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NONMULTIPLICATIVE MEASURES IN LOCALLY COMPACT PRODUCTS

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1. We are concerned in this paper with set functions in the product of two locally compact spaces which are a generalization of the product of two regular Borel measures treated in [4, p. 113], [2, p. 139] and [1, p. 203, Ex. 13]. Some integration in the product of two locally compact spaces with respect to such set functions is also developed.

For the rest of the paper, S and T denote the locally compact (Hausdorff) spaces. The Borel sets of S, T, and $S \times T$ are the σ-ring $\mathscr{B}(S)$, $\mathscr{B}(T)$ and $\mathscr{B}(S \times T)$ generated by the compact subsets of S, T, and $S \times T$, respectively. Let us denote

$$\mathscr{B}(S) \text{ o } \mathscr{B}(T) = \{E \times F : E \in \mathscr{B}(S), F \in \mathscr{B}(T)\}.$$

Further, $\mathscr{B}(S) \hat{\text{o}} \mathscr{B}(T)$ will denote the ring generated by $\mathscr{B}(S) \text{ o } \mathscr{B}(T)$. It is well known [3, Theorem 33 E] that the ring $\mathscr{B}(S) \hat{\text{o}} \mathscr{B}(T)$ consists of all sets of the form

$$G = \bigcup_{i=1}^{k} E_i \times F_i,$$

where $E_i \times F_i$, $i = 1, \ldots, k$, is a finite family of the mutually disjoint sets from $\mathscr{B}(S) \text{ o } \mathscr{B}(T)$. The product σ-ring of $\mathscr{B}(S)$ and $\mathscr{B}(T)$, denoted $\mathscr{B}(S) \times \mathscr{B}(T)$, is the σ-ring generated by $\mathscr{B}(S) \text{ o } \mathscr{B}(T)$ or equivalently by $\mathscr{B}(S) \hat{\text{o}} \mathscr{B}(T)$. Since $\mathscr{B}(S) \times \mathscr{B}(T)$ is in fact generated by rectangles with compact sides [1, 35.2], we have

$$\mathscr{B}(S) \times \mathscr{B}(T) \subset \mathscr{B}(S \times T),$$

the inclusion being, in general, proper. For the definitions of Baire, Borel measure and regular Borel measure, the reader is referred to [1, Chapter 8]. Let μ and ν be any regular Borel measures on S and T, respectively. The product [3, 35.B], $\mu \times \nu$, of μ and ν may fail to be a Borel measure, because its domain is too small. However, by Johnson [4, Theorem 2.1] the following was proved (cf. also [2, p. 139]).

Theorem A. *If μ and ν are regular Borel measures on the locally compact spaces S and T, respectively, then there exists one and only one regular Borel measure ϱ on $S \times T$ which extends the product $\mu \times \nu$ of μ and ν.*

By Berberian [2, p. 139] a slightly more general result than that contained in Theorem A was proved.

Theorem B. *Let S and T be locally compact spaces, and suppose that τ is a measure on the σ-ring $\mathscr{B}(S) \times \mathscr{B}(T)$ such that* (i) *for each compact set C in S, the correspondence*

$$F \to \tau(C \times F) \quad (F \in \mathscr{B}(T))$$

is a regular Borel measure on T, and (ii) *for each compact set D in T, the correspondence*

$$E \to \tau(E \times D) \quad (E \in \mathscr{B}(S))$$

is a regular Borel measure on S. Then τ may be extended to one and only one regular Borel measure ϱ on $S \times T$.

The measure ϱ in Theorem A extending $\mu \times \nu$ is multiplicative in the sense that

$$\varrho(E \times F) = \mu(E)\nu(F)$$

for all Borel sets E and F in S and T, respectively.

2. If we apply Theorem B we can derive some interesting results concerning nonmultiplicative set functions defined on $\mathscr{B}(S) \circ \mathscr{B}(T)$.

We shall prove the following theorem (cf. also [5] for a different approach).

Theorem 1. *Let λ be a non negative extended real valued set function on $\mathscr{B}(S) \circ \mathscr{B}(T)$ such that*

(i) *$\lambda(C \times D) < \infty$ for all compact sets $C \in \mathscr{B}(S)$ and $D \in \mathscr{B}(T)$ and for each rectangle $E \times F$ in $\mathscr{B}(S) \circ \mathscr{B}(T)$ we have*

$$\lambda(E \times F) = \sup \{\lambda(C \times D) \colon C, D \text{ compact}, C \subset E, D \subset F\},$$

(ii) *for each E in $\mathscr{B}(S)$ the correspondence $F \to \lambda(E \times F)$ is additive on $\mathscr{B}(T)$,*

(iii) *for each F in $\mathscr{B}(T)$ the correspondence $E \to \lambda(E \times F)$ is additive on $\mathscr{B}(S)$.*

Then λ is σ-additive on $\mathscr{B}(S) \circ \mathscr{B}(T)$ and may be extended to one and only one regular Borel measure ϱ on $S \times T$.

Proof. Denote by $\bar{\lambda}$ the unique additive extension of λ to the ring $\mathscr{R} = \mathscr{B}(S)\ \hat{\circ}\ \mathscr{B}(T)$. Take an arbitrary $\varepsilon > 0$ and an arbitrary decreasing sequence G_n, $G_n \in \mathscr{R}$, $n = 1, 2, \ldots$, of the form

$$G_n = \bigcup_{i=1}^{k_n} E_i^n \times F_i^n$$

with $0 < \varepsilon < \bar{\lambda}(G_n) < \infty$, $n = 1, 2, \ldots$

From the assumption (i) it follows that for every n there exist compact sets $C_i^n \subset E_i^n$ and $D_i^n \subset F_i^n$, $i = 1, \ldots, k_n$, such that

$$\bar{\lambda}(G_n - Y_n) < \frac{\varepsilon}{2^n}, \quad n = 1, 2, \ldots,$$

where

$$Y_n = \bigcup_{i=1}^{k_n} C_i^n \times D_i^n.$$

Denote

$$X_n = \bigcap_{i=1}^{n} Y_i.$$

Then $m \leqq n$ implies $X_n \subset X_m$ and we have

$$\bar{\lambda}(G_n - X_n) = \bar{\lambda}\left(\bigcup_{i=1}^{n} (G_n - Y_i)\right) \leqq \sum_{i=1}^{n} \bar{\lambda}(G_i - Y_i) < \varepsilon.$$

It follows that $\bar{\lambda}(X_n) > 0$, $n = 1, 2, \ldots$, that is, the sets X_n are non empty and $X_{n+1} \subset X_n$. Since X_n are compact, we have

$$\bigcap_{n=1}^{\infty} G_n \supset \bigcap_{n=1}^{\infty} X_n \neq \emptyset.$$

We have thus proved that if $G_n \in \mathscr{R}$, $G_n \downarrow \emptyset$, $\bar{\lambda}(G_n) < \infty$, $n = 1, 2, \ldots$, then $\bar{\lambda}(G_n) \downarrow 0$.

Take $G, G_n \in \mathscr{B}(S) \circ \mathscr{B}(T)$, $n = 1, 2, \ldots$, with G_n mutually disjoint and $G = \bigcup_{n=1}^{\infty} G_n$. If $\lambda(G) < \infty$, then

$$\lim_{k \to \infty} \bar{\lambda}\left(\bigcup_{n=k}^{\infty} G_n\right) = 0$$

and hence

$$\lambda(G) = \sum_{n=1}^{\infty} \lambda(G_n).$$

Let now $\lambda(E \times F) = \infty$, $E \times F = \bigcup_{i=1}^{\infty} E_i \times F_i$ with $E_i \times F_i$ in $\mathscr{B}(S) \circ \mathscr{B}(T)$, $i = 1, 2, \ldots$ and mutually disjoint. If $\lambda(E_i \times F_i) = \infty$ for some i, we are ready. Let $\lambda(E_i \times F_i) < \infty$, $i = 1, 2, \ldots$ For any positive number K there are compact sets $C \subset E$ and $D \subset F$ such that

$$0 < K < \lambda(C \times D) < \infty.$$

We have

$$0 < K < \lambda(C \times D) = \lambda(C \times D \cap E \times F) =$$

$$= \lambda(C \times D \cap \bigcup_{i=1}^{\infty} E_i \times F_i) = \sum_{i=1}^{\infty} \lambda(C \times D \cap E_i \times F_i) \leqq$$

$$\leqq \sum_{i=1}^{\infty} \lambda(E_i \times F_i),$$

and since K is arbitrary, we have

$$\lambda(E \times F) = \sum_{i=1}^{\infty} \lambda(E_i \times F_i).$$

The σ-additive extension $\bar{\lambda}$ of λ to the ring $\mathscr{B}(S)\ \hat{o}\ \mathscr{B}(T)$ is a σ-finite measure and so has the unique extension τ to the σ-ring generated by $\mathscr{B}(S)\ \hat{o}\ \mathscr{B}(T)$, that is, to $\mathscr{B}(S) \times \mathscr{B}(T)$. The measure τ is defined on the σ-ring $\mathscr{B}(S) \times \mathscr{B}(T)$ and it follows from (i) that for each compact set C in S, the correspondence $F \to \tau(E \times F)$ $(F \in \mathscr{B}(T))$ is an inner regular and hence a regular [1, p. 197] Borel measure on T, and for each compact set D in T, the correspondence $E \to \tau(E \times D)$ is an inner regular and hence a regular Borel measure on S. According to Theorem B the measure τ (and also λ) may be extended to one and only one regular Borel measure ϱ on $S \times T$.

From Theorem 1 we obtain Theorem A.

Let $\mathscr{B}_0(S)$ and $\mathscr{B}_0(T)$ denote the σ-ring of Baire sets in S and T, respectively. The following holds.

Theorem 2. *Let μ be a non negative extended real valued set function on $\mathscr{B}_0(S)$ o $\mathscr{B}_0(T)$ such that*

(i) *$\mu(C \times D) < \infty$ for all compact sets G_δ, $C \in \mathscr{B}_0(S)$ and $D \in \mathscr{B}_0(T)$ and we have*

$$\mu(E \times F) = \sup \{\mu(C \times D) \colon C, D \text{ compact } G_\delta,\ C \subset E,\ D \subset F\},$$

(ii) *for each $E \in \mathscr{B}_0(S)$, the correspondence $F \to \mu(E \times F)$ is additive on $\mathscr{B}_0(T)$,*

(iii) *for each $F \in \mathscr{B}_0(T)$ the correspondence $E \to \mu(E \times F)$ is additive on $\mathscr{B}_0(S)$.*

Then μ is σ-additive on $\mathscr{B}_0(S)$ o $\mathscr{B}(T)$ and on $\mathscr{B}_0(S \times T)$ there exists one and only one Baire measure v which extends μ.

The proof is similar as in the Borel case, simpler however, because for Baire sets we have $\mathscr{B}_0(S) \times \mathscr{B}_0(T) = \mathscr{B}_0(S \times T)$.

3. We shall give some connections of the preceding results with integration in product spaces.

Let λ, τ and ϱ have the same meaning as in Theorem 1. If φ and ψ are Borel functions on S and T, respectively, then the function $\varphi \otimes \psi \colon (s, t) \to \varphi(s)\psi(t)$ is $\mathscr{B}(S) \times \mathscr{B}(T)$-measurable and if $\varphi \otimes \psi$ is, moreover, ϱ-integrable, then it is also τ-integrable [1, p. 218] and we have

$$\int \varphi(s)\psi(t)\mathrm{d}\varrho(s,t)=\int \varphi(s)\psi(t)\mathrm{d}\tau(s,t).$$

For example, we may take bounded Borel functions φ and ψ with compact support.

Let now φ be a non negative bounded Borel function with compact support defined on S. There exists a compact set $C\subset S$ such that $\varphi=0$ on $S-C$, and a sequence of simple Borel functions φ_n such that

$$0\leqq\varphi_n\uparrow\varphi,\quad \operatorname{supp}\varphi_n\subset\operatorname{supp}\varphi\subset C,$$

$$\varphi_n=\sum_{i=1}^{k_n} a_i^n c_{A_i^n},\quad A_i^n\subset C,\ i=1,\dots,k_n;\quad n=1,2,\dots$$

We suppose that the sets A_i^n are mutually disjoint and hence $a_i^n>0$. Since C is a compact set, the correspondence $F\to\lambda(C\times F)$ is a regular Borel measure on T and since $A_i^n\subset C$, the correspondences

$$F\to\lambda(A_i^n\times F),\quad i=1,\dots,k_n;\quad n=1,2,\dots$$

are regular Borel measures on T. It follows that the correspondences

$$F\to\nu_n(F)=\sum_{i=1}^{k_n} a_i^n\lambda(A_i^n\times F),\quad n=1,2,\dots$$

are regular Borel measures on T.

Observe that for every relatively compact set $D\subset T$ the map $E\to\lambda(E\times D)$, $E\in\mathscr{B}(S)$, is a regular Borel measure on S. For each compact set $D\subset T$ we have

$$\nu_n(D)=\sum_{i=1}^{k_n} a_i^n\lambda(A_i^n\times D)\leqq\int_S\varphi(s)\lambda(\mathrm{d}s\times D)<\infty,$$

$$n=1,2,\dots$$

We can see that ν_n form an increasing sequence of regular Borel measures on T such that for every compact set $D\subset T$ the sequence $\nu_n(D)$ is bounded. It follows [1, p. 203, Ex. 11] that its limit $\nu=\lim_{n\to\infty}\nu_n$ is a regular Borel measure on T. Of course, if D is compact, then we have

$$\nu(D)=\int_S\varphi(s)\lambda(\mathrm{d}s\times D).$$

This equality can be extended for all Borel sets H in T which are relatively compact,

$$\nu(H)=\int_S\varphi(s)\lambda(\mathrm{d}s\times H).$$

However, we may write

$$\nu(F)=\int_S \varphi(s)\lambda(\mathrm{d}s\times F)$$

for any $F\in\mathscr{B}(T)$ and for every non negative bounded Borel function φ with compact support, since if φ is $\lambda(\cdot\times F)$-integrable, we may use the monotone convergence theorem, if not (in which case we put $\nu(F)=\infty$), we may use its corollary [1, p. 94 and 95, Ex. 6].

Thus the function

$$F\to\nu(F)=\int_S \varphi(s)\lambda(\mathrm{d}s\times F)$$

is a regular Borel measure on T for any non negative bounded Borel function φ with compact support. Therefore every bounded Borel function ψ on T with compact support is integrable with respect to ν.

Similarly we could take a non-negative bounded Borel function ψ with compact support and obtain a regular Borel measure μ on S,

$$E\to\mu(E)=\int_T \psi(t)\lambda(E\times \mathrm{d}t),\quad E\in\mathscr{B}(S).$$

Hence every bounded Borel function φ on S with compact support is integrable with respect to μ.

From the preceding arguments we may derive the following theorem.

Theorem 3. (i) *Let λ be as in Theorem 1. Let φ be a bounded non negative real valued Borel function on S with compact support. Then the set function*

$$F\to\nu(F)=\int_S \varphi(s)\lambda(\mathrm{d}s\times F),\quad F\in\mathscr{B}(T),$$

is a regular Borel measure on T.

(ii) *Let ψ be a bounded real valued Borel function on T with compact support. Then ψ is ν-integrable.*

Now we may define an "iterated" integral with respect to λ as follows (for φ and ψ as in Theorem 3)

$$\int_T\int_S \varphi\otimes\psi\ \lambda(\mathrm{d}s\,\mathrm{d}t)=\int_T \psi\,\mathrm{d}\nu.$$

Similarly we may define an "iterated" integral

$$\int_S\int_T \varphi\otimes\psi\ \lambda(\mathrm{d}t\,\mathrm{d}s)=\int_S \varphi\,\mathrm{d}\mu.$$

Let $\varphi = c_A$ and $\psi = c_B$ be the characteristic functions of bounded Borel sets A and B, respectively, in S and T. Then we have

$$\int_T c_A \otimes c_B \,\mathrm{d}\tau = \tau(A \times B) =$$

$$= \int_T c_B(t)\tau(A \times \mathrm{d}t) = \int_T \int_S c_A(s) c_B(t) \tau(\mathrm{d}s\,\mathrm{d}t).$$

If

$$\varphi = \sum_{i=1}^{n} a_i c_{A_i}, \qquad \psi = \sum_{j=1}^{m} b_j c_{B_j}, \qquad a_i \geqq 0, \quad b_j \geqq 0,$$

are the simple Borel functions with compact support, we have

$$\int \varphi \otimes \psi \,\mathrm{d}\tau = \int \left(\sum_{i=1}^{n} a_i c_{A_i}\right)\left(\sum_{j=1}^{m} b_j c_{B_j}\right) \mathrm{d}\tau =$$

$$= \int_T \int_S \left(\sum_{i=1}^{n} a_i c_{A_i}(s)\right)\left(\sum_{j=1}^{m} b_j c_{B_j}(t)\right) \tau(\mathrm{d}s\,\mathrm{d}t) =$$

$$= \int_T \int_S \varphi(s)\psi(t)\tau(\mathrm{d}s\,\mathrm{d}t).$$

If φ and ψ are bounded non negative Borel functions with compact support on S and T, respectively, then they are uniform limits of a sequence of Borel simple functions with compact support,

$$\varphi = \lim_{n\to\infty} \varphi_n, \qquad \psi = \lim_{m\to\infty} \psi_m.$$

We have

$$\int \varphi \otimes \psi \,\mathrm{d}\tau = \lim_{n\to\infty} \lim_{m\to\infty} \int \varphi_n \otimes \psi_m \,\mathrm{d}\tau =$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \int_T \int_S \varphi_n(s)\psi_m(t)\tau(\mathrm{d}s\,\mathrm{d}t) =$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{k_m} b_j^m \sum_{i=1}^{k_n} a_i^n \tau(A_i^n \times B_j^m) =$$

$$= \lim_{m\to\infty} \sum_{j=1}^{k_m} b_j^m \int \varphi(s)\tau(\mathrm{d}s \times B_j^m) = \lim_{m\to\infty} \sum_{j=1}^{k_m} b_j^m \nu(B_j^m) =$$

$$= \int_T \psi(t)\,\mathrm{d}\nu(t) = \int_T \int_S \varphi(s)\psi(t)\tau(\mathrm{d}s\,\mathrm{d}t).$$

Similarly we could show that the equality

$$\int \varphi \otimes \psi \, d\tau = \int_S \int_T \varphi(s)\psi(t)\tau(dt\,ds)$$

holds. We have thus proved the following.

Theorem 4. *Let λ, φ and ψ be as in Theorem 3. Then both iterated integrals are equal, that is we have*

$$\int_S \int_T \varphi(s)\psi(t)\lambda(dt\,ds) = \int_T \int_S \varphi(s)\psi(t)\lambda(ds\,dt).$$

From the preceding we can see that the following theorem is also true.

Theorem 5. *Let ϱ be a regular Borel measure on $S \times T$. Let φ and ψ be bounded non negative Borel functions with compact support defined on S and T, respectively. Then we have*

$$\int_{S\times T} \varphi \otimes \psi \, d\varrho = \int_T \int_S \varphi \otimes \psi \varrho(ds\,dt) = \int_S \int_T \varphi \otimes \psi \varrho(dt\,ds).$$

Remark 1. It is easy to see that Theorem 5 remains true if the measure ϱ is replaced with the measure τ obtained from λ as in Theorem 1.

Remark 2. In this paper we started with a set function λ satisfying conditions of Theorem 1. We have extended the λ to the measure τ. This made it possible to prove the equality of both iterated integrals with respect to λ. It can be expected that another approach could start with iterated integrals with respect to λ without extending λ to a measure on $\mathcal{B}(S) \times \mathcal{B}(T)$ or on $\mathcal{B}(S \times T)$ as indicated above.

Remark 3. In theorems 3 through 5 functions φ and ψ are supposed to be non negative. It can be shown that we may take also real valued not necessarily non negative functions φ and ψ such that, for example, $\nu_2(F) = = \int \varphi^-(s)\ \lambda(ds \times F)$ is finite for all $F \in \mathcal{B}(T)$ and $\mu_2(E) = = \int \psi^-(t)\ \lambda(E \times dt)$ is finite for all $E \in \mathcal{B}(S)$.

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НЕМУЛЬТИПЛИКАТИВНЫЕ МЕРЫ В ЛОКАЛЬНО КОМПАКТНЫХ ПРОДУКТАХ

Милослав Духонь

Резюме

Пусть S и T — локально компактные (хаусдорфовы) пространства. Пусть $\mathscr{B}(S)$, $\mathscr{B}(T)$ и $\mathscr{B}(S\times T)$ — σ-кольца борелевских множеств соответсвенно в S, T, $S\times T$ (т.е. σ-кольца порожденные компактными множествами) и

$$\mathscr{B}(S)\ o\ \mathscr{B}(T)=\{E\times F\colon E\in\mathscr{B}(S),\ F\in\mathscr{B}(T)\}.$$

В статьи доказывается, кроме других, следующая теорема.

Пусть λ — неотрицательная конечная или бесконечная действительная функция множества на $\mathscr{B}(S)\ o\ \mathscr{B}(T)$ и такая, что

(1) $\lambda(C\times D)<\infty$ для всех компактных множеств $C\subset S$ и $D\subset T$ и кроме того для всякого борелевского прямоугольника $E\times F$ имеет место равенство

$$\lambda(E\times F)=\sup\{\lambda(C\times D)\colon C, D \text{ компактные, } C\subset E, D\subset F\},$$

(2) для всякого борелевского E соответствие $F\to\lambda(E\times F)$ аддитивно на $\mathscr{B}(T)$,

(3) для всякого борелевского F соответствие $E\to\lambda(E\times F)$ аддитивно на $\mathscr{B}(S)$.

Тогда λ σ-аддитивно на $\mathscr{B}(S)$ o $\mathscr{B}(T)$ и однозначно продолжимо до регулярной борелевской меры ϱ на $S\times T$.

Далее определены повторные интегралы по λ для продукта ограниченных неотрицательных борелевских функций φ на S и ψ на T с компактным носителем и показано, что имеет место равенство обеих повторных интегралов.