty$.

P r o o f. We define the functions

$$R(n,s) = \sum_{q=1}^{\infty} K^{(q)}(n,s), \qquad I(n) = \sum_{s=n_0}^{n-1} R(n,s) f(s)$$

and

$$\overline{y}(n) = f(n) + I(n) \qquad \text{for} \quad n \geq s \geq n_0$$

and state that $\overline{y}(n)$ satisfies (5).

Next, we choose a number a satisfying the inequality $\max(\lambda, \mu) < a < 1$. Then for some $n_1 \geq n_0$ we have

$$\sum_{s=n_0}^{n-1} |K^{(1)}(n,s)| \leq \sum_{s=n_0}^{n-1} \big(|N(n,s)| + |\psi(s)|\big) \leq a \qquad \text{for} \quad n \geq n_1 .$$

We shall prove by induction the inequality

$$|K^{(q)}(n,s)| \leq a^{q-1} F^{1-\alpha}(s) |N(n,s)|^{\alpha} + (q-1) a^{q-2} \psi_1(n,s) + a^{q-1} |\psi(s)| \quad (6)$$

for $n \geq n_1$, $n \geq s \geq n_0$, $q = 1, 2, \dots$, where

$$\psi_1(n,s) = F^{1-\alpha}(s) \sum_{i=s}^{n-1} |N(i,s)|^{\alpha} |\psi(i)| .$$

We immediately verify that (6) is true for $q = 1$.

Suppose now that it is true for the index $q-1$ ($q \geq 2$). Then observing that $\psi_1(n,s)$ is increasing function of the variable $n \geq n_1$ for $n \geq s \geq n_1$, we have

$$\begin{aligned}
&|K^{(q)}(n,s)| \\
&\leq \sum_{i=s+1}^{n-1} |N(n,i) + \psi(i)| \big\{ a^{q-2} F^{1-\alpha}(s) |N(i,s)|^{\alpha} \\
&\qquad\qquad + (q-2) a^{q-3} \psi_1(i,s) + a^{q-2} |\psi(s)| \big\} \\
&\leq a^{q-1} |\psi(s)| + (q-2) a^{q-2} \psi_1(n,s) + a^{q-2} F^{1-\alpha}(s) \sum_{i=s+1}^{n-1} |N(n,i)| |N(i,s)|^{\alpha} \\
&\leq a^{q-1} |\psi(s)| + (q-1) a^{q-2} \psi_1(n,s) + a^{q-1} F^{1-\alpha}(s) |N(n,s)|^{\alpha} .
\end{aligned}$$

Hence (6) follows. Therefore the series $\sum_{q=1}^{\infty} K^{(q)}(n,s)$ is uniformly convergent for $n \geq s \geq n_0$. Taking $\sum_{q=1}^{\infty} K^{(q)}(n,s) = R(n,s)$ we obtain from (6) for $n \geq s \geq n_1$

$$\begin{aligned}
|R(n,s)| &\leq \sum_{q=1}^{\infty} \big\{ a^{q-1} |\psi(s)| + (q-1) a^{q-2} \psi_1(n,s) + a^{q-1} F^{1-\alpha}(s) |N(n,s)|^{\alpha} \big\} \\
&\leq \frac{1}{1-a} |\psi(s)| + \frac{1}{(1-a)^2} \psi_1(n,s) + \frac{1}{1-a} F^{1-\alpha}(s) |N(n,s)|^{\alpha} .
\end{aligned}$$

We have

$$\sum_{s=n_0}^{n-1} \psi_1(n,s) = \sum_{s=n_0}^{n-1} F^{1-\alpha}(s) \sum_{i=s}^{n-1} |N(i,s)|^\alpha |\psi(i)|$$
$$= \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s-1} F^{1-\alpha}(i) |N(s,i)|^\alpha |\psi(s)|\,.$$

From this and from assumptions $2^\circ - 4^\circ_{\mathrm{a}}$ it follows that $\overline{\lim\limits_{n\to\infty}} |\overline{y}(n)| < \infty$. In the case 4°_{b} it is easy to verify that the function

$$y_1(n) = \overline{y}(n) - (E-\sigma)^{-1} s_1$$

where $s_1 = s + \sum\limits_{n=n_0}^{\infty} \psi(n)\overline{y}(n)$ satisfies the equation

$$y_1(n) = f_1(n) + \sum_{s=n_0}^{n-1} N(n,s) y_1(s) \qquad (n \geq n_0) \tag{7}$$

where

$$f_1(n) = f(n) + \sum_{s=n_0}^{n-1} \psi(s)\overline{y}(s) + \left(\sum_{s=n_0}^{n-1} N(n,s) - E \right) (E-\sigma)^{-1} s_1\,. \tag{8}$$

From this and from the assumptions we have

$$\lim_{n\to\infty} f_1(n) = 0\,.$$

We demonstrate that $\lim\limits_{n\to\infty} y_1(n) = 0$. Assume that the last condition does not hold. Let $M = \overline{\lim\limits_{n\to\infty}} |y_1(n)| > 0$, then there exists $n_1 \geq n_0$ such that $|y_1(n)| < M$ for $n \geq n_1$. From (7) and assumption 4°_{b} we infer that

$$\begin{aligned}
M &= \overline{\lim_{n\to\infty}} |y_1(n)| \\
&\leq \overline{\lim_{n\to\infty}} |f_1(n)| + \overline{\lim_{n\to\infty}} \sum_{s=n_0}^{n-1} |N(n,s)||y_1(s)| \\
&\leq \overline{\lim_{n\to\infty}} |f_1(n)| + \lim_{n\to\infty} \sum_{s=n_0}^{n_1} |N(n,s)||y_1(s)| + \lim_{n\to\infty} \sum_{s=n_1+1}^{n-1} |N(n,s)||y_1(s)| \\
&\leq M\mu
\end{aligned}$$

what contradicts 3°.

Therefore $\lim\limits_{n\to\infty} y_1(n) = 0$ and $\lim\limits_{n\to\infty} y(n) = (E-\sigma)^{-1} s_1$. □

THEOREM 2. *Let the assumptions* $1^\circ - 3^\circ$ *of Theorem* 1 *hold. Suppose in addition that*

$$\lim_{n\to\infty} f(n) = s \qquad (\,|s| < \infty\,),$$

$$\lim_{n\to\infty} \sum_{s=n_0}^{n-1} N(n,s) = 0\,; \qquad \lim_{n\to\infty} \sum_{s=n_0}^{n_1} |N(n,s)| = 0$$

for fixed $n_1 \geq n_0$. *We set* $K^{(1)}(n,s) = N(n,s) + \psi(s)$ *for* $n \geq s \geq n_0$. *Then for* $\varepsilon > 0$ *there exists an* $n^* \geq n_0$ *such that the unique solution* $\overline{y}(n)$ *of the equation*

$$y(n) = f(n) + \sum_{s=n^*}^{n-1} K^{(1)}(n,s)y(s) \tag{9}$$

satisfies the relation $\lim_{n\to\infty} \overline{y}(n) = s_1$, *where* $|s - s_1| \leq \varepsilon$.

P r o o f. Under our hypotheses we have for $n \geq n_1$ (with some $n_1 \geq n_0$)

$$|f(n)| \leq c_1 = 1 + |s|,$$

$$\sum_{s=n_1}^{n-1} F^{1-\alpha}(s)|N(n,s)|^\alpha \leq c_2,$$

$$\sum_{s=n_1}^{n-1} \psi_1(n,s) = \sum_{s=n_1}^{n-1} |\psi(s)| \sum_{l=n_1}^{s-1} F^{1-\alpha}(l)|N(s,l)|^\alpha \leq c_2\mu,$$

$$\sum_{s=n_1}^{n-1} |K^{(1)}(n,s)| \leq a,$$

where $\psi_1(n,s) = F^{1-\alpha}(s) \sum_{i=s}^{n-1} |N(i,s)|^\alpha |\psi(i)|$ and a satisfies the inequality $\max(\lambda,\mu) < a < 1$.

We choose an arbitrary $n_2 \geq n_1$. The above inequalities remain true for $n \geq n_2$ if we replace n_1 by n_2 (since $F(n) \geq 0$). As in Theorem 1 we obtain

$$\sum_{s=n_2}^{n-1} |R(n,s)| \leq \frac{1}{1-a}\mu + \frac{1}{(1-a)^2}c_2\mu + \frac{1}{1-a}c_2 = c_3$$

for $n \geq n_2$. We denote by $\overline{y}(n)$ the unique solution of the equation (9). With n_2 instead of n^* we have

$$\overline{y}(n) = f(n) + \sum_{s=n_2}^{n-1} R(n,s)f(s)$$

and

$$|\overline{y}(n)| \le c_1 + c_1 c_3 = c \qquad \text{for} \quad n \ge n_2 .$$

Let us observe that c is independent of n_2 (if $n_2 \ge n_1$).

For a given $\varepsilon > 0$ we choose a fixed $n^* \ge n_1$ such that

$$\sum_{n=n^*}^{\infty} |\psi(n)| \le \frac{\varepsilon}{c} .$$

Obviously $|\overline{y}(n)| \le c$ for $n_1 \ge n^*$. As in the proof of Theorem 1 we find that $\lim_{n\to\infty} \overline{y}(n) = s_1$ (we replace σ by 0), where

$$|s - s_1| \le \sum_{s=n^*}^{\infty} |\psi(s)\overline{y}(s)| \le \varepsilon .$$

□

Now we consider the scalar situation.

THEOREM 3. *Suppose that*

1° *the function $g(n)$ has property: $g(n) \ne 0$, $|g(n)|$ is monotone for $n \ge n_0$,*

2° *$h(n)$, $\varphi(n)$, $f(n)$ and $\psi(n)$ are defined on $\mathbb{N}$,*

   2°_{a} $\overline{\lim}_{n\to\infty} |f(n)| = M < \infty$

or

   2°_{b} $\overline{\lim}_{n\to\infty} f(n) = s$ $(|s| < \infty)$,

3° $\sum_{n=n_0}^{\infty} |\psi(n)| < \infty$,

4° $\lim_{n\to\infty} \varphi(n) = 0$,

5° $\sum_{n=n_0}^{\infty} |h(n)| \le h_0 < \infty$.

Let $K^{(1)}(n,s) = \frac{h(s)}{g(n)}\varphi(s) + \psi(s)$ where $N(n,s) = \frac{h(n)}{g(n)}\varphi(s)$ for $n \ge n_0$, $s \ge n_0$.

Then in the case of $\lim_{n\to\infty} g(n) = \infty$ the unique solution $\overline{y}(n)$ of the equatio

$$y(n) = f(n) + \sum_{s\ n_1}^{n-1} K^{(1)}(n,s) y(s) \qquad (n \ge n_1) \tag{10}$$

($n_1 \ge n_0$ is chosen so large that $\sum_{n=n_1}^{\infty} |\psi(n)| < 1$) remains bounded for $n \to \infty$.

In case 2°_{b}, $\overline{y}(n)$ is convergent as $n \to \infty$.

P r o o f. $\lim_{n\to\infty} |g(n)| = \infty$. We chosen a fixed $\alpha \in (0,1)$ and small enough $\delta > 0$. Next, we choose $n_2 \ge n_0$ such that $|\varphi(n)| \le \delta$ holds for $n > n_2$. We prove the inequality in hypothesis 1°, hypothesis 2° and the first hypothesis in 3° of Theorem 1. We obtain by 1°, 4° and 5° for $n \geq s \geq n_2$

$$\sum_{l=s}^{n-1} |N(n,l)||N(l,s)|^\alpha = \sum_{l=s}^{n-1} \left|\frac{h(l)}{g(n)}\varphi(l)\right|\left|\frac{h(s)}{g(l)}\varphi(s)\right|^\alpha$$

$$= \left|\frac{h(s)\varphi(s)}{g(n)}\right|^\alpha |g(n)|^{\alpha-1} \sum_{l=s}^{n-1} \frac{|h(l)||\varphi(l)|}{|g(l)|^\alpha} \leq K\delta |N(n,s)|^\alpha,$$

$$0 < K = \frac{h_0}{|g(n_0)|}.$$

Then the inequality in hypothesis 1° of Theorem 1 is satisfied with $\lambda = K\delta < 1$.

Next, we state that hypothesis 2° of Theorem 1 is satisfied for

$$F(s) = \left|\frac{h(s)\varphi(s)}{g(s)}\right| \qquad \text{for} \quad s \geq n_2 .$$

We shall show that the first hypothesis in 3° of Theorem 1 is also satisfied. We have

$$\sum_{s=n_2}^{n-1} F^{1-\alpha}(s)|N(n,s)|^\alpha = \sum_{s=n_2}^{n-1} \left|\frac{h(s)\varphi(s)}{g(s)}\right|^{1-\alpha} \left|\frac{h(s)\varphi(s)}{g(n)}\right|^\alpha$$

$$= \frac{1}{|g(n)|^\alpha} \sum_{s=n_2}^{n-1} \frac{|h(s)\varphi(l)|}{|g(s)|^{1-\alpha}} \leq \frac{\delta}{|g(o)|} \sum_{s=n_2}^{n-1} h(s) < \infty$$

for $\quad n \geq n_2$.

Furthermore, we have

$$\overline{\lim_{n\to\infty}} \sum_{s=n_1}^{n-1} |K^{(1)}(n,s)| \leq \overline{\lim_{n\to\infty}} \sum_{s=n_1}^{n-1} (|N(n,s)| + |\psi(s)|)$$

$$\leq \lim_{n\to\infty} \frac{1}{|g(n)|} \sum_{s=n_1}^{n-1} |h(s)\varphi(s)| + \sum_{n=n_1}^{\infty} |\psi(n)| = A$$

where $A = \sum\limits_{n=n_1}^{\infty} |\psi(n)|$.

The second hypothesis in 3° and 4°_b of Theorem 1 are then satisfied for $n_0 = n_1$, $\mu = A$ and $\sigma = A_1$ where $A_1 = \sum\limits_{n=n_1}^{\infty} \psi(n)$. □

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Poznań University of Technology
Institute of Mathematics
Piotrowo 3a
PL-60-965 Poznań
POLAND
E-mail: jmorchal@math.put.poznan.pl