Mathematica Slovaca

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Mathematica Slovaca, Vol. 42 (1992), No. 3, 317--324

Persistent URL: http://dml.cz/dmlcz/132061

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THE OSCILLATION OF A DIFFERENTIAL EQUATION OF SECOND ORDER WITH DEVIATING ARGUMENT

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ABSTRACT. Our aim in this paper is to present criteria for oscillation and asymptotic behaviour of the equation

$$\big(r(t)u'(t)\big)' + p(t)f\big(u[g(t)]\big) = 0 .$$

The considered equation is in canonical or noncanonical form.

We consider the second order differential equation with deviating argument

$$\big(r(t)u'(t)\big)' + p(t)f\big(u[g(t)]\big) = 0 , \qquad t \ge t_0 . \tag{1}$$

We suppose throughout the paper that the following conditions hold.

(i) $p \in C\big([t_0,\infty)\big), \quad p(t) > 0,$

(ii) $r \in C\big([t_0,\infty)\big), \quad r(t) > 0,$

(iii) $g \in C\big([t_0,\infty)\big), \quad g(t) \to \infty \quad \text{as} \quad t \to \infty,$

(iv) $f \in C\big((-\infty,\infty)\big), \quad xf(x) > 0 \quad \text{for} \quad x \neq 0.$

In the sequel we will restrict our attention to those solutions of the equations considered which are non-trivial in any neighbourhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

Equation (1) is said to be in canonical form if

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty , \tag{2}$$

and to be in canonical form if

AMS Subject Classification (1991): Primary 34C10.
Key words: Oscillation, Deviating argument.

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} < \infty . \tag{3}$$

For the sake of brevity we use the following notation

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)}, \qquad t \ge t_0 ,$$

for canonical equations,

$$\varrho(t) = \int_{t}^{\infty} \frac{\mathrm{d}s}{r(s)}, \qquad t \ge t_0 ,$$

for noncanonical equations and

$$S_f = \max\left\{ \limsup_{y\to-\infty} \frac{y}{f(y)} , \limsup_{y\to\infty} \frac{y}{f(y)} \right\} ,$$

for all equations and further we assume that $S_f < \infty$. Second order equations are most important in applications. Thus there is much literature regarding this kind of equations, mostly for canonical form (see e.g. [3], [4] and [6]). In the paper we deal with both canonical and noncanonical equations.

Consider the linear case of equation (1), namely, the equation

$$\big(r(t)u'(t)\big)' + p(t)\big(u[g(t)]\big) = 0 . \tag{4}$$

The following theorem is a simple consequence of [1, Theorem 11].

THEOREM 1. *Assume that* (2) *hold. Let*

$$Q \in C^1\big([t_0,\infty)\big) , \quad Q'(t) > 0 , \quad Q(t) \le \min\{g(t),t\} , \quad Q(t) \to \infty \ \textit{as} \ t \to \infty . \tag{5}$$

(i) *Equation* (4) *is oscillatory if*

$$\liminf_{t\to\infty} R[Q(t)] \int_{t}^{\infty} p(s)\,\mathrm{d}s > \frac{1}{4} .$$

(ii) *If*

$$\liminf_{t\to\infty} R[Q(t)] \int_t^\infty p(s)\,\mathrm{d}s > 0\,,$$

then every nonoscillatory solution $u(t)$ *of* (4) *satisfies*

$$\begin{aligned} &\lim_{t\to\infty} |u(t)| = \infty\,, \\ &\lim_{t\to\infty} r(t)\,u'(t) = 0\,. \end{aligned} \tag{P$_1$}$$

The following theorem is intended for extending the previous result to equation (1)

THEOREM 2. *Assume that* (2) *and* (5) *hold.*

(i) *The condition*

$$\liminf_{t\to\infty} R[Q(t)] \int_t^\infty p(s)\,\mathrm{d}s > \frac{1}{4} S_f \tag{6}$$

is sufficient for equation (1) *to be oscillatory.*

(ii) *If*

$$\liminf_{t\to\infty} R[Q(t)] \int_t^\infty p(s)\,\mathrm{d}s > 0\,, \tag{7}$$

then every nonoscillatory solution $u(t)$ *of* (1) *satisfies* (P_1).

Proof.

(i): Let $u(t)$ be a nonoscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality we may assume that $u(t)$ is positive. Then by a well-known lemma of Kiguradze [2, Lemma 3] we obtain that $u'(t) > 0$ for all $t \ge t_1$ $(\ge t_0)$. Let us assume that $u(t)$ is bounded. Then there exist some positive constants c_1 and c_2 such that for all $t \ge t_1$

$$\begin{aligned} c_1 &\ge u(t) \ge c_2\,, \\ c_1 &\ge u[g(t)] \ge c_2\,. \end{aligned}$$

An integration of (1) yields

$$u(t) \ge \int_{t_1}^{t} \frac{1}{r(s_1)} \int_{s_1}^{\infty} p(s_2)\, f\big(u[g(s_2)]\big)\,\mathrm{d}s_2\,\mathrm{d}s_1\,, \qquad t \ge t_1\,. \tag{8}$$

Denote $f_0 = \min\limits_{[c_1,c_2]} f(u)$, then from (8) one gets

$$c_1 \geq u(t) \geq f_0 \int\limits_{t_1}^{t} p(s_2) \int\limits_{t_1}^{s_2} \frac{1}{r(s_1)} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \,, \qquad t \geq t_1 \,.$$

Hence, for all $t \geq t_1$

$$c_1 \geq f_0 \int\limits_{t_1}^{t} p(s_2) \big(R(s_2) - R(t_1)\big) \, \mathrm{d}s_2 \,. \tag{9}$$

Since (6) implies $\int^{\infty} p(s)R(s) \, \mathrm{d}s = \infty$, from (9) we have a contradiction for $t \to \infty$.

Now, let us suppose that $u(t)$ is unbounded as $t \to \infty$. Let t_2 ($\geq t_1$) be chosen so that $g(t) \geq t_0$ for $t \geq t_2$. Then arguing exactly as in the proof of [5, Theorem 1], it is easy to see that $u(t)$ is a solution of the linear equation

$$\big(r(t)z'(t)\big)' + \hat{p}(t)u[g(t)] = 0 \,, \qquad t \geq t_0 \,, \tag{10}$$

where

$$\hat{p}(t) = \begin{cases} p(t)\dfrac{f\big(u[g(t)]\big)}{u[g(t)]} \,, & \text{for } t \geq t_2 \\ \hat{p}(t_2) \,, & \text{for } t \in [t_0, t_2]. \end{cases}$$

and moreover

$$\int^{\infty} p(s) \, \mathrm{d}s \leq \left\{ \sup_{y \geq u[g(t)]} \frac{y}{f(y)} \right\} \int^{\infty} \hat{p}(s) \, \mathrm{d}s \tag{11}$$

and

$$\liminf_{t \to \infty} R[Q(t)] \int\limits_{t}^{\infty} p(s) \, \mathrm{d}s \leq \frac{S_f}{4} \,,$$

which contradicts (6).

(ii): Let us suppose that $u(t)$ be a positive solution of (1) on $[t_0, \infty)$. Since (7) implies $\int^{\infty} p(s)R(s) \, \mathrm{d}s = \infty$, then exactly as in the part (i) it can be shown that

$u(t)$ cannot be bounded. Hence, we may suppose that $u(t)$ is unbounded as $t \to \infty$. Similarly as above we can verify that $u(t)$ is a solution of (10). Let us suppose that $u(t)$ does not satisfy (P_1). Then by Theorem 1 there must be

$$\liminf_{t\to\infty} R[Q(t)] \int_t^\infty \hat{p}(s)\,\mathrm{d}s = 0 .$$

Taking (11) into account and proceeding as above we obtain

$$\liminf_{t\to\infty} R[Q(t)] \int_t^\infty p(s)\,\mathrm{d}s \le 0 , \qquad S_f = 0 ,$$

which contradicts (7). The proof is complete.

Theorem 2 provides for the superlinear case of equation (1), namely, for the equation

$$\big(r(t)u'(t)\big)' + p(t)|u[g(t)]|^\alpha \operatorname{sgn} u[g(t)] = 0 , \qquad \alpha > 1 , \tag{12}$$

the following result:

COROLLARY 1. *Assume that* (2) *and* (5) *hold. The condition*

$$\liminf_{t\to\infty} R[Q(t)] \int_t^\infty p(s)\,\mathrm{d}s > 0$$

is sufficient for equation (12) *to be oscillatory.*

Note that Theorem 1 is an extension of [3, Corollary 4.5.2] and generalizes the result of J. Ohriska [4, Theorem 2.3] concerning the equation $\big(r(t)u'(t)\big)' + p(t)\big(u(t)\big) = 0$. On the other hand, Theorem 2 extends the result of Ch. G. Philos and Y. G. Sficas [5, Theorem 1']. For another result similar to Corollary 1 the reader is referred to [3, Corollary 4.5.1], where $\alpha \in (0,1)$.

Now we are prepared to extend the previous result to noncanonical equations.

THEOREM 3. *Assume that* (3) *and* (5) *hold.*

(i) *Equation* (4) *is oscillatory if*

$$\liminf_{t\to\infty} \frac{1}{\varrho[Q(t)]} \int_t^\infty \varrho[g(s)]\varrho(s)p(s)\,\mathrm{d}s > \frac{1}{4} . \tag{13}$$

(ii) *If*

$$\liminf_{t\to\infty} \frac{1}{\varrho[Q(t)]} \int_t^\infty \varrho[g(s)]\varrho(s)p(s)\,ds > 0\,, \tag{14}$$

then every nonoscillatory solution $u(t)$ *of (1) satisfies* $\lim_{t\to\infty} u(t) = 0$.

Proof.

(i): According to the general theory of W. F. Trench [7, Lemma 1], equation (4) can be rewritten in canonical form as

$$q_2(t)\big(q_1(t)\big(q_0(t)u(t)\big)'\big)' + p(t)u[g(t)] = 0\,, \tag{4_c}$$

where functions $q_i(t) \in C\big([t_0,\infty)\big)$ and are defined as follows:

$$\frac{1}{q_0(t)} = \frac{1}{q_2(t)} = \int_t^\infty \frac{ds}{r(s)}\,,$$

$$q_1(t) = r(t)\left(\int_t^\infty \frac{ds}{r(s)}\right)^2$$

The transformation $q_0 u = y$ reduces equation (4_c) to

$$\big(q_1(t)y'(t)\big)' + \frac{p(t)}{q_2(t)q_0[g(t)]}y[g(t)] = 0\,, \tag{15}$$

and furthermore equation (4_c) (as well as (4)) is oscillatory if and only if equation (15) is oscillatory. On the other hand, by Theorem 1 we know that equation (15) is oscillatory if

$$\liminf_{t\to\infty}\left(\int_{t_0}^{Q(t)} \frac{ds}{q_1(s)}\right)\left(\int_t^\infty \frac{p(s)}{q_2(s)q_0[g(s)]}\,ds\right) > \frac{1}{4}\,. \tag{16}$$

A simple computation shows that (16) is equivalent to (13).

(ii): From the first part of the proof we know that $y(t)$ is a solution of (15) if and only if $u(t) = y(t)/q_0(t)$ is a solution of (4). Theorem 1 implies that if

$$\liminf_{t\to\infty}\left(\int_{t_0}^{Q(t)} \frac{ds}{q_1(s)}\right)\left(\int_t^\infty \frac{p(s)}{q_2(s)q_0[g(s)]}\,ds\right) > 0\,,$$

(which is equivalent to (14)), then every eventually positive solution of (15) satisfies

$$\lim_{t\to\infty} q_1(t)y'(t) = 0 .$$

Therefore every eventually positive solution $u(t)$ of (4) satisfies

$$\lim_{t\to\infty} r(t)\varrho(t)^2 \left(\frac{u(t)}{\varrho(t)}\right)' = 0 .$$

Hence, for every sufficiently small $\varepsilon > 0$, there exists a t_1 such that for all $t \geq t_1$

$$\left(\frac{u(t)}{\varrho(t)}\right)' \leq \frac{\varepsilon}{r(t)\varrho^2(t)} .$$

Integrating this inequality from t_1 to t ($\geq t_1$), we have

$$\frac{u(t)}{\varrho(t)} - \frac{u(t_1)}{\varrho(t_1)} \leq \varepsilon \left(\frac{1}{\varrho(t)} - \frac{1}{\varrho(t_1)}\right),$$

that is

$$0 < u(t) \leq k\varrho(t) + \varepsilon , \qquad t \geq t_1 , \tag{17}$$

where k is a positive constant. Letting $t \to \infty$ in (17) we obtain $\lim\limits_{t\to\infty} u(t) = 0$. The proof is complete.

Proceeding similarly as above and taking Corollary 1 into account we obtain the following result, which is related to [3, Corollary 4.7.3].

THEOREM 4. *Assume that* (3) *and* (5) *hold. Then the condition*

$$\liminf_{t\to\infty} \frac{1}{\varrho[Q(t)]} \int_t^\infty \varrho^\alpha[g(s)]\varrho(s)p(s)\,\mathrm{d}s > 0$$

is sufficient for equation (12) *to be oscillatory.*

In literature we can find many sufficient conditions for canonical equation (1) to be oscillatory. Applying the above-mentioned technique to such results we can obtain sufficient conditions for noncanonical equation (1) to be oscillatory.

We would like to point out that Theorems 1 – 4 work no matter if equations (1) and (4) are delay equations or advanced or even a mixed type.

Example 1. Consider the noncanonical differential equation

$$\left(t^2y'(t)\right)' + ay(t/3) = 0 , \qquad t \geq 1 . \tag{18}$$

Put $g(t) = Q(t) = t/3$ and $\varrho(t) = 1/t$. By Theorem 3 this equation is oscillatory if $a > 1/4$ and every nonoscillatory solution $y(t)$ of (18) satisfies $\lim_{t\to\infty} y(t) = 0$ if $a > 0$. For example for $a = 1/(4\sqrt{3})$ equation (18) has a solution $y(t) = t^{-1/2}$, which vanishes in infinity.

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Received August 30, 1990

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