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THEOREMS OF BOHR-NEUGEBAUER-TYPE FOR ALMOST-PERIODIC DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to present Bohr-Neugebauer theorems (every bounded solution is almost-periodic) for linear nonhomogeneous differential systems, where the nonhomogenity is assumed to be essentially bounded and almost-periodic in various metrics (in the sense of Bohr, Stepanov, Weyl, Besicovitch).

1. Introduction

Almost-periodic solutions of ordinary linear differential equations with constant coefficients and a uniformly almost-periodic nonhomogenity were studied first by H. Bohr and O. Neugebauer in [BN]. They proved the following result:

Any bounded solution on the whole real axis is almost-periodic.

The proof can be found (as well as the above quoted paper) in other monographs or textbooks, for example, [C1], [Fi], [KBK]. Their result was extended to systems by C. Corduneanu [C1], (cf. also [De]).

Following Bohr and Neugebauer's initiation of this research field, many results were obtained in the next decades. Later on [AP], L. Amerio extended the results to the case of almost-periodic (finite-dimensional) nonlinear systems. He has provided a criterion for the existence of almost-periodic solutions under the assumption of the existence of a semitrajectory that belongs to a compact set in $\mathbb{R}^n$. Now, the author's interest is not only related to linear ordinary equations, but also to nonlinear ones, to partial differential equations, to differential inclusions and to differential equations in abstract spaces (see e.g. [An], [AB1], [ABL], [DK], [Za]).

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The problem of the existence of almost-periodic solutions to linear equations in a Banach space, with constant or almost-periodic operators, has been investigated by several authors. For example, Yu. L. Daleckii and M. Krein [DK] considered the equation $x' = \mathcal{A}x + f(t)$ where $\mathcal{A}$ is a linear stationary operator on a Banach space and f is an almost-periodic map with values in the given Banach space. They established spectral conditions, under which the existence of almost-periodic solutions is assured. In their book [MS], J. L. Massera and J. J. Schaeffer obtained criteria for linear equations with a nonstationary operator $\mathcal{A}$. For more recent related monographs, see [NG], [Za].

M. A. Krasnoselskii et al. have used the method of integral equations in [KBK]. The main idea relies on the use of Green functions. Using these functions, the case of nonlinear almost-periodic equations can be reduced to the problem of nonlinear integral equations.

Another method in the study of almost-periodic differential equations is the method of Liapunov functions. The main contributor to the topic was T. Yoshizawa. Some of the results obtained by this method are included in [Yo]. For further results, see e.g. [C2], [Fi], [De], [Fa], [LZ].

In this paper, we will concentrate only on linear systems of the form $x' = Ax + f(t)$, where A is a constant real matrix and f is measurable, essentially bounded and almost-periodic in various metrics.

2. Definitions and elementary properties

In this part, we define different types of almost-periodicity and fundamental relations among functions almost-periodic (a.p.) in given senses (cf. [Be], [Le]).

DEFINITION 1. Let us introduce the metrics:

(*Bohr*)

$$D_{\mathrm{u}}(f,g) := \sup_{t\in\mathbb{R}} \|f(t) - g(t)\|,$$

(*Stepanov*)

$$D_{\mathrm{S}_l}(f,g) := \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} \|f(t) - g(t)\| \, \mathrm{d}t,$$

(*Weyl*)

$$D_{\mathrm{W}}(f,g) := \lim_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} \|f(t) - g(t)\| \, \mathrm{d}t = \lim_{l\to\infty} D_{\mathrm{S}_l}(f,g),$$

(*Besicovitch*)

$$D_{\mathrm{B}}(f,g) := \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)-g(t)\|\,\mathrm{d}t\,,$$

where f,g are measurable functions from $\mathbb{R}$ into $\mathbb{R}^n$. Denote by D_G any of the above (pseudo-) metrics. The metric space (G,D_G) we understand the related quotient space in the sense that we identify such elements f_1, f_2 for which $D_G(f_1,f_2)=0$.

Definition 2. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ is said to be *G-almost-periodic* (G-a.p.) if

$$(\forall\,\varepsilon>0)\,(\exists\,k>0)\,(\forall\,a\in\mathbb{R})\,(\exists\,\tau\in[a,a{+}k])\Big(D_G\big(f(t+\tau),f(t)\big)<\varepsilon\Big)\,.$$

Then τ is called an *ε-almost-period* in the given sense.

For the sake of simplicity, we call a D_{u}-a.p. or D_{S_1}-a.p. or D_{W}-a.p. or D_{B}-a.p. function u.a.p. or S_1-a.p. or W-a.p. or B-a.p., respectively.

The following definition surprisingly uses the Stepanov metric for the a.p. notion in the sense of Weyl.

Definition 3. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ is said to be *equi-Weyl almost periodic* (equi-W-a.p.) if

$$(\forall\,\varepsilon>0)\,(\exists\,k,l_0(\varepsilon)>0)\,(\forall\,a\in\mathbb{R})\,(\exists\,\tau\in[a,a{+}k])\,(\forall\,l\ge l_0(\varepsilon))$$
$$\Big(D_{\mathrm{S}_l}\big(f(t+\tau),f(t)\big)<\varepsilon\Big)\,.$$

Remark 1. One can easily check that one can take $l_0 \ge 1$ without loss of generality in Definition 3.

Definition 4. For an S_1-a.p. function f, we define the *Bochner translation* as follows (cf. [AP])

$$f^b(t) := f(t+\eta)\,,\qquad \eta\in[0,1]\,,\ \ t\in\mathbb{R}\,.$$

Properties. (*cf.* [AP]):

a) *A function f is S_1-a.p. if and only if f^b is u.a.p..*

b) *A function f is bounded in the Stepanov metric if and only if f^b is bounded in the metric induced by the sup-norm.*

Remark 2. Observe that a vector function $f\colon \mathbb{R}\to\mathbb{R}^n$ is G-a.p. or equi-W-a.p. if it is G-a.p. or equi-W-a.p. in each of its components, respectively. The reverse implication does not need to hold for W-a.p. or B-a.p. vector functions, because the spaces of W-a.p. functions and B-a.p. functions do not seem to be linear.

DEFINITION 5. (cf. [ABG]) A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ is said to be *G-normal* if, for every sequence $\{h_n\}$ of real numbers, there corresponds a subsequence $\{h_{n_i}\}$ such that the sequence of functions $\{f(x + h_{n_i})\}$ is fundamental in the G-metric. Of course, if (G, D_G) is complete, the subsequence $\{f(x + h_{n_i})\}$ is required to be convergent.

DEFINITION 6. (cf. [ABG]) A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ is said to be *equi-W-normal* if the family of functions $\{f(x + h)\}$, $h \in \mathbb{R}$, is precompact in the D_{S_l}-metric for sufficiently large l, i.e. if, for each sequence $f(x + h_1)$, $f(x+h_2), \dots$, one can choose a fundamental subsequence in the D_{S_l}-metric, for sufficiently large l.

The numbers h_i are called *translation numbers*.

Since the spaces of W-a.p. functions and B-a.p. functions do not seem to be linear, we must add some further properties of them which will be suitable for applications. Following arguments e.g. in [Be] or [Le], one can fortunately show that uniform continuity of at least one term is sufficient for the sum of two terms to be W-a.p. or B-a.p., respectively, as observed by J. Andres and A. M. Bersani [AB2].

Let $f(t)$ be a real (or a complex) function, defined for all real values of t and belonging to the space of W-a.p. functions or to the space of B-a.p. functions (for the sake of simplicity, we will indifferently denote either of the two spaces by $G^\star_{\mathrm{ap}}$). Let $E\{\varepsilon, f(t)\}$ be the set of ε-almost-periods of $f(t)$. Let us recall that if τ_1 and τ_2 are almost-periods respectively related to ε_1 and to ε_2, then $\tau_1 \pm \tau_2$ is an $(\varepsilon_1 + \varepsilon_2)$-almost period.

DEFINITION 7. $f(t)$ is called *uniformly G-continuous* if,

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)\Big(|h| < \delta \implies \|f(t+h) - f(t)\|_G < \varepsilon\Big).$$

If, in particular, $G = D_{\mathrm{u}}$, then we simply speak about *uniform continuity* of f.

We will denote by $\tilde{G}$ the space of G-a.p. functions which are uniformly G-continuous.

Remark 3. Every G-normal function is G-a.p.. On the other hand, it is possible to find W-a.p. or B-a.p. functions which are not W-normal or B-normal (see [ABG] and the references therein). Under the assumption of uniform G-continuity, the spaces of G-normal and G-a.p. functions are equivalent. The spaces of equi-W-a.p. functions and equi-W-normal functions are equivalent (see [ABG]).

PROPOSITION 1. *If $f(t) \in \tilde{G}$, then, for every $\varepsilon > 0$, there corresponds $\delta = \delta(\varepsilon) > 0$ such that $E\{\varepsilon, f(t)\}$ contains all numbers in the interval $(-\delta, \delta)$.*

Proof. The proof is trivial. □

Lemma 1. ([AB2]) *For every $f(t) \in \tilde{G}$, and for every $\varepsilon_2 > \varepsilon_1 > 0$, there exists $\delta = \delta(\varepsilon_1, \varepsilon_2) > 0$ such that $E\{\varepsilon_2, f(t)\}$ contains every number whose distance from $E\{\varepsilon_1, f(t)\}$ is less than $\delta(\varepsilon_1, \varepsilon_2)$.*

Proof. Since $f(t) \in \tilde{G}$, according to Proposition 1 there exists $\delta(\varepsilon_2 - \varepsilon_1) > 0$ such that $E\{\varepsilon_2 - \varepsilon_1, f(t)\}$ contains all numbers of the interval $(-\delta, \delta)$. By the above property of the translation numbers, taking $\tau_1 \in E\{\varepsilon_1, f(t)\}$ and $\tau_2 \in (-\delta, \delta) \subset E\{\varepsilon_2 - \varepsilon_1, f(t)\}$, we obtain

$$\tau_1 + \tau_2 \in E\{\varepsilon_2, f(t)\}.$$

□

The following Lemma holds for every function belonging to $G^{\star}_{\mathrm{ap}}$ without requiring uniform continuity.

Lemma 2. ([AB2]) *Let $f_1, f_2 \in G^{\star}_{\mathrm{ap}}$. For every $\varepsilon > 0$, $\delta > 0$, the set of numbers belonging to $E\{\varepsilon, f_1(t)\}$ whose distance from $E\{\varepsilon, f_2(t)\}$ is less than δ, is r.d..*

Proof. Consider the sets $E\left\{\frac{1}{2}\varepsilon, f_1(t)\right\}$ and $E\left\{\frac{1}{2}\varepsilon, f_2(t)\right\}$ and let $l = k\delta$ ($k \in \mathbb{N}$) be an inclusion interval for both these sets (i.e. in every real interval of length l there can be found numbers τ_1, τ_2 such that $\tau_i \in E\left\{\frac{1}{2}\varepsilon, f_i(t)\right\}$ for $i = 1, 2$).

Divide $\mathbb{R}$ into intervals $[(n-1)l, nl]$, where $n \in \mathbb{Z}$. Inside every interval $[(n-1)l, nl]$, we can find

$$\tau_1^{(n)} \in E\left\{\tfrac{1}{2}\varepsilon, f_1(t)\right\}, \qquad \tau_2^{(n)} \in E\left\{\tfrac{1}{2}\varepsilon, f_2(t)\right\}.$$

We have

$$-k\delta = -l < \tau_1^{(n)} - \tau_2^{(n)} < l = k\delta.$$

Denote by λ_i the interval $(i-1)\delta \leq x < i\delta$, so that $\tau_1^{(n)} - \tau_2^{(n)} \in \lambda_i$ for some $i = -k+1, \ldots, k$. Moreover, there exists n_0 such that, to any $n \in \mathbb{Z}$, there corresponds $n' \in \mathbb{Z}$ ($-n_0 \leq n' \leq n_0$) for which $\tau_1^{(n')} - \tau_2^{(n')}$ belongs to the same interval λ_i of $\tau_1^{(n)} - \tau_2^{(n)}$.

Thus,

$$\tau_1^{(n)} - \tau_2^{(n)} = \tau_1^{(n')} - \tau_2^{(n')} + \vartheta\delta \qquad (-1 < \vartheta < 1),$$

i.e.

$$\tau_1^{(n)} - \tau_1^{(n')} = \tau_2^{(n)} - \tau_2^{(n')} + \vartheta\delta.$$

By the above property, since $\tau_1^{(n')} \in E\left\{\frac{1}{2}\varepsilon, f_1(t)\right\}$ and $\tau_2^{(n')} \in E\left\{\frac{1}{2}\varepsilon, f_2(t)\right\}$, then $\left(\tau_1^{(n)} - \tau_1^{(n')}\right)$ and $\left(\tau_2^{(n)} - \tau_2^{(n')}\right)$ belong respectively to $E\{\varepsilon, f_1(t)\}$ and $E\{\varepsilon, f_2(t)\}$.

Thus, the distance from $E\{\varepsilon, f_2(t)\}$ of every number $\left(\tau_1^{(n)} - \tau_1^{(n')}\right)$ $(n \in \mathbb{Z})$ is less than δ.

Let us compute

$$\left|\left(\tau_1^{(n)} - \tau_1^{(n')}\right) - \left(\tau_1^{(n+1)} - \tau_1^{(n+1)'}\right)\right|.$$

Since

$$\left|\tau_1^{(n)} - \tau_1^{(n+1)}\right| \le 2l$$

and

$$\left|\tau_1^{(n')} - \tau_1^{(n+1)'}\right| \le (2n_0 + 1)l,$$

then

$$\left|\left(\tau_1^{(n)} - \tau_1^{(n')}\right) - \left(\tau_1^{(n+1)} - \tau_1^{(n+1)'}\right)\right| \le \left|\tau_1^{(n)} - \tau_1^{(n+1)}\right| + \left|\tau_1^{(n')} - \tau_1^{(n+1)'}\right| \le (2n_0+3)l.$$

Since the distance between two consecutive numbers is less than $(2n_0 + 3)l$, then, taking as inclusion interval the number $l' = (2n_0 + 3)l$, the set of numbers $\left(\tau_1^{(n)} - \tau_1^{(n')}\right)$ $(n \in \mathbb{Z})$ is r.d.. □

The following theorem requires uniform continuity only for f_1.

PROPOSITION 2. ([AB2]) *For every $\varepsilon > 0$, $f_1 \in \tilde{G}$, $f_2 \in G^\star_{\mathrm{ap}}$, the set $E\{\varepsilon, f_1(t)\} \cap E\{\varepsilon, f_2(t)\}$ is r.d..*

P r o o f. Let us take $\varepsilon > \varepsilon_1 > 0$. By Lemma 1, there exists $\delta > 0$ such that the set of all numbers whose distance from $E\{\varepsilon_1, f_1(t)\}$ is less than δ belongs to $E\{\varepsilon, f_1(t)\}$. By Lemma 2, the set of all numbers belonging to $E\{\varepsilon_1, f_2(t)\}$ whose distance from $E\{\varepsilon, f_1(t)\}$ is less than δ, which we denote by $\tilde{E}$ and which belongs to $E\{\varepsilon, f_1(t)\}$, is r.d.. Consequently, since $\tilde{E} \subset E\{\varepsilon, f_1(t)\}$ and $\tilde{E} \subset E\{\varepsilon_1, f_2(t)\}$,

$$\tilde{E} \subset E\{\varepsilon, f_1(t)\} \cap E\{\varepsilon_1, f_2(t)\} \subset E\{\varepsilon, f_1(t)\} \cap E\{\varepsilon, f_2(t)\}.$$

Thus also $E\{\varepsilon, f_1(t)\} \cap E\{\varepsilon, f_2(t)\}$ is r.d.. □

Finally, we state an important result about summability of the space of $G^\star_{\mathrm{ap}}$-functions and $\tilde{G}$-functions .

PROPOSITION 3. ([AB2]) *The sum of a function $f_1 \in \tilde{G}$ and a function $f_2 \in G^\star_{\mathrm{ap}}$ belongs to $G^\star_{\mathrm{ap}}$.*

P r o o f. For every $\varepsilon > 0$, take $\tau \in E\left\{\frac{1}{2}\varepsilon, f_1(t)\right\} \cap E\left\{\frac{1}{2}\varepsilon, f_2(t)\right\}$. Thus,

$$\|(f_1 + f_2)(x + \tau) - (f_1 + f_2)(x)\|_G \le \varepsilon.$$

Consequently, $\tau \in E\{\varepsilon, f_1(t) + f_2(t)\}$, and

$$E\left\{\tfrac{1}{2}\varepsilon, f_1(t)\right\} \cap E\left\{\tfrac{1}{2}\varepsilon, f_2(t)\right\} \subset E\{\varepsilon, f_1(t) + f_2(t)\},$$

from which $E\{\varepsilon, f_1 + f_2\}$ is r.d.. □

Remark 4. In order to guarantee at least one function in the sum to be uniformly G-a.p., it is sufficient to assume only its continuity in the G-norm (see [LZ; p. 2]). Since Carathéodory bounded solutions of differential equations are understood to be locally absolutely continuous and so, because of the assumptions below, (in particular) essential boundedness of nonhomogenities, they become uniformly continuous. Consequently, they will be also continuous in the G-metric. So, they can be used as those which are uniformly G-continuous.

Remark 5. In [Le; pp. 206–207] (cf. also [C1]), we can find sufficient and necessary conditions for almost periodicity of the integral over $\mathbb{R}$ of u.a.p. or S_1-a.p. functions, namely the integral of u.a.p. or S_1-a.p. function is u.a.p. if and only if it is uniformly continuous and bounded on $\mathbb{R}$ in the given metric, respectively.

PROPOSITION 4. *If the integral $F(x) = \int_0^x f(t)\,\mathrm{d}t$, where $F\colon \mathbb{R} \to \mathbb{R}$, of a G-a.p. or equi-W-a.p. function $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous and bounded on $\mathbb{R}$, then it is G-a.p. or equi-W-a.p., respectively.*

Proof. We follow some ideas in [St] (cf. also [De]), where they were however used for another goal. Since the integral

$$F(x) = \int_0^x f(t)\,\mathrm{d}t$$

is bounded, for every $\varepsilon > 0$ there exist numbers x_1, x_2 for which $F(x_1) < g + \varepsilon$ and $F(x_2) > G - \varepsilon$, where $g = \inf F(x)$ and $G = \sup F(x)$. Let us put $\xi = \min\{x_1, x_2\}$, $\eta = \max\{x_1, x_2\}$ and $d = |x_1 - x_2| > 0$. Since f is G-a.p., so in every interval of length $k > 0$ there exists at least one ε-almost period of f (in the given sense).

Let us choose τ, the ε-almost period of the function f such that the number $\xi + \tau$ lies in an interval $(\alpha, \alpha{+}k)$. Then the numbers $z_1 = x_1 + \tau$, $z_2 = x_2 + \tau$ are contained in the interval $(\alpha, \alpha{+}K)$, $K = k + d > 0$ and we obtain

$$\begin{aligned} F(z_2) - F(z_1) &= F(x_2) - F(x_1) + \int_{x_1}^{x_2} \big[f(t) - f(t+\tau)\big]\,\mathrm{d}t \\ &> G - g - 2\varepsilon + \int_{x_1}^{x_2} \big[f(t) - f(t+\tau)\big]\,\mathrm{d}t\,. \end{aligned}$$

So, we proved that in every interval of length K, there exist numbers z_1, z_2 such that

$$F(z_1) < g+2\varepsilon-\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t, \qquad F(z_2) > G-2\varepsilon+\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t.$$

Let us choose in the interval $(x, x+K)$, where x is an arbitrary real number, some z_1 such that $F(z_1) < g+2\varepsilon-\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t$. We can write

$$\begin{aligned} F(x+\tau)-F(x) &= F(z_1+\tau)-F(z_1)+\int_{x}^{z_1}[f(t)-f(t+\tau)]\,\mathrm{d}t \\ &> -2\varepsilon+\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t+\int_{x}^{z_1}[f(t)-f(t+\tau)]\,\mathrm{d}t. \end{aligned}$$

And again, choosing in the interval $(x, x+K)$, where x is an arbitrary real number, a z_2 such that $F(z_2) > G-2\varepsilon-\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t$ we obtain in the same way the following estimate:

$$F(x+\tau)-F(x) < 2\varepsilon+\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t+\int_{x}^{z_2}[f(t)-f(t+\tau)]\,\mathrm{d}t,$$

i.e.

$$\begin{aligned} |F(x+\tau)-F(x)| &< 2\varepsilon+\left|\int_{x_1}^{x_2}[f(t)-f(t+\tau)]\,\mathrm{d}t\right|+\left|\int_{x}^{z}|f(t)-f(t+\tau)|\,\mathrm{d}t\right| \\ &\le 2\varepsilon+\int_{\xi}^{\eta}|f(t+\tau)-f(t)|\,\mathrm{d}t+\int_{x}^{z}|f(t+\tau)-f(t)|\,\mathrm{d}t, \end{aligned}$$

where $z=\max\{z_1, z_2\}$.

Since the cases of u.a.p. and S_1-a.p. functions were pointed out in Remark 5, we restrict ourselves to the remaining cases.

So, if f is equi-W-a.p. (i.e., for given $\varepsilon > 0$ there exists $l_0 \ge 1$ (cf. Remark 1) such that, for every $l \ge l_0$, $\sup_{a\in\mathbb{R}} \frac{1}{l}\int_{a}^{a+l}|f(t+\tau)-f(t)|\,\mathrm{d}t < \varepsilon$), one can estimate

$$\int_{\xi}^{\eta}|f(t+\tau)-f(t)|\,\mathrm{d}t < \left(\frac{d}{l_0}+1\right) l_0\varepsilon$$

and

$$\begin{aligned}
\sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} \left(\int_x^{z} |f(t+\tau) - f(t)| \,\mathrm{d}t \right) \mathrm{d}x
&\le \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} \left(\int_x^{x+K} |f(t+\tau) - f(t)| \,\mathrm{d}t \right) \mathrm{d}x \\
&= \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} \left(\int_0^{K} |f(t+x+\tau) - f(t+x)| \,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_0^{K} \left(\sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t+x+\tau) - f(t+x)| \,\mathrm{d}x \right) \mathrm{d}t < K\varepsilon .
\end{aligned}$$

It follows (cf. Remark 1) that

$$\begin{aligned}
\sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |F(x+\tau) - F(x)| \,\mathrm{d}x < 2\varepsilon + \left(\frac{d}{l_0} + 1 \right) \varepsilon + K\varepsilon = \varepsilon \left(2 + \frac{d}{l_0} + 1 + K \right) \\
\le \varepsilon(3 + d + K) .
\end{aligned}$$

Clearly, F is equi-W-a.p.

Using the estimates introduced above, we can deduce the following inequalities for a W-a.p. function f:

$$\begin{aligned}
&\lim_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |F(x+\tau) - F(x)| \,\mathrm{d}x \\
&< 2\varepsilon + \varepsilon \left(\frac{d}{l_0} + 1 \right) \lim_{l\to\infty} \frac{l_0}{l} + \lim_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l} \int_0^{K} \left(\int_a^{a+l} |f(x+t+\tau) - f(x+t)| \,\mathrm{d}x \right) \mathrm{d}t \\
&= 2\varepsilon + \int_0^{K} \left(\lim_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(x+t+\tau) - f(x+t)| \,\mathrm{d}x \right) \mathrm{d}t \\
&< 2\varepsilon + K\varepsilon = \varepsilon(2 + K) .
\end{aligned}$$

Thus, a bounded integral of a W-a.p. function is W-a.p.

An analogous result holds when f is B-a.p.:

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |F(x+\tau) - F(x)| \, \mathrm{d}x \\
&< 2\varepsilon + \limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \left(\int\limits_{\xi}^{\eta} |f(t+\tau) - f(t)| \, \mathrm{d}t \right) \mathrm{d}x \\
&\quad + \limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \left(\int\limits_{x}^{z} |f(t+\tau) - f(t)| \, \mathrm{d}t \right) \mathrm{d}x \\
&< 2\varepsilon + \varepsilon \left(\limsup_{T\to\infty} \left[\frac{d}{2T} \right] + 1 \right) \\
&\quad + \limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \left(\int\limits_{0}^{K} |f(x+t+\tau) - f(x+t)| \, \mathrm{d}t \right) \mathrm{d}x \\
&< 3\varepsilon + \int\limits_{0}^{K} \left(\limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |f(x+t+\tau) - f(x+t)| \, \mathrm{d}x \right) \mathrm{d}t < \varepsilon(3+K) \,,
\end{aligned}$$

which implies the B-almost-periodicity of F. □

3. Main results for a.p. solutions

In this section, almost-periodicity results will be proved for almost-periodic nonhomogenities in various metrics, on the basis of the integral representation of entirely bounded solutions.

A) A.p. solutions of a scalar equation.

First, consider the equation

$$x' = ax + f(t) \tag{1}$$

and assume that $a \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is essentially bounded and G-a.p. or equi-W-a.p.. Let $x\colon \mathbb{R} \to \mathbb{R}$ solve (1) and let x be bounded. Now, we shall state theorems about almost-periodic solutions.

THEOREM 1. *Every bounded solution $x\colon \mathbb{R} \to \mathbb{R}$ of the differential equation* (1) *with $a \in \mathbb{R}$ is G-normal* (*and subsequently, G-a.p.*) *or equi-W-a.p.* (*and subsequently, equi-W-normal*), *provided $f(t)$ is essentially bounded and G-a.p., where G can be any of the given metrics, or equi-W-a.p., respectively.*

Proof. A general solution of the equation takes the form (cf. [C1], [De], [NG])

$$x(t) = \mathrm{e}^{at}\left(x(0) + \int_0^t f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\right).$$

We can distinguish three possibilities according to the sign of the constant a.

a) If $a > 0$, in order to have a bounded solution for $t \to +\infty$, we must take $x(0) = -\int_0^{+\infty} f(\tau)\,\mathrm{e}^{-a\tau}\,\mathrm{d}\tau$, i.e. a particular solution of the equation (1) is

$$x(t) = -\,\mathrm{e}^{at}\int_t^{+\infty} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\,. \tag{2}$$

Let f be G-a.p. Suppose $x(t)$ is a solution of the equation (1) and $x(t)$ is bounded on $\mathbb{R}$ ($\Longrightarrow$ $x(t)$ is bounded in the G-metric). First, we would like to prove its G-almost-periodicity. Obviously,

$$x(t+\tau) = \mathrm{e}^{a(t+\tau)}\int_{t+\tau}^{+\infty} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\,.$$

Thus,

$$\begin{aligned}
|x(t+\tau) - x(t)| &= \left|\mathrm{e}^{a(t+\tau)}\int_{t+\tau}^{+\infty} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s - \mathrm{e}^{at}\int_t^{+\infty} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\right| \\
&= \left|\mathrm{e}^{a(t+\tau)}\int_t^{+\infty} f(s+\tau)\,\mathrm{e}^{-a(s+\tau)}\,\mathrm{d}s - \mathrm{e}^{at}\int_t^{+\infty} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\right| \\
&= \left|\mathrm{e}^{at}\int_t^{+\infty} \big(f(s+\tau) - f(s)\big)\,\mathrm{e}^{-as}\,\mathrm{d}s\right|.
\end{aligned}$$

Furthermore, for applying the G-metric, we need the following estimate:

$$
\begin{aligned}
\int_\alpha^\beta |x(t+\tau)-x(t)|\,\mathrm{d}t &= \int_\alpha^\beta \left| \mathrm{e}^{at} \int_t^{+\infty} \big(f(s+\tau)-f(s)\big)\,\mathrm{e}^{-as}\,\mathrm{d}s \right| \mathrm{d}t \\
&= \int_\alpha^\beta \left| \mathrm{e}^{at} \int_0^{+\infty} \big(f(s+t+\tau)-f(s+t)\big)\,\mathrm{e}^{-a(s+t)}\,\mathrm{d}s \right| \mathrm{d}t \\
&= \int_\alpha^\beta \left| \int_0^{+\infty} \big(f(s+t+\tau)-f(s+t)\big)\,\mathrm{e}^{-as}\,\mathrm{d}s \right| \mathrm{d}t \\
&\le \int_\alpha^\beta \int_0^{+\infty} \left| \big(f(s+t+\tau)-f(s+t)\big)\,\mathrm{e}^{-as} \right| \mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{+\infty} \int_\alpha^\beta \left| \big(f(s+t+\tau)-f(s+t)\big)\,\mathrm{e}^{-as} \right| \mathrm{d}t\,\mathrm{d}s \\
&= \int_0^{+\infty} |\mathrm{e}^{-as}| \left(\int_\alpha^\beta |f(s+t+\tau)-f(s+t)|\,\mathrm{d}t \right) \mathrm{d}s\,.
\end{aligned}
$$

Using the Stepanov metric, we get ($\alpha := u$, $\beta := u+1$):

$$
\begin{aligned}
&\sup_{u\in\mathbb{R}} \int_u^{u+1} |x(t+\tau)-x(t)|\,\mathrm{d}t \\
&\quad\le \sup_{u\in\mathbb{R}} \int_0^{+\infty} |\mathrm{e}^{-as}| \left(\int_u^{u+1} |f(s+t+\tau)-f(s+t)|\,\mathrm{d}t \right) \mathrm{d}s \\
&\quad= \int_0^{+\infty} |\mathrm{e}^{-as}| \left(\sup_{u\in\mathbb{R}} \int_u^{u+1} |f(s+t+\tau)-f(s+t)|\,\mathrm{d}t \right) \mathrm{d}s \\
&\quad< \widehat{\varepsilon} \int_0^{+\infty} \mathrm{e}^{-as}\,\mathrm{d}s = \frac{\widehat{\varepsilon}}{a} = \varepsilon\,.
\end{aligned}
$$

Hence, every $\widehat{\varepsilon}$-almost period of function f (in the sense of Stepanov) corresponds to an ε-almost period of solution x (in the sense of Stepanov).

For the equi-Weyl case, one can obtain ($\alpha := u$, $\beta := u + l$):

$$\begin{aligned}
&\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |x(t+\tau) - x(t)| \,\mathrm{d}t \\
&\quad \le \sup_{u\in\mathbb{R}} \frac{1}{l} \int_0^{+\infty} |\,\mathrm{e}^{-as}\,| \left(\int_u^{u+l} |f(s+t+\tau) - f(s+t)| \,\mathrm{d}t \right) \mathrm{d}s \\
&\quad = \int_0^{+\infty} |\,\mathrm{e}^{-as}\,| \left(\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |f(s+t+\tau) - f(s+t)| \,\mathrm{d}t \right) \mathrm{d}s \\
&\quad < \widehat{\varepsilon} \int_0^{+\infty} \mathrm{e}^{-as} \,\mathrm{d}s = \frac{\widehat{\varepsilon}}{a} = \varepsilon \,.
\end{aligned}$$

If one uses the estimate introduced above, one obtains the following inequalities for the W-almost-periodicity:

$$\begin{aligned}
&\lim_{l\to\infty} \left[\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |f(t+\tau) - f(t)| \,\mathrm{d}t \right] \\
&\quad \le \int_0^{+\infty} |\,\mathrm{e}^{-as}\,| \left(\lim_{l\to\infty} \left[\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |f(s+t+\tau) - f(s+t)| \,\mathrm{d}t \right] \right) \mathrm{d}s \\
&\quad < \widehat{\varepsilon} \int_0^{+\infty} \mathrm{e}^{-as} \,\mathrm{d}s = \frac{\widehat{\varepsilon}}{a} = \varepsilon \,,
\end{aligned}$$

which implies the W-almost-periodicity of the solution x.

The proof for B-almost-periodicity is again based on the application of the inequality which was derived above. Consequently, ($\alpha := -T$, $\beta := T$):

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t+\tau) - x(t)| \,\mathrm{d}t \\
&\quad = \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \mathrm{e}^{at} \int_0^{+\infty} \mathrm{e}^{-a(s+t)} \big(f(s+t+\tau) - f(s+t) \big) \,\mathrm{d}s \right| \mathrm{d}t \\
&\quad = \limsup_{T\to\infty} \frac{1}{2T} \int_0^{+\infty} \int_{-T}^{T} \left| \mathrm{e}^{-as} \big(f(s+t+\tau) - f(s+t) \big) \right| \mathrm{d}t \,\mathrm{d}s
\end{aligned}$$

$$= \int_0^{+\infty} |\,\mathrm{e}^{-as}\,| \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \big|\big(f(s+t+\tau) - f(s+t)\big)\big| \,\mathrm{d}t \,\mathrm{d}s$$

$$< \widehat{\varepsilon} \int_0^{+\infty} |\,\mathrm{e}^{-as}\,| \,\mathrm{d}s = \frac{\widehat{\varepsilon}}{|a|} = \varepsilon\,.$$

b) If $a < 0$, we take the initial condition

$$x(0) = \int_{-\infty}^{0} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\,.$$

Instead of (2), the particular solution of the equation (1) can be written as follows:

$$x(t) = \mathrm{e}^{at} \int_{-\infty}^{t} f(s)\,\mathrm{e}^{-as}\,\mathrm{d}s\,.$$

The procedure and the conclusion are similar to the proof of the case **a**).

c) For $a = 0$, the equation (1) simplifies to

$$x'(t) = f(t)\,.$$

For arbitrary $t \in \mathbb{R}$, we obtain the solution

$$x(t) = x(0) + \int_0^t f(s)\,\mathrm{d}s\,.$$

Therefore,

$$x(t+\tau) - x(t) = \int_0^{t+\tau} f(s)\,\mathrm{d}s - \int_0^t f(s)\,\mathrm{d}s\,.$$

By Proposition 4 and Remark 4, G-a. periodicity or equi-W-a. periodicity of the solution x follows now from the boundedness of x and from the G-a. periodicity or equi-W-a. periodicity of the function f, respectively.

The G-normality of the above G-a.p. solutions follows from Remarks 3 and 4. The equi-W-normality of the above equi-W-a.p. solutions follows from Remark 3.

$\square$

B) A.p. solutions of a system.

Now consider the system

$$\boldsymbol{x}' = \mathbf{A}\boldsymbol{x} + \boldsymbol{f}(t)\,, \tag{3}$$

where $\mathbf{A} = \{a_{i,j}\}_{i,j=1}^{n}$ is a real constant matrix of the type $n \times n$ and the function $\boldsymbol{f}\colon \mathbb{R} \to \mathbb{R}^n$ is measurable and essentially bounded. We can also write the system (3) in the vector form:

$$\begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

THEOREM 2. *Every bounded solution $\boldsymbol{x}\colon \mathbb{R} \to \mathbb{R}^n$ of system (3) is G-normal or equi-W-normal, provided $\boldsymbol{f}$ is essentially bounded and G-a.p. or equi-W-a.p., respectively.*

P r o o f . We follow the ideas in [C1], [De]. For our constant matrix $\mathbf{A}$, there exists a matrix $\mathbf{B}$ such that putting

$$\boldsymbol{y} = \mathbf{B}\boldsymbol{x}\,,$$

then

$$\boldsymbol{y}' = \mathbf{C}\boldsymbol{y} + \boldsymbol{g}(t)\,, \tag{4}$$

where

$$\mathbf{C} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1} = \big[\mathbf{I}_{q_1}(\lambda_1), \dots, \mathbf{I}_{q_p}(\lambda_p)\big]$$

is the constant matrix, λ_j, $j = 1, \dots p$, are the eigenvalues of $\mathbf{A}$ and

$$\boldsymbol{g}(t) = \mathbf{B}\boldsymbol{f}(t)$$

is a vector of G-a.p. functions or equi-W-a.p. functions. Here the indices q_i satisfy the equality $\sum\limits_{i=1}^{p} q_i = n$ and $\mathbf{I}_{q_i}(\lambda_i)$ denotes the matrix (of type $q_i \times q_i$)

$$\mathbf{I}_{q_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix}.$$

Using the transformation $\boldsymbol{y} = \mathbf{B}\boldsymbol{x}$, we obtain system (4). Starting from the scalar equation in the last line of (4), which satisfies the assumptions of Theorem 1, we have its G-normal solution or its equi-W-normal solution, i.e. also a G-a.p. solution or equi-W-a.p. solution, respectively. Substituting this solution to the equation on the line above, we get again the scalar equation whose nonhomogenity consists of a $\tilde{G}$-function (the one after the substitution) and a

G_{ap}-function or two equi-W-a.p. functions, respectively. Thus, (see also Proposition 3 and Remarks 4 and 5), the same conclusion follows. Repeating this procedure, we obtain a solution $\boldsymbol{y}$ of (4) which is G-normal (and subsequently, G-a.p.) or equi-W-normal (and subsequently, equi-W-a.p.). By applying the inverse transformation and according to Proposition 3, Remarks 4 and 5, we obtain that $\boldsymbol{x}$ is the G-normal (and subsequently, G-a.p.) or equi-W-normal (and subsequently, equi-W-a.p.) solution of the system (3), respectively. This completes the proof. □

4 Concluding remarks

Some of the assertions in Theorem 2 can be proved alternatively by means of different methods (cf. [AP], [C2], [Fa]). A part of the results can also be extended to differential equations in Banach spaces (cf. [AB1], [BT], [DK], [MS], [NG], [Za]). We will treat Bohr-Neugebauer-type theorems for linear systems with a time dependent matrix $\mathbf{A}(t)$ or for nonlinear perturbations of systems of the form (3) elsewhere.

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